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Kyoto University
Omae’s knot and $12_{a990}$ are ribbon

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Abstract. The purpose of this note is twofold: First, we prove that Omae’s knot is ribbon, which was known to be homotopically slice. Second, we give a sufficient condition for a given knot to be ribbon. As a corollary, we show that the knot $12_{a990}$ is ribbon, which was known to be slice.

1. Omae’s knot is ribbon

A knot $K$ in the 3-sphere $S^3 = \partial D^4$ is slice if there exists a smoothly embedded disk $D^2 \subset D^4$ such that $\partial D^2 = K$. A knot $K$ is ribbon if there exists a smoothly immersed disk $D^2 \subset S^3$ with only ribbon singularities such that $\partial D^2 = K$. It is easy to see that every ribbon knot is slice. The slice-ribbon conjecture due to Fox [5] states that every slice knot is ribbon, which has been a long-standing unsolved problem in knot theory.

In the positive direction, the slice-ribbon conjecture was conformed for two-bridge knots [19, Lisca], certain pretzel knots [11, Greene-Jabuka], certain Montesinos knots [17, Lecuona] and simple slice knots [23, Shibuya]. On the other hand, potential counterexamples to the slice-ribbon conjecture are demonstrated through the study of the 4-dimensional smooth Poincaré conjecture [2, 6, 7, 9].

Omae [22] studied the knot depicted in the left of Figure 1. The first author and Jong [1] observed that Omae’s knot bounds a smoothly embedded disk in a homotopy 4-ball $W$ which is represented by the handle diagram as in the right of Figure 1 (see also Section 4). In this note, we prove the following.

Theorem 1.1. The 4-manifold $W$ is diffeomorphic to the standard 4-ball.
Proof. Handle calculus in Figure 2 implies that $W$ is diffeomorphic to the standard 4-ball. 

Corollary 1.2. Omae’s knot is slice. Furthermore, it is ribbon.

Proof. Theorem 1.1 implies that Omae’s knot is slice. Recall that Omae’s knot is isotopic to the boundary of cocore disk of the 2-handle (colored Grey) of the top left handle diagram in Figure 2. By chasing Omae’s knot in handle diagrams in Figure 2, we obtain a ribbon presentation of Omae’s knot as in Figure 3.

Remark 1.3. Another potential counterexample to the slice-ribbon conjecture is the $(2,1)$-cable of the figure eight knot. Livingston and Melvin [18] and Kawauchi [14] proved that it is algebraically slice. Furthermore Kawauchi [15] showed that it is rationally slice. On the other hand, by the theorem of Casson-Gordon [4], Miyazaki [21] proved that it is not ribbon. Until now, it is not known whether the $(2,1)$-cable of the figure eight knot is slice or not. See also Gomp-Miyazaki [8].

2. The Knot $12_{a990}$ is Ribbon

The simplest slice knot which might not be ribbon is $12_{a990}$. Indeed, Herald, Kirk and Livingston [12] showed that the connected sum of $12_{a990}$ and right- and left-handed trefoils is ribbon, implying that $12_{a990}$ is slice. However it was unknown whether $12_{a990}$ is ribbon.

A $t_n$-move is a tangle replacement as in Figure 4. In this section, we show the following.

\footnote{C. Livingston (e-mail communication) informed us that they knew that $12_{a990}$ is ribbon, however they did not write that $12_{a990}$ is ribbon in [12].}
Figure 2. Handle diagrams which represent $W$.

**Theorem 2.1.** Let $K$ be a knot. If we obtain the 3-component unlink from $K$ by applying a $t_{2n+1}^-$ and $t_{-(2n+1)}$-move, then $K$ is ribbon.

We denote by $T(p, q)$ the torus knot of type $(p, q)$. First, we show the following.

**Lemma 2.2.** Let $K$ be a knot. If we obtain the 3-component unlink from $K$ by applying a $t_{2n+1}^-$ and $t_{-(2n+1)}$-move, then $K \# T(2, 2n + 1) \# T(2, -(2n + 1))$ is ribbon, where $\#$ denotes the connected sum.

**Proof.** We may assume that a $t_{2n+1}^-$-move and a $t_{-(2n+1)}$-move are done simultaneously. In other words, there exist two trivial tangles $(B_+, T_+) and
FIGURE 3. A ribbon presentation of Omae's knot.

FIGURE 4. The definition of a $t_n$-move for $n > 0$ (left) and for $n < 0$ (right).

(B_, T_) with $B_+ \cap B_- = \emptyset$ such that if we apply a $t_{2n+1}$-move for $(B_+, T_+)$ and a $t_{-(2n+1)}$-move for $(B_-, T_-)$, then we obtain the 3-component unlink. Now we consider $K \# T(2, 2n+1) \# T(2, -(2n+1))$ as in Figure 5. If we add two bands along dotted arcs in Figure 5, then the resulting 3-component link is trivial by the assumption. Therefore $K \# T(2, 2n+1) \# T(2, -(2n+1))$ is ribbon. \[\square\]
Now we prove Theorem 2.1.

Proof of Theorem 2.1. By the assumption, there exist two trivial tangles $(B_+, T_+)$ and $(B_-, T_-)$ with $B_+ \cap B_- = \emptyset$ such that if we apply a $t_{2n+1}$-move for $(B_+, T_+)$ and a $t_{-(2n+1)}$-move for $(B_-, T_-)$, then we obtain the 3-component unlink. If we need, by choosing another 3-balls, we may assume that two trivial tangles $(B_+, T_+)$ and $(B_-, T_-)$ are connected as in Figure 6. Now we consider again $K \# T(2, 2n+1) \# T(2, -(2n+1))$ as in Figure 5 with two bands attached along dotted arcs. Then we deform $T(2, -(2n+1))$ as in Figure 7 with the band. We can see the knot $T(2, 2n+1) \# T(2, -(2n+1))$ in $B_+$ which is known to be ribbon. We concentrate on $B_+$ and deform the tangle (in $B_+$) as in Figure 8. Then we obtain a ribbon presentation of $K$. \qed
As a corollary of Theorem 2.1, we obtain the following.

**Corollary 2.3.** The knot $K_n$ in the left of Figure 9 is ribbon. In particular, $K_1 = 12_{a990}$ is ribbon.

**Proof.** We choose two 3-balls $B_+$ and $B_-$ as in the left of Figure 10. We apply a $t_{2n+1}$-move for $(B_+, K_n \cap B_+)$ and a $t_{-(2n+1)}$-move for $(B_-, K_n \cap B_-)$. Then we obtain the 3-component link as in the right of Figure 10 which is trivial. Therefore $K_n$ is ribbon by Theorem 2.1. \qed
3. ON THE RIBBON FUSION NUMBER

A ribbon knot $K$ is of $m$-fusions if $K$ is isotopic to

$$
\bigcup_{i=0}^{m} S_{i}^{1} - \text{int}(\bigcup_{j=1}^{m} b_{j}(\partial I \times I)) \cup \bigcup_{j=1}^{m} b_{j}(I \times \partial I)
$$

where $\bigcup_{i=0}^{m} S_{i}^{1}$ is the $(m + 1)$-component unlink and $b_{j} : I \times I \rightarrow S^{3}$ ($j = 1, 2, \ldots, m$) are disjoint embeddings such that

$$
S_{i}^{1} \cap b_{j} = \begin{cases} 
    b_{j}([0] \times I) & \text{if } i = 0, \\
    b_{j}([1] \times I) & \text{if } i = j, \\
    \emptyset & \text{otherwise.}
\end{cases}
$$

It is known that a ribbon knot is of $m$-fusions for some $m$ [20, 25]. The ribbon fusion number of a ribbon knot is defined to be the minimal number of such $m$. For the study of the ribbon fusion number, see [3, 13, 24].

**Question 1.** Is the ribbon fusion number of Omae's knot two?

**Question 2.** Is the ribbon fusion number of the knot 12_{a990} two?

4. HOMOTOPY 4-SPHERES ASSOCIATED TO UNKNOTTING NUMBER ONE RIBBON KNOTS

In the conference, Intelligence of Low-dimensional Topology, the first author talked on annulus twist, diffeomorphic 4-manifolds, and slice knots. In this section, we assume some terminologies in [1]. The first author and Jong showed the following.
Proposition 4.1 ([1]). Let $K$ be an unknotted number one knot, $(A, b, c, \epsilon)$ the associated band presentation and $K_n$ the knot obtained from $K$ by applying an annulus twist $n$ times. If $K$ is ribbon, then there exists a homotopy 4-ball $W_n$ with $\partial W_n = S^3$ such that $K_n$ bounds a smoothly embedded disk in $W_n$. In particular, we can associate a homotopy 4-sphere for each $n$.

Let $K$ be the knot $8_{20}$. Note that the unknotted number of $8_{20}$ is one and the associated band presentation of $K$ is depicted in Figure 11. Let $K_n$ the knot obtained from $K$ by applying an annulus twist $n$ times. Then $K_1$ is Omae's knot. Since $8_{20}$ is ribbon, we can associate a homotopy 4-sphere $\Sigma_n$ for each $n$ by Proposition 4.1. Theorem 1.1 implies that $\Sigma_1$ is standard.

![Figure 11. The associated band presentation for $8_{20}$.

Conjecture 4.2. The homotopy 4-sphere $\Sigma_n$ is standard for each $n$.

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