Some reduced expressions of the classical Weyl groups and the Weyl groupoids of the Lie superalgebras $\text{osp}(2m|2n)$ (Hopf algebras and quantum groups: their possible applications)

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Abstract

We give some reduced expressions of the classical Weyl groups $W(A_{N-1}), W(B_N) = W(C_N), W(D_N)$ and the Weyl groupoid of the Lie superalgebra $osp(2m|2n)$.

1 Some reduced expressions of the classical Weyl groups

For $m, n \in \mathbb{Z}$, let $J_{n,m} := \{k \in \mathbb{Z} | m \leq k \leq n\}$.

Let $N \in \mathbb{N}$. Let $M_N(\mathbb{R})$ be the $\mathbb{R}$-algebra of $N \times N$-matrices. For $k, r \in J_{1,N}$, let $E_{k,r} := [\delta_{k,k'}\delta_{r,r'}]_{k',r' \in J_{1,N}} \in M_N(\mathbb{R})$, that is $E_{k,r}$ is the matrix unite such that its $(k, r)$-component is 1 and the other components is 0. Then $M_N(\mathbb{R}) = \oplus_{k,r \in J_{1,N}} \mathbb{R}E_{k,r}$. Let $\mathbb{R}^N$ denote the $\mathbb{R}$-linear space of $N \times 1$-matrices. For $k \in J_{1,N}$, let $e_k$ is the element of $\mathbb{R}^N$ such that its $(k,1)$-component is 1 and the other components is 0. That is $\{e_k | k \in J_{1,N}\}$ is the standard basis of $\mathbb{R}^N$. The $\mathbb{R}$-algebra $M_N(\mathbb{R})$ acts on $\mathbb{R}^N$ in the ordinal way, that is $E_{k,r}e_p = \delta_{r,p}e_r$. Let $GL_N(\mathbb{R})$ be the group of invertible $N \times N$-matrices, that is $GL_N(\mathbb{R}) = \{X \in M_N(\mathbb{R}) | \det X \neq 0\}$. Let $(\cdot, \cdot) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be the $\mathbb{R}$-bilinear map defined by $(e_k, e_r) := \delta_{kr}.$

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**Definition 1.1.** For \( v \in \mathbb{R}^N \setminus \{0\} \), define \( s_v \in GL_N(\mathbb{R}) \) by 
\[
 s_v(u) := u - \frac{2(u,v)}{(v,v)} v 
\]
\( (u \in \mathbb{R}^N) \), that is \( s_v \) is the reflection with respect to \( v \).

Note that

\[
 s_v^2 = 1.
\]

We say that a subset \( R \) of \( \mathbb{R}^N \setminus \{0\} \) is a root system (in \( \mathbb{R}^N \)) if 
\( |R| < \infty \), \( s_v(R) = R \) and \( \mathbb{R}v \cap R = \{ v, -v \} \) for all \( v \in R \), see [Hum, 1.1].

Let \( R \) be a root system in \( \mathbb{R}^N \). We say that a subset \( \Pi \) of \( R \) is a root basis of \( \Pi \) if \( \Pi \) is a (set) basis of \( \text{Span}_\mathbb{R}(\Pi) \) as an \( \mathbb{R} \)-linear space and 
\( R \subset \text{Span}_{\mathbb{R}_{\geq 0}}(\Pi) \cup -\text{Span}_{\mathbb{R}_{\geq 0}}(\Pi) \) (this is called a simple system in [Hum, 1.3]).

Let \( R \) be a root system in \( \mathbb{R}^N \). Let \( \Pi \) be a root basis of \( R \). Let \( R^+(\Pi) := R \cap \text{Span}_{\mathbb{R}_{\geq 0}}(\Pi) \). We call \( R^+(\Pi) \) a positive root system of \( R \) associated with \( \Pi \) (this is called a positive system in [Hum, 1.3]).

**Definition 1.2.** (See [Hum, 2.10].) Let \( R \) be a root system in \( \mathbb{R}^N \). Let \( \Pi \) be a root basis of \( R \).

1. Assume \( N \geq 2 \). We call \( R \) the \( A_{N-1} \)-type root system if

\[
 R = \{ e_x - e_y \mid x, y \in J_{1,N}, x \neq y \}.
\]

We call \( \Pi \) the \( A_{N-1} \)-type standard root basis if

\[
 \Pi = \{ e_x - e_{x+1} \mid x \in J_{1,N-1} \}.
\]

2. Assume \( N \geq 2 \). We call \( R \) the \( B_N \)-type standard root system if

\[
 R = \{ ce_x + c'e_y \mid x, y \in J_{1,N}, x < y, c, c' \in \{1, -1\} \} \cup \{ c''e_x \mid c'' \in \{1, -1\} \}.
\]

We call \( \Pi \) the \( B_N \)-type standard root basis if

\[
 \Pi = \{ e_x - e_{x+1} \mid x \in J_{1,N-1} \} \cup \{ e_N \}.
\]

3. Assume \( N \geq 2 \). We call \( R \) the \( C_N \)-type root system if

\[
 R = \{ ce_x + c'e_y \mid x, y \in J_{1,N}, x < y, c, c' \in \{1, -1\} \} \cup \{ 2c''e_x \mid c'' \in \{1, -1\} \}.
\]

We call \( \Pi \) the \( C_N \)-type standard root basis if

\[
 \Pi = \{ e_x - e_{x+1} \mid x \in J_{1,N-1} \} \cup \{ 2e_N \}.
\]
(4) Assume $N \geq 4$. We call $R$ the $D_N$-type root system if

$$R = \{ ce_x + c'e_y \mid x, y \in J_{1,N}, x < y, c, c' \in \{1, -1\} \}.$$  

We call $\Pi$ the $D_N$-type standard root basis if

$$\Pi = \{ e_x - e_{x+1} \mid x \in J_{1,N-1} \} \cup \{ e_{N-1} + e_N \}.$$  

Let $R$ be a root system in $\mathbb{R}^N$. Let $\Pi$ be a root basis of $R$. We call $W(\Pi)$ the Coxeter group associated with $(R, \Pi)$. Let $S(\Pi) := \{ s_v \in W(\Pi) \mid v \in \Pi \}$. We call $(W(\Pi), S(\Pi))$ the Coxeter system associated with $(R, \Pi)$, see [Hum, 1.9 and Theorem 1.5]. Define the map $\ell : W(\Pi) \to \mathbb{Z}_{\geq 0}$ in the following way, see [Hum, 1.6]. Let $\ell(1) := 0$, where 1 is a unit of $W(\Pi)$. Note that an arbitrary $w \in W(\Pi)$ can be written as a product of finite $s_v$'s with some $v \in \Pi$, say $w = s_{v_1} \cdots s_{v_r}$ for some $r \in \mathbb{N}$ and some $v_x \in \Pi$ ($x \in J_{1,r}$). If $w \neq 1$, let $\ell(w)$ be the smallest $r$ for which such an expression exists, and call the expression reduced. For $w \in W(\Pi)$, we call $\ell(w)$ the length of $w$. Let

$$\mathcal{L}(w) := \{ v \in R^+(\Pi) \mid w(v) \in -R^+(\Pi) \}. $$

It is well-known that

(1.2) \quad $\ell(w) = |\mathcal{L}(w)|$

(see [Hum, Corollary 1.7]). It is also well-known that for $v \in \Pi$,

(1.3) \quad $s_v(R^+(\Pi) \setminus \{v\}) = R^+(\Pi) \setminus \{v\}$

(see [Hum, Propsoition 1.4]), and

(1.4) \quad $\ell(ws_v) = \begin{cases} \ell(w) + 1 & \text{if } w(v) \in R^+(\Pi), \\ \ell(w) - 1 & \text{if } w(v) \in -R^+(\Pi) \end{cases}$

(see [Hum, Lemma 1.6 and Corollary 1.7]). Assume that $|R| < \infty$. By the above properties, we can see that there exists a unique $w_o \in W(\Pi)$ such that $w_o(\Pi) = -\Pi$, see [Hum, 1.8]. It is well-known that

(1.5) \quad $\ell(w_o) = |R^+(\Pi)|$, 

which can easily be proved by (1.2), (1.3) and (1.4). Note that \( w_o \) is the only element \( W(\Pi) \) that \( \ell(w) \leq \ell(w_o) \) for all \( w \in W(\Pi) \), and \( \ell(w) = \ell(w_o) - \ell(w_o w^{-1}) \) for all \( w \in W(\Pi) \). We call \( w_o \) the longest element of the Coxeter system of \( (W(\Pi), S(\Pi)) \).

Let \( k, r \in J_{1,N} \) be such that \( k \leq r \). For \( z_p \in J_{k,r} \cup (-J_{k,r}) \) \((p \in J_{k,r})\) with \(|u_p| \neq |u_t| \) \((p \neq t)\), let

\[
\left\{ \begin{array}{cc}
k & k+1 \\
k & z_{k+1} \\
\vdots & \vdots \\
r & z_r
\end{array} \right\} := \sum_{p \in J_{k,r}} \frac{z_p}{|z_p|} E_{|z_p|,p} + \sum_{t \in J_{1,N \setminus J_{k,r}}} E_{t,t} \in GL_N(\mathbb{R}).
\]

We have

(1.6) \hspace{1cm} s_{e_k} = \left\{ \begin{array}{cc} k \\
-k \end{array} \right\} \hspace{1cm} (k \in J_{1,N}),

(1.7) \hspace{1cm} s_{e_k-e_{k+1}} = \left\{ \begin{array}{cc}
k & k+1 \\
+1 & k
\end{array} \right\} \hspace{1cm} (k \in J_{1,N-1}),

and

(1.8) \hspace{1cm} s_{e_k+e_{k+1}} = \left\{ \begin{array}{cc}
k & k+1 \\
-(k+1) & -k
\end{array} \right\} \hspace{1cm} (k \in J_{1,N-1}).

Let \( k, p, r \in J_{k,r} \) with \( k < r \) and \( k \leq p \leq r \), let

\[
\left\{ \begin{array}{cc}
k & \ldots & p \\
 & z_{k} \\
 & \ldots \\
p+1 & \ldots & r
\end{array} \right\} := \left\{ \begin{array}{cc}
k & \ldots & p \\
 & z_{k} \\
 & \ldots \\
p+1 & \ldots & r
\end{array} \right\}
\]

Let \( k, r \in J_{1,N-1} \) with \( k \leq r \). Define \( s_{(k,r)} \) inductively by

(1.9) \hspace{1cm} s_{(k,r)} := \left\{ \begin{array}{cl} 1 & \text{if } k = r \\
s_{(k,r-1)} s_{e_{r-1} - e_r} & \text{if } k < r. \end{array} \right.

Then, if \( r > k \), we have

(1.10) \hspace{1cm} s_{(k,r)} = \left\{ \begin{array}{cc} k & \ldots & p \\
k+1 & \ldots & p+1 \\
r-1 & \ldots & r \end{array} \right\},

since (if \( r \geq k + 2 \))

\[
s_{(k,r)} = s_{(k,r-1)} s_{e_{r-1} - e_r} = \left\{ \begin{array}{cc}
k & \ldots & p \\
k+1 & \ldots & p+1 \\
r-2 & \ldots & r-1 \end{array} \right\} \left\{ \begin{array}{cc} r-1 & r \\
r & r-1 \end{array} \right\} = \left\{ \begin{array}{cc}
k & \ldots & p \\
k+1 & \ldots & p+1 \\
r-1 & \ldots & r \end{array} \right\}.
\]

(by (1.7) and an induction)
Define $s_{(r,k)}$ inductively by $s_{(r,k)} := s_{e_{r-1} - e_r} s_{(r-1,k)}$ if $r \geq k + 1$. Clearly (if $r > k$) we have

\[(1.12) \quad s_{(r,k)} = s_{(k,r)}^{-1} = \left\{ \begin{array}{llllllllll} k & k+1 & \ldots & p & \ldots & r \\ r & k & \ldots & p-1 & \ldots & r-1 \end{array} \right\}.\]

**Lemma 1.3.** Let $\Pi$ be the $A_{N-1}$-type standard root basis. Let $w_o$ be the longest element of $(W(\Pi), S(\Pi))$. Let $s_k := s_{e_k - e_{k+1}} \in S(\Pi)$ for $k \in J_{1,N-1}$.

1. We have

\[(1.13) \quad w_o = \left\{ \begin{array}{llllll} 1 & \ldots & p & \ldots & N \\ N & \ldots & N-p+1 & \ldots & 1 \end{array} \right\}.

Moreover

\[(1.14) \quad w_o = \left( s_1 s_2 \cdots s_{N-1} \right) \left( s_1 s_2 \cdots s_{N-2} \right) \cdots \left( s_1 s_2 \right).\]

Furthermore RHS of (1.14) is the reduced expression of $w_o$.

2. Let $m \in J_{2,N-1}$. Then

\[(1.15) \quad w_o = \left( s_1 s_2 \cdots s_{m-1} \right) \left( s_1 s_2 \cdots s_{m-2} \right) \cdots \left( s_1 s_2 \right) \left( s_{m+1} s_{m+2} \cdots s_{N-1} \right) \left( s_{m+1} s_{m+2} \cdots s_{N-2} \right) \cdots \left( s_{m+1} s_{m+2} \right).\]

and RHS of (1.15) is a reduced expression of $w_o$.

**Proof.** By (1.5), we have

\[(1.16) \quad \ell(w) = \frac{N(N-1)}{2}.\]

Let $k, r \in J_{1,n}$ with $k < r$. Let

\[x_{(k,r)} := \left\{ \begin{array}{llllll} k & \ldots & p & \ldots & r \\ r & \ldots & r-p+k & \ldots & k \end{array} \right\}.\]
Then

\[(1.17) \quad s_{(k,r)}s_{(k,r-1)}\cdots s_{(k,k+1)} = x_{(k,r)}, \]

since, if \( r \geq k + 2 \), we have

\[
\begin{align*}
s_{(k,r)}(s_{(k,r-1)}\cdots s_{(k,k+1)}) &= \left\{ \begin{array}{lllllll}
k & \cdots & p & \cdots & r-1 & ; & r \\
k+1 & \cdots & p+1 & \cdots & r & ; & k
\end{array} \right\} \cdot x_{(k,r-1)} \\
&= x_{(k,r)}.
\end{align*}
\]

We have

\[(1.18) \quad x_{(k,r)} \in W(\Pi) \quad \text{and} \quad \ell(x_{(k,r)}) = \frac{(k-r+1)(k-r)}{2}, \]

where the first claim follows from \((1.17)\) and the second claim follows from \((1.2)\), since \( \mathfrak{L}(x_{(k,r)}) = \{e_x - e_y | k \leq x < y \leq r\} \).

We obtain the claim \((1)\) from \((1.16)\). \((1.17)\) and \((1.18)\) for \( k = 1 \) and \( r = N \).

For \( k, r, t \in J_{1,N-1} \) with \( k < r \leq t \), let

\[(1.19) \quad y_{(k,r-1;r,t)} := \left\{ \begin{array}{llllllll}
k & \cdots & z & \cdots & r-1 & ; & r & \cdots & y & \cdots & t \\
k+1 & \cdots & z+1 & \cdots & t-1 & ; & k & \cdots & y+k-r & \cdots & t+k-r
\end{array} \right\} \]

We have

\[(1.20) \quad s_{(k+t-r,t)}s_{(k+t-r-1,t-1)}\cdots s_{(k+1,r+1)}s_{(k,r)} = y_{(k-1,r;r,t)} \]

since, if \( t > r \),

\[
\begin{align*}
(s_{(k+t-r,t)}s_{(k+t-r-1,t-1)}\cdots s_{(k+1,r+1)})s_{(k,r)} &= y_{(k+1,r;r+1,t)} \cdot \left\{ \begin{array}{lllllll}
k & \cdots & p & \cdots & r-1 & ; & r \\
k+1 & \cdots & p+1 & \cdots & r & ; & k
\end{array} \right\} \\
&= y_{(k,r-1;r,t)}.
\end{align*}
\]
We have
\[(1.21) \quad y_{(k,r-1;r,t)} \in W(\Pi) \quad \text{and} \quad \ell(y_{(k,r-1;r,t)}) = (t - r + 1)(r - k),\]
where the first claim follows from (1.20) and the second claim follows from by (1.2), since \( \mathcal{L}(x_{(k,r)}) = \{e_x - e_y | x \in J_{k,r-1}, x \in J_{r,t}\} \).

Let \( m \in J_{2,N-1} \). By (1.13), we have
\[(1.22) \quad w_o = x_{(1,m)}x_{(m+1,N)}y_{(1,N-m;N-m+1,N)}.\]

Then we obtain the claim (2) from (1.16), (1.18), (1.21) and (1.22), since \( \frac{m(m-1)}{2} + \frac{(N-m)(N-m-1)}{2} + (N - m)m = \frac{N(N-1)}{2} \).

Let \( k, r \in J_{1,N} \) with \( k \leq r \). Let
\[(1.23) \quad b_{(k,r)} := s_{e_k} \cdots s_{e_r} = \begin{cases} k & \ldots & p & \ldots & r \\ -k & \ldots & -p & \ldots & -r \end{cases},\]
see also (1.6). By (1.10), we have
\[(1.24) \quad (s_{(k,r)})^{r-k+1} = 1.\]

By (1.6) and (1.10), we have
\[(1.25) \quad s_{e_t}s_{(k,r)} = s_{(k,r)}s_{e_{t-1}}\]

By (1.23), (1.24) and (1.25), for \( t \in J_{k+1,r} \), we have
\[(1.26) \quad (s_{(k,r)}s_{e_r})^{r-k+1} = (s_{(k,r)})^{r-k+1}s_{e_k} \cdots s_{e_r} = b_{(k,r)}.\]

By (1.6), (1.10) and (1.12), we have
\[(1.27) \quad s_{e_k-e_{k+1}} \cdots s_{e_r-e_r} s_{e_r} s_{e_{r-1}-e_r} \cdots s_{e_k-e_{k+1}} = s_{(k,r)}s_{e_r}s_{(r,k)} = s_{e_k}.\]

**Lemma 1.4.** Let \( \Pi \) be the \( B_N \)-type standard root basis. Let \( w_o \) be the longest element of \( (W(\Pi), S(\Pi)) \). Let \( s_k := s_{e_k-e_{k+1}} \in S(\Pi) \) for \( k \in J_{1,N-1} \) and let \( s_N := s_{e_N} \in S(\Pi) \).

\( 1 \) We have
\[(1.28) \quad w_o = b_{(1,N)} = (s_1s_2 \cdots s_N)^N.\]
Moreover the rightmost hand side of (1.28) is a reduced expression of $w_o$.

(2) Let $k, r \in J_{1,N}$ with $k \leq r$. Then

\begin{equation}
(1.29) \quad b_{(k,r)} = \left(\frac{s_k s_{k+1} \cdots s_{N-1} s_N s_{N-1} \cdots s_{r+1} s_r}{2N-k-r+1}\right)^{r-k+1}.
\end{equation}

Moreover RHS of (1.29) is a reduced expression of $b_{(k,r)}$.

(3) Let $k_1, k_2, \ldots, k_{r-1} \in J_{1,N}$ with $k_1 < k_2 < \ldots < k_{r-1}$. Let $b'_y := b_{(k_y-1,k_y)}$ ($y \in J_{1,r}$), where let $k_0 := 1$ and $k_r := N + 1$. Then we have $w_o = b'_1 b'_2 \cdots b'_r$ and $\ell(w_o) = \sum_{y=1}^r \ell(b'_y)$. Moreover $b'_y b'_z = b'_z b'_y$ for $y, z \in J_{1,r}$.

(4) Let $m \in J_{1,N-1}$. Then

\begin{equation}
(1.30) \quad w_o = \left(\frac{s_{N-m+1} s_{N-m+2} \cdots s_N}{m}\right)^m \left(\frac{s_1 s_2 \cdots s_{N-1} s_N \cdots s_{N-m+1} s_{N-m}}{N+m}\right)^{N-m}.
\end{equation}

Moreover RHS of (1.30) is a reduced expression of $w_o$.

Proof. We can easily show (1.29) by (1.26) and (1.27).

Let $k, r \in J_{1,N}$ be such that $k \leq r$. Note that

$$\mathcal{L}(b_{(k,r)}) = \{ e_t \mid t \in J_{k,r} \} \cup \{ e_t + ce_t' \mid c \in \{-1, 1\}, t \in J_{k,r}, t' \in J_{t',N} \}.$$ 

Hence by (1.2), we have

\begin{equation}
(1.31) \quad \ell(b_{(k,r)}) = (r - k + 1) + 2 \sum_{t=k}^r (N - t) \\
= (r - k + 1) + 2N(r - k + 1) - 2\left(\frac{r(r+1)}{2} - \frac{k(k-1)}{2}\right) \\
= (r - k + 1)(1 + 2N - (r + k)) \\
= (2N - k - r + 1)(r - k + 1).
\end{equation}

Hence we obtain the second claim of the claim (2). We also obtain the claim (1) since $|R^+(\Pi)| = N^2$.

Let $k, t, r \in J_{1,N}$ be such that $k \leq t < r$. By (1.23), we have

\begin{equation}
(1.32) \quad b_{(k,t)} b_{(t+1,r)} = b_{(k,r)}.
\end{equation}
By (1.31), we have

\[
\ell(b_{(k,t)}) + \ell(b_{(t+1,r)})
= (2N - k - t + 1)(t - k + 1) + (2N - t - r)(r - t)
= 2N(r - k + 1) - (k + t - 1)(t - k + 1) - (t + r)(r - t)
= 2N(r - k + 1) - (-k^2 + t^2 + 2k - 1) - (r^2 - t^2)
= 2N(r - k + 1) + (k^2 - r^2 - 2k + 1)
= 2N(r - k + 1) + (k - 1 + r)(k - 1 - r)
= (2N - r - k - 1)(r - k + 1)
= \ell(b_{(k,r)}).
\]

(1.33)

By (1.32), (1.32) and the claim (1), we get the claim (3).

The claim (4) follows immediately from the claims (1) and (2). \[\square\]

Using Lemma 1.4, we have

**Lemma 1.5.** Let $\Pi$ be the $D_N$-type standard root basis. Let $w_o$ be the longest element of $(W(\Pi), S(\Pi))$. Let $s_k := s_{e_k - e_{k+1}} \in S(\Pi)$ for $k \in J_{1,N-1}$ and let $s_N := s_{e_k + e_{k+1}} \in S(\Pi)$. For $k \in J_{1,N-1}$, let

\[
(1.34) \quad d_{(k)} := \left( s_k \cdots s_{N-2}s_{N-1}s_N \right)^{N-k}.
\]

Then

\[
(1.35) \quad \ell(d_{(k)}) = (N - k)(N - k + 1)
\]

and

\[
(1.36) \quad d_{(k)} = \begin{cases} 
  b_{(k,N)} & \text{if } N - k \text{ is odd}, \\
  b_{(k,N-1)} & \text{if } N - k \text{ is even}.
\end{cases}
\]

In particular,

\[
(1.37) \quad w_o = d_{(1)}.
\]
Proof. By (1.6), (1.7) and (1.8), we have

\begin{equation}
 s_{N-1}s_N = \begin{pmatrix} N-1 & N \\ -(N-1) & -N \end{pmatrix} = s_{e_{N-1}}s_{e_N}.
\end{equation}

Then we have

\begin{align*}
\text{RHS of (1.34)} & = (s_{(k,N-1)}s_{e_{N-1}}s_{e_N})^{N-k} \quad \text{(by (1.38))} \\
& = (s_{(k,N-1)}s_{e_{N-1}}s_{e_N})^{N-k}s_{e_N}^{N-k} \quad \text{(by (1.6) and (1.10))} \\
& = b_{(k,N-1)}s_{e_N}^{N-k} \quad \text{(by (1.26))} \\
& = \text{RHS of (1.36)}
\end{align*}

By (1.36), we have

\[ \mathcal{L}(d_{(k)}) = \{ e_t + ce_{t'} | c \in \{-1, 1\}, t \in J_{k,r}, t' \in J_{k',r'} \}. \]

Hence by (1.2), we have (1.35) and (1.37). This completes the proof. $\square$

2 Weyl groupoids of super CD-type

Let $m \in J_{1,N-1}$. Let $\mathcal{D}_{m|N-m}$ be the set of maps $a : J_{1,n} \rightarrow J_{0,1}$ with $|a^{-1}(\{0\})| = m$.

Let $a \in \mathcal{D}_{m|N-m}$. Let $(,)^a : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ be the $\mathbb{R}$-bilinear map defined by $(e_i, e_j)^a := \delta_{ij} \cdot (-1)^{a(i)}$. For $v \in \mathbb{R}^N$ with $(v, v)^a \neq 0$, define $s_v \in \text{GL}_N(\mathbb{R})$ by $s_v^a(u) := u - \frac{2(u,v)^a}{(v,v)^a} v \ (u \in \mathbb{R}^N)$.

Let

\[ \hat{\mathcal{D}}_{m|N-m} := \{ (a, d) \in \mathcal{D}_{m|N-m} \times J_{0,1} | d \in J_{0,a(N)} \}. \]
For \( i \in J_{1,N} \), define the bijection \( \tau_{i} : \dot{\mathcal{D}}_{m|N-m} \rightarrow \dot{\mathcal{D}}_{m|N-m} \) by

\[
\tau_{i}(a, d) := \begin{cases} 
(a \circ s_{e_i-e_{i+1}}, d) & \text{if } i \in J_{1,N-2} \text{ and } a(i) \neq a(i+1), \\
(a \circ s_{e_{N-1}-e_{N}}, d) & \text{if } i = N - 1, \ d = 0 \text{ and } a(N - 1) \neq b(N), \\
(a \circ s_{e_{N-1}-e_{N}}, 1) & \text{if } i = N, \ a(N - 1) = 1, \ a(N) = 0, \\
(a \circ s_{e_{N-1}-e_{N}}, 0) & \text{if } i = N, \ a(N - 1) = 0, \ a(N) = 1 \text{ and } d = 1, \\
(a, d) & \text{otherwise.}
\end{cases}
\]

Then \( \tau_{i}^{2} = \text{id}_{\mathbb{R}^{N}} \).

Let \((a, d) \in \dot{\mathcal{D}}_{m|N-m}\).

\[
R_{+}^{(a,d)} := \{ e_{x} + te_{y} | x, y \in J_{1,N}, \ x < y, \ t \in \{1, -1\} \} 
\]

\[
\cup \{ 2e_{z} | z \in J_{1,N}, \ a(z) = 1 \},
\]

and \( R^{(a,d)} := R_{+}^{(a,d)} \cup -R_{+}^{(a,d)} \). Then

\[
|R_{+}^{(a,d)}| = N(N - 1) + (N - m) = N^2 - m.
\]

For \( i \in J_{1,N} \), let

\[
\alpha_{i}^{(a,d)} := \begin{cases} 
\alpha_{i} & \text{if } i \in J_{1,N-2}, \\
e_{N-1} - e_{N} & \text{if } i = N - 1 \text{ and } d = 0, \\
2e_{N} & \text{if } i = N - 1 \text{ and } d = 1, \\
e_{N-1} + e_{N} & \text{if } i = N, \ a(N) = 0 \text{ and } d = 0, \\
2e_{N} & \text{if } i = N, \ a(N) = 1 \text{ and } d = 0, \\
e_{N-1} - e_{N} & \text{if } i = N, \ d = 1.
\end{cases}
\]

Let \( \Pi^{(a,d)} := \{ \alpha_{i}^{(a,d)} | i \in J_{1,N} \} \). Then \( \Pi^{(a,d)} \) is an \( \mathbb{R} \)-basis of \( \mathbb{R}^{N} \). Moreover

\[
\Pi^{(a,d)} \subset R_{+}^{(a,d)} \subset (\bigoplus_{i=1}^{N} \mathbb{Z}_{\geq 0} \alpha_{i}^{(a,d)}) \setminus \{0\}.
\]
Note that
\[ \tau_i(a, d) = (a, d) \text{ if and only if } (\alpha_i^{(a,d)}, \alpha_i^{(a,d)})^a \neq 0. \]

For \( i \in J_{1,N} \), define \( s_i^{(a,d)} \in \text{GL}_N(\mathbb{R}) \) by
\[ s_i^{(a,d)}(\alpha_i^{(a,d)}) :=
\begin{cases}
-\alpha_i^{\tau_i(a,d)} & \text{if } i = j, \\
\alpha_j^{\tau_j(a,d)} & \text{if } i \neq j \text{ and } (\alpha_i^{(a,d)}, \alpha_i^{(a,d)})^a \neq 0, \\
\alpha_j^{\tau_j(a,d)} + \alpha_i^{\tau_i(a,d)} & \text{if } i \neq j, (\alpha_i^{(a,d)}, \alpha_i^{(a,d)})^a = 0 \text{ and } (\alpha_i^{(a,d)}, \alpha_j^{(a,d)})^a \neq 0.
\end{cases} \]

We can directly see

**Lemma 2.1.** Let \((a, d) \in \mathcal{D}_{m|N-m}\) and \( i \in J_{1,N} \). Assume that \( d = 0 \). Assume that \( i \in J_{1,N-1} \) if \( a(N-1) = 1 \) and \( a(N) = 0 \). Then \( s_i^{(a,d)} = s_{\alpha_i^{(a,d)}} \), where \( s_{\alpha_i^{(a,d)}} \) is the one of Definition 1.1.

**Notation.** Let \((a, d) \in \mathcal{D}_{m|N-m}\). Let \( \text{Map}_0^N \) be a set with \(|\text{Map}_0^N| = 1 \). For \( r \in \mathbb{N} \), let \( \text{Map}_r^N \) be the set of all maps from \( J_{1,r} \) to \( J_{1,N} \). Let \( \text{Map}_\infty^N \) be the set of all maps from \( \mathbb{N} \) to \( J_{1,N} \). For \( r \in \mathbb{Z}_{\geq 0}, \ f \in \text{Map}_r^N \cup \text{Map}_\infty^N \) and \( t \in J_{1,r} \), let
\[ (a, d)_{f,0} := (a, d), \quad 1^{(a,d)}s_{f,0} := \text{id}_{\mathbb{R}^N} \]
\[ (a, d)_{f,t} := \tau_i((a, d)_{f,t-1}), \quad 1^{(a,d)}s_{f,t} := 1^{(a,d)}s_{f,t-1}s_{f(t)}^{(a,d)_{f,t}}. \]

**Proposition 2.2.** Let \((a, d) \in \mathcal{D}_{m|N-m}\) be such that \( d = 0 \), \( b(z) = 1 \) (\( z \in J_{1,N-m} \)) and \( b(z') = 0 \) (\( z' \in J_{N-m+1,N} \)). Let \( n := |P_+^{(a,d)}| \). Define \( f \in \text{Map}_n^N \) by
\[
(2.2) \quad f(t) := \begin{cases}
N - m + t & (\text{if } t \in J_{1,m}), \\
f(t - m) & (\text{if } t \in J_{m+1,m(m-1)}), \\
t - m(m - 1) & (\text{if } t \in J_{m(m-1)+1,m(m-1)+N}), \\
2N + m(m - 1) - t & (\text{if } t \in J_{m(m-1)+N+1,m^2+N}), \\
f(t - (N + m)) & (\text{if } t \in J_{m^2+N+1,n}).
\end{cases}
\]
Then

\[(2.3) \quad 1^{(a,d)} s_{f,n} = \begin{cases} b_{(1,N)} & \text{if } m \text{ is odd}, \\ b_{(1,N-1)} & \text{if } m \text{ is even}. \end{cases}\]

**Proof.** For \(y \in J_{1,m}\), define \(a^{(y)} \in \mathcal{D}_{m|N-m}\) by

\[
a^{(y)}(z) := \begin{cases} 1 & \text{if } z \in J_{1,N-m-1} \cup \{N - m + y\}, \\ 0 & \text{if } z \in J_{N-m,N-m+y-1} \cup J_{N-m+y+1,N}. \end{cases}
\]

Then we can directly see that for \(t \in J_{1,n}\),

\[
(a, d)_{f,t} = \begin{cases} (a, d) & \text{if } t \in J_{1,m(m-1)+N-m-1}, \\ (a^{(t-(N-m-1))}, 0) & \text{if } t \in J_{m(m-1)+N-m(m-1)+N-1}, \\ (a^{(m-(t-(m(m-1)+N))}), 0) & \text{if } t \in J_{m(m-1)+N,m(m-1)+N+m}, \\ (a, d)_{f,t-(N+m)} & \text{if } t \in J_{m^2+N+1,n}. \end{cases}
\]

So we see that for \(t \in J_{1,n}\),

\[(2.4) \quad s_{f(t)}^{(a,d)} = \begin{cases} s_{e_{f(t)}-e_{f(t)+1}} & \text{if } f(t) \in J_{1,N-1}, \\ s_{e_{N-1}+e_N} & \text{if } t \in J_{1,m(m-1)} \text{ and } f(t) = N, \\ s_{2e_N} (= s_{e_N}) & \text{if } t \in J_{m(m-1)+1,n} \text{ and } f(t) = N. \end{cases}
\]

Define \(f' \in \text{Map}_{n-m(m-1)}^{N}\) by \(f'(t) := f(t + m(m - 1))\), so

\[(2.5) \quad 1^{(a,d)} s_{f,n} = 1^{(a,d)} s_{f,m(m-1)} 1^{(a,d)} f_{n-m(m-1)}.\]

By (1.29) and (1.36), \(1^{(a,d)} s_{f,m(m-1)} \) equals \(b_{(N-m+1,N)}\) (resp. \(b_{(N-m+1,N-1)}\)) if \(m\) is odd (resp. even). By (1.29) and (2.4), \(1^{(a,d)} s_{f',n-m(m-1)} = b_{(1,N-m)}\). Hence by (1.22) and (2.5), we have (2.3), as desired. \(\square\)

For \((a, d) \in \dot{\mathcal{D}}_{m|N-m}\) and \(i, j \in J_{1,N}\), define \(C^{(a,d)} = [c_{ij}^{(a,d)}]_{i,j \in J_{1,N}} \in M_N(\mathbb{Z})\) by

\[
s_i^{(a,d)}(\alpha_j^{(a,d)}) = \alpha_j^{(a,d)} - c_{ij}^{(a,d)} \alpha_i^{(a,d)}.\]
Then $C^{(a,d)}$ is a generalized Cartan matrix, i.e., (M1) and (M2) below hold.

(M1) $c_{ii}^{(a,d)} = 2$ ($i \in J_{1,N}$).
(M2) $c_{jk}^{(a,d)} \leq 0, \delta_{c_{jk}^{(a,d)},0} = \delta_{c_{kj}^{(a,d)},0}$ ($j, k \in J_{1,N}, j \neq k$).

Then the data

$$\dot{C}_{m|N-m} := C(J_{1,N}, \dot{D}_{m|N-m}, (\tau_{i})_{i \in J_{1,N}}, (C^{(a,d)})_{(a,d) \in \dot{D}_{m|N-m}})$$

is a (rank-N) Cartan scheme, i.e., (C1) and (C2) below hold.

(C1) $\tau_{i}^{2} = id_{\dot{D}_{m|N-m}}$ ($i \in J_{1,N}$).
(C2) $c_{ij}^{\tau_{i}(a,d)} = c_{ij}^{(a,d)}$ ($i \in J_{1,N}$).

Note that

$$-c_{ij}^{(a,d)} = |R_{+}^{(a,d)} \cap (\mathbb{Z}\alpha_{i}^{(a,d)} \oplus \mathbb{Z}\alpha_{j}^{(a,d)})| (i, j \in J_{1,N}, i \neq j).$$

The data

$$\dot{\mathcal{R}}_{m|N-m} := \mathcal{R}(\dot{C}_{m|N-m}, (R_{+}^{(a,d)})_{(a,d) \in \dot{D}_{m|N-m}})$$

is a generalized root system of type $C$, i.e., (R1)-(R4) below hold.

(R1) $R^{(a,d)} = R_{+}^{(a,d)} \cup -R_{+}^{(a,d)}$ ($(a, d) \in \dot{D}_{m|N-m}$).
(R2) $R^{(a,d)} \cap \mathbb{Z}\alpha_{i} = \{ \alpha_{i}, -\alpha_{i} \}$ ($(a, d) \in \dot{D}_{m|N-m}, i \in J_{1,N}$).
(R3) $s_{i}^{(a,d)}(R^{(a,d)}) = R^{\mathcal{T}i(a,d)}$ ($(a, d) \in \dot{D}_{m|N-m}, i \in J_{1,N}$).
(R4) $(\tau_{i}\tau_{j})^{-c_{ij}^{(a,d)}}(a, d) = (a, d)$ ($(a, d) \in \dot{D}_{m|N-m}, i, j \in J_{1,N}$).

For $(a, d) \in \dot{D}_{m|N-m}$, let

$$W^{(a,d)} := \{ 1^{(a,d)}s_{f,r} \in GL_{N}(\mathbb{R}) | r \in \mathbb{Z}_{\geq 0}, f \in \text{Map}_{r}^{N} \},$$

and define the map $\ell^{(a,d)} : W^{(a,d)} \to \mathbb{Z}_{\geq 0}$ by

$$\ell^{(a,d)}(w) := \min\{ r \in \mathbb{Z}_{\geq 0} | \exists f \in \text{Map}_{r}^{N}, w = 1^{(a,d)}s_{f,r} \}.$$

By [HY08, Lemma 8 (iii)], we see that

$$1^{(a,d)}s_{f,r} = 1^{(a,d)}s_{f',r'} \text{ implies } (a, d)_{f,r} = (a, d)_{f',r'},$$
and that 

(2.7) \[ \ell^{(a,d)}(w) = |w^{-1}(R_{+}^{(a,d)}) \cap - \oplus_{i=1}^{N} \mathbb{Z}_{\geq 0}\alpha_{i}|. \]

For \((a, d) \in \mathcal{D}_{m|N-m}, w \in W^{(a,d)}, f \in \text{Map}_{\ell(a,d)}^{N}(w),\) if \(w = 1^{(a,d)}s_{f,\ell(a,d)}(w)\), we call \(f\) a reduced word map of \(w\).

By (2.6) and (2.7), we have formulas for \(W^{(a,d)}\) similar to (1.3) and (1.4). In particular, for each \((a, d) \in \mathcal{D}_{m|N-m},\) there exists a unique \(w_{0}^{(a,d)} \in W^{(a,d)}\) such that

\[ \ell^{(a,d)}(w_{0}^{(a,d)}) = |R_{+}^{(a,d)}|, \]

and we call \(w_{0}^{(a,d)}\) the longest element of \(W^{(a,d)}\).

By Proposition 2.2, we have

**Theorem 2.3.** Let \((a, d) \in \mathcal{D}_{m|N-m}\) be such that \(d = 0, a(z) = 1 (z \in J_{1,N-m})\) and \(a(z') = 0 (z' \in J_{N-m+1,N})\). Then a reduced word map of \(w_{0}^{(a,d)}\) is given by (2.2). Moreover,

(2.8) \[ w_{0}^{(a,d)} = \begin{cases} b_{(1,N)} & \text{if } m \text{ is odd}, \\ b_{(1,N-1)} & \text{if } m \text{ is even}. \end{cases} \]

**Definition 2.4.** For \((a, d), (a', d') \in \mathcal{D}_{m|N-m}\), let \(W^{(a,d)}_{(a',d')}\) be the subset of \(W^{(a,d)}\) composed of all the elements \(1^{(a,d)}s_{f,r}\) with \(r \in \mathbb{Z}_{\geq 0}, f \in \text{Map}_{r}^{N}\) and \((a, d)_{f,r} = (a', d')\), and \(\mathcal{H}^{(a,d)}_{(a',d')} := \{(a, d)\} \times W^{(a,d)}_{(a',d')} \times \{(a', d')\}(\subset \mathcal{D}_{m|N-m} \times \text{GL}_{N}(\mathbb{R}) \times \mathcal{D}_{m|N-m})\). Let

\[ (\mathcal{W}_{m|N-m})' := \bigcup_{(a,d),(a',d') \in \mathcal{D}_{m|N-m}} \mathcal{H}^{(a,d)}_{(a',d')}, \]

and \(\dot{\mathcal{W}}_{m|N-m} := (\mathcal{W}_{m|N-m})' \cup \{o\}\), where \(o\) is an element such that \(o \notin (\mathcal{W}_{m|N-m})'\). We regard \(\mathcal{W}_{m|N-m}\) as the semigroup by \(o\omega := \omega o := o (\omega \in \mathcal{W}_{m|N-m})\) and

\[ ((a_{1}, d_{1}), w_{1}, (a_{2}, d_{2}))(a_{3}, d_{3}), w_{2}, (a_{4}, d_{4})) := \begin{cases} ((a_{1}, d_{1}), w_{1}w_{2}, (a_{4}, d_{4})) & \text{if } (a_{2}, d_{2}) = (a_{3}, d_{3}), \\ o & \text{if } (a_{2}, d_{2}) \neq (a_{3}, d_{3}). \end{cases} \]

We call \(\dot{\mathcal{W}}_{m|N-m}\) the Weyl groupoid of the Lie superalgebra \(\text{osp}(2m|2(N-m))\).
For \((a, d) \in \dot{\mathcal{D}}_{m|N-m}\), let 
\(\epsilon^{(a,d)} := ((a, d), \text{id}_{\mathbb{R}^{N}}, (a, d)) \in \mathcal{H}_{(a,d)}^{(a,d)}\). For \((a, d) \in \dot{\mathcal{D}}_{m|N-m}\) and \(i \in J_{1,N}\), let 
\(\sigma_{i}^{(a,d)} := (\tau_{i}(a, d), s_{i}^{(a,d)}, (a, d)) \in \mathcal{H}_{\tau_{i}(a,d)}^{(a,d)}\). For \(r \in \mathbb{Z}_{\geq 0}\), \(t \in J_{0,r}\) and \(f \in \text{Map}_{r}^{N}\), let 
\(1^{(a,d)}s_{f,r} := ((a, d), 1^{(a,d)}s_{f,r}, (a, d)_{f,r}) \in \mathcal{H}_{(a,d)_{f,r}}^{(a,d)}(t \in \mathbb{N})\).

For \(i, j \in J_{1,N}\), define 
\(f_{ij} \in \text{Map}_{\infty}^{N}\) by 
\(f_{ij}(2t-1) := i, f_{ij}(2t) := j, (t \in \mathbb{N})\).

By [HY08, Theorem 1], we have

**Theorem 2.5.** The semigroup \(\dot{\mathcal{W}}_{m|N-m}\) can also be defined by the generators 
\(0, \epsilon^{(a,d)}, \sigma_{i}^{(a,d)} ((a, d) \in \dot{\mathcal{D}}_{m|N-m}, i \in J_{1,N})\), and relations 
\(\omega \epsilon = \epsilon \omega = \omega \quad (\omega \in \dot{\mathcal{W}}_{m|N-m}), \epsilon^{(a,d)}\epsilon^{(a,d)} = \epsilon^{(a,d)}, \epsilon^{(a,d)}\epsilon^{(a',d')} = o \quad ((a, d) \neq (a', d'))
\(\epsilon^{\tau_{i}(a,d)}\sigma_{i}^{(a,d)} = \sigma_{i}^{(a,d)}\epsilon^{(a,d)} = \sigma_{i}^{(a,d)}, \sigma_{i}^{(a,d)}\sigma_{i}^{(a,d)}} = \epsilon^{(a,d)}, \)
\(1^{(a,d)}\sigma_{ij} = 1^{(a,d)}\sigma_{ij} = \epsilon^{(a,d)}, (i \neq j)).\)

Let \((a, d) \in \dot{\mathcal{D}}_{m|N-m}\), \(r \in \mathbb{Z}_{\geq 0}\) and \(f, f' \in \text{Map}_{r}^{N}\). We write \(f \sim_{r}^{(a,d)} f'\) if there exist \(i, j \in J_{1,N}\) and \(t \in J_{0,r}\) such that \(f(k_{1}) = f'(k_{1}) \quad (k_{1} \in J_{1,t} \cup J_{t-c_{ij}^{(a,d)}_{f,k}+1,r}), f(k_{2}) = i, f'(k_{2}) = j \quad (k_{2} \in J_{t+1,t-c_{ij}^{(a,d)}_{f,k} \cap 2N - 1})\). We write \(f \sim_{r}^{(a,d)} f'\) if \(f = f'\) or there exists \(t \in \mathbb{N}\) and \(f_{k} \in \text{Map}_{r}^{N}\) \((k \in J_{1})\) such that \(f \sim_{t}^{(a,d)} f_{1}, f_{k} \sim_{r}^{(a,d)} f_{k+1} \quad (k \in J_{1,t-1})\) and \(f_{t} \sim_{r}^{(a,d)} f'\).

By [HY08, Theorem 5, Corollary 6], we have

**Theorem 2.6.** Let \((a, d) \in \dot{\mathcal{D}}_{m|N-m}\) and \(w \in \dot{W}^{(a,d)}.\)

1. Let \(f, f' \in \text{Map}_{(a,d)}^{N}(w)\) be such that \(1^{(a,d)}s_{f,\ell^{(a,d)}(w)} = 1^{(a,d)}s_{f',\ell^{(a,d)}(w)} = w.\) Then \(f \sim_{\ell^{(a,d)}(w)}^{(a,d)} f'.\)

2. Let \(r \in \mathbb{N}\) and \(f \in \text{Map}_{r}^{N}\) be such that \(r > \ell^{(a,d)}(w)\) and \(1^{(a,d)}s_{f,r} = w.\) Then there exist \(f' \in \text{Map}_{r}^{N}\) and \(t \in J_{1,r-1}\) such that \(f \sim_{r}^{(a,d)} f'\) and \(f'(t) = f'(t+1)\).
References

