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UNIQUENESS OF POSITIVE RADIAL SOLUTIONS OF
\[ \Delta u + g(r)u + h(r)u^p = 0 \]
AND ITS APPLICATIONS
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UNIQUENESS OF POSITIVE RADIAL SOLUTIONS OF 
$\Delta u + g(r)u + h(r)u^p = 0$ AND ITS APPLICATIONS

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1. INTRODUCTION AND MAIN RESULTS

We consider the problem

$$
\begin{align*}
\begin{cases}
u_{rr} + \frac{n-1}{r} u_r + g(r)u + h(r)u^p = 0, & 0 < r < R, \\
u(0) \in (0, \infty), & u(R) = 0,
\end{cases}
\end{align*}
$$

(1.1)

where $n \geq 2, R \in (0, \infty], p \in (1, \infty)$ and $g, h : (0, R) \to \mathbb{R}$ are appropriate functions. Here, $u(R) = 0$ in the case $R = \infty$ means $u(x) \to 0$ as $|x| \to \infty$. Such a problem has been studied by many researchers; see [1, 3, 5, 6, 8, 9, 12-18, 20-27, 30, 32-36] and others.

In this note, we introduce a result obtained in [28].

**Theorem 1.** Let $0 < R \leq \infty, n \in \mathbb{R}$ with $n \geq 2$ and $p \in (1, \infty)$. Let $g \in C([0, R]) \cap C^1((0, R))$ and $h \in C^2([0, R]) \cap C^3((0, R))$ such that $h$ is positive on $[0, R)$. Assume the following.

(i) In the case of $R < \infty$, $g \in C([0, R]), h \in C^2([0, R])$ and $h(R) > 0$ are also satisfied.

(ii) There exists $\kappa \in [0, R]$ such that

$$
G(r) \geq 0 \text{ on } (0, \kappa) \quad \text{and} \quad G(r) \leq 0 \text{ on } (\kappa, R),
$$

where

$$
G(r) = \frac{2(n-1)(p+1)}{2(p+3)^3h(r)^{2/p+3}} \left(4(n-1)\frac{n+2-(n-2)p}{[n-4+(n-2)p]}h(r)^3 + \left[2(n-1)(p-1)(p+3)^2r^2h(r)^3 - 4(p+3)^2r^3h(r)^2h_r(r)\right]g(r)
\right.
$$

$$+ (p+3)^3rgh(r)^3 + (n-1)[(2n-3)p(6-p) + 6n - 33]rh(r)^2h_r(r)
$$

$$+ 3(n-1)(p-1)(p+5)r^2h(r)h_r(r)^2 - 2(p+4)(p+5)r^3h_r(r)^3
$$
-3(n - 1)(p - 1)(p + 3)r^2 h(r)^2 h_{rr}(r) + 3(p + 3)(p + 5)r^3 h(r) h_{r}(r) h_{rr}(r) - (p + 3)^2 r^3 h(r)^2 h_{rrr}(r).

(iii) In the case of $R = \infty$, $G^{-} \neq 0$ is satisfied.

Then in the case of $R < \infty$, problem (1.1) has at most one positive solution, and in the case of $R = \infty$, problem (1.1) has at most one positive solution $u$ which satisfies $J(r; u) \to 0$ as $r \to \infty$, where

$$a(r) = r^{2(\frac{2(n-1)(p+1)}{p+3}-1)} h(r)^{\frac{2}{p+3}},$$
$$b(r) = \frac{r^{2(\frac{2(n-1)(p+1)}{p+3}-2)} (2(n - 1)h(r) + rh_{r}(r))}{(p + 3)^2 h(r)^{\frac{2(p+4)}{p+3}}} \left(2(n - 1)[n + 2 - (n - 2)p]h(r)^2 + (p + 5)r^2 h_{r}(r)^2ight. - (n - 1)(p - 5)rh(r) h_{r}(r) - (p + 3)^2 h(r)^2 h_{rr}(r)),
$$
$$c(r) = \frac{r^{2(\frac{2(n-1)(p+1)}{p+3}-2)} (2(n - 1)[n + 2 - (n - 2)p]h(r)^2 + (p + 5)r^2 h_{r}(r)^2}{(p + 3)^2 h(r)^{\frac{2(p+4)}{p+3}}}(2(n - 1)(p - 1)h_{r}(r) + rh_{r}(r)),$$

$$J(r; u) = \frac{1}{2} a(r) u_{r}(r)^2 + b(r) u_{r}(r) u(r) + \frac{1}{2} c(r) u(r)^2 + \frac{1}{2} a(r) g(r) u(r)^2 + \frac{1}{p + 1} a(r) h(r) u(r)^{p+1}.$$

Remark 1. In [32, Theorems 2.1 and 2.2], Yanagida obtained a closely related result.

By the theorem above, we can obtain the following; see [13, Theorem 0.1].

Corollary 1 (Kabeya-Tanaka). Let $n \in \mathbb{N}$ with $n \geq 2$. Let $p > 1$ and $g \in C^2([0, \infty))$ such that $-\infty < \inf_{r \in [0, \infty)} g(r) \leq \sup_{r \in [0, \infty)} g(r) < 0$, and set

$$L = \frac{2(n - 1)[(n - 2)p + n - 4]}{(p + 3)^2} \quad \text{and} \quad \beta = \frac{2(n - 1)(p - 1)}{p + 3}.$$

Assume that

$$g_{r}(r)r^3 + \beta g(r)r^2 - (\beta - 2)L < 0 \quad \text{for each} \quad r \geq 0$$

in the case of $n = 2$, and that $p < (n + 2)/(n - 2)$ and

$$\sup_{r > 0} (g_{r r}(r)r^2 + (3 + \beta)g_{r}(r)r + 2\beta g(r)) < 0$$

in the case of $n \geq 3$. Then the problem

$$(1.2) \quad u \in H^1(\mathbb{R}^n), \quad \Delta u(x) + g(|x|) u(x) + u(x)^p = 0 \quad \text{in} \ \mathbb{R}^n$$

has a unique positive radial solution.
Next, we consider the problem
\begin{equation}
\begin{cases}
  u_{rr}(r) + \frac{n-1}{r}u_{r} + g(r)u(r) + h(r)u(r)^{p} = 0, & R' < r < R, \\
  u(R') = 0, & u(R) = 0.
\end{cases}
\end{equation}

The uniqueness of a positive solution of such a problem was studied in [4, 6, 7, 10, 11, 19, 24, 29–31].

The following is also obtained in [28].

**Theorem 2.** Let $0 < R' < R \leq \infty$, $n \in \mathbb{R}$, $p \in (1, \infty)$, $g \in C([R', R]) \cap C^{1}((R', R))$, $h \in C^{2}([R', R]) \cap C^{3}((R', R))$ such that $h$ is positive on $[R', R)$. Let $a$, $b$, $c$, $G$ and $J$ be the functions given in Theorem 1. Assume the following.

(i) In the case of $R < \infty$, $g \in C([R', R])$, $h \in C^{2}([R', R])$ and $h(R) > 0$ are also satisfied.

(ii) There exists $\kappa \in [R', R]$ such that

\[ G(r) \geq 0 \text{ on } (R', \kappa) \quad \text{and} \quad G(r) \leq 0 \text{ on } (\kappa, R). \]

Then in the case of $R < \infty$, problem (1.3) has at most one positive solution, and in the case of $R = \infty$, problem (1.3) has at most one positive solution $u$ which satisfies $J(r; u) \rightarrow 0$ as $r \rightarrow \infty$.

**Remark 2.** For the case $h(r) \equiv 1$, a similar result is obtained by Felmer-Martínez-Tanaka; see [10, Theorem 1.1].

## 2. Applications

In this section, we give examples of Theorem 1. First, we give a comment on the scalar field equation
\[ \Delta u(x) - u(x) + u(x)^{p} = 0 \quad \text{in } \mathbb{R}^{n}, \quad u(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \]

The unique existence of its positive solution was established by Kwong [18]. Since the uniqueness of its positive solution can be derived from Corollary 1, of course, it can be also done by Theorem 1.

Next, we consider the following Brezis-Nirenberg problem.
\begin{equation}
\begin{cases}
  \Delta_{S^{n}}u + \lambda u + u^{p} = 0 \quad \text{in } D, \\
  u = 0 \quad \text{on } \partial D.
\end{cases}
\end{equation}

Here, $n$ is a natural number with $n \geq 3$, $S^{n}$ is the unit sphere in $\mathbb{R}^{n+1}$, $\Delta_{S^{n}}$ is the Laplace-Beltrami operator on $S^{n}$, $D = \{X \in S^{n} : X_{n+1} > \cos \theta_{1}\}$ with $\theta_{1} \in (0, \pi)$,
1 < p \leq (n+2)/(n-2) and \lambda < \lambda_1, where \lambda_1 is the first eigenvalue of \(-\Delta_{S^n}\) on \(D\) with the Dirichlet boundary condition.

Let \(P : S^n \setminus \{(0, \ldots, 0, -1)\} \to \mathbb{R}^n\) be the stereographic projection defined by
\[
P(X_1, \ldots, X_n, X_{n+1}) = \frac{1}{X_{n+1} + 1}(X_1, \ldots, X_n)
\]
for \(X \in S^n \setminus \{(0, \ldots, 0, -1)\}\).

Then we can see \(P(D) = B_R\), where \(B_R = \{x \in \mathbb{R}^n : |x| < R\}\) with
\[
R = \frac{\sin \theta_1}{1 + \cos \theta_1}.
\]

Let \(u\) be a positive solution of (2.1) and define \(v : \overline{B_R} \to \mathbb{R}\) by \(u(P^{-1}x) = (1 + |x|^2)^{\frac{n-2}{2}}v(x)\) for \(x \in \overline{B_R}\). Then we see that \(v\) is a positive solution of
\[
\begin{align*}
\Delta v + \frac{n(n-2) + 4\lambda}{(1+|x|^2)^2}v + 4(1+|x|^2)^{(n-2)p-(n+2)}v^p &= 0 \quad \text{in } B_R, \\
v &= 0 \quad \text{on } \partial B_R.
\end{align*}
\]

We set
\[
g(r) = \frac{n(n-2) + 4\lambda}{(1+r^2)^2} \quad \text{and} \quad h(r) = 4(1+r^2)^{(n-2)p-(n+2)} \quad \text{for } r \geq 0.
\]

We can see that \(G\) in Theorem 1 is given by
\[
G(r) = \frac{2^{p+1}(n-1)}{(p+3)^3} r^{2(n-1)(p+1)-3}(1+r^2)^{n+2-(n-2)p-3}(1-r^2)(Ar^4 + Br^2 + A),
\]
where
\[
A = (n-2)^2 \left( \frac{n+2}{n-2} - p \right) \left( \frac{n+4}{n-2} \right)
= (p+3)[3n^2 - 6n - (n^2 - 4n + 4)p] - 6(n-1)^2,
B = (p+3)[-6n^2 + 12n + (2n^2 + 4\lambda - 4)p + 2\lambda p^2 - 6\lambda - 12] + 16(n-1)^2.
\]

Then we can infer the following. For the details, see [28].

**Theorem 3.** Let \(n \in \mathbb{N}\) with \(n \geq 3\), \(1 < p \leq (n+2)/(n-2)\) and \(\theta_1 \in (0, \pi)\). Assume that one of the following conditions:

(i) \(\theta_1 \in (0, \pi/2]\) and \(\lambda < \lambda_1\),
(ii) \(\theta_1 \in (\pi/2, \pi)\) and
\[
\frac{6 + (6-4p)p}{(p+3)(p-1)} \leq \lambda < \lambda_1.
\]

Then (2.1) has at most one positive radial solution. Moreover, if \(\lambda \geq -n(n-2)/4\) is also satisfied, then (2.1) has at most one positive solution.
Remark 3. It holds that

\[
\frac{6 + (6 - 4n)p}{(p + 3)(p - 1)} \leq -\frac{n(n - 2)}{4},
\]

and if \( p = \frac{(n + 2)}{(n - 2)} \) then the constants in the both sides in the inequality above coincide.

Remark 4. In the case of \( n = 3 \), Bandle-Benguria obtained a sharper result. For the details, see [2].

Remark 5. In the case of \( R > 1 \), we cannot apply Yanagida’s uniqueness theorem [32, Theorem 2.1]. Indeed, by his notation, we have

\[
G(r; n - 2) = \frac{2(4\lambda + n(n - 2))r^{n-1}(1-r^{2})}{(r^{2}+1)^{3}}.
\]

So one of his assumptions \( G(r; n - 2) \leq 0 \) on \((0, R)\) is not satisfied even if \( \lambda > -n(n - 2)/4 \).

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