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Examples of spaces with branching geodesics satisfying the curvature-dimension condition

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Abstract

In this short note, we give two simple examples of metric measure spaces containing branching geodesics and satisfying Lott, Sturm and Villani’s curvature-dimension condition. In the first example, only geodesics in a subspace of co-dimension 1 can branch. The second one is positively curved in the sense of the curvature-dimension condition. We also discuss related open problems.

1 Background

The aim of this half-expository note is to draw attention to examples of spaces satisfying the curvature-dimension condition and containing branching geodesics. We first explain the background of this subject.

1.1 Curvature-dimension condition

The curvature-dimension condition for a (complete, separable) metric measure space $(X, d, m)$ is a generalized notion of lower Ricci curvature bounds introduced independently by Sturm ([St1], [St2]) and Lott and Villani ([LV1], [LV2]). Given $K \in \mathbb{R}$ and $N \in (1, \infty]$, a Riemannian manifold $(M, g)$ equipped with the Riemannian volume measure satisfies the curvature-dimension condition $CD(K, N)$ if and only if $\text{Ric}_g \geq K$ (i.e., $\text{Ric}_g(v, v) \geq K(v, v)$ for all $v \in TM$) and $\dim M \leq N$. Metric measure spaces satisfying $CD(K, N)$ are known to behave like enjoying \textit{‘Ric} $\geq K$ and $\dim \leq N’$ in geometric and analytic respects (see [St1], [St2], [LV1], [LV2] and [Vi, Part III]).

The curvature-dimension condition is a certain convexity condition of an entropy functional on the space of probability measures on $X$. Let us give the precise definition (in the sense of Sturm [St1]) only for $N = \infty$. Denote by $\mathcal{P}(X)$ the set of all Borel probability measures on $(X, d)$, and by $\mathcal{P}^2(X)$ the subset consisting of $\mu \in \mathcal{P}(X)$ with finite second moment. Define the $L^2$-Wasserstein distance between $\mu, \nu \in \mathcal{P}^2(X)$ by

$$W_2(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \left( \int_{X \times X} d(x, y)^2 \pi(dx, dy) \right)^{1/2},$$

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where $\Pi(\mu, \nu)$ is the set of all $\pi \in \mathcal{P}(X \times X)$ whose first and second projections to $X$ coincide with $\mu$ and $\nu$, respectively. We call $(\mathcal{P}^2(X), W_2)$ the $L^2$-Wasserstein space over $(X, d)$. The relative entropy $\text{Ent}_m: \mathcal{P}^2(X) \to [-\infty, \infty]$ associated with a Borel measure $m$ on $X$ is defined by

$$\text{Ent}_m(\mu) := \int_X \rho \log \rho \, dm$$

if $\mu = \rho m \ll m$ and $\int_{(\rho > 1)} \rho \log \rho \, dm < \infty$, otherwise we set $\text{Ent}_m(\mu) := \infty$. Then $(X, d, m)$ is said to satisfy $\text{CD}(K, \infty)$ if any pair $\mu, \nu \in \mathcal{P}^2(X)$ is connected by a minimal geodesic $(\mu_t)_{t \in [0, 1]} \subset \mathcal{P}^2(X)$ (i.e., $\mu_0 = \mu$, $\mu_1 = \nu$, $W_2(\mu_s, \mu_t) = |s - t| W_2(\mu, \nu)$ for all $s, t \in [0, 1]$) satisfying

$$\text{Ent}_m(\mu_t) \leq (1 - t) \text{Ent}_m(\mu) + t \text{Ent}_m(\nu) - \frac{K}{2} (1 - t) t W_2(\mu, \nu)^2 \quad (1.1)$$

for all $t \in [0, 1]$. A minimal geodesic $(\mu_t)_{t \in [0, 1]}$ in $(\mathcal{P}^2(X), W_2)$ can be described via the push-forward of $\mu_0$ along geodesics in $X$, so that the behavior of $\text{Ent}_m(\mu_t)$ is naturally controlled by the Ricci curvature in the Riemannian case. The definition of $\text{CD}(K, N)$ for $N < \infty$ is more involved (especially for $1 < N < \infty$), here we only mention that $\text{CD}(K, N)$ implies $\text{CD}(K', N')$ for any $K' \in (-\infty, K]$ and $N' \in [N, \infty]$.

### 1.2 Branching geodesics

The equivalence between the curvature-dimension condition and lower Ricci curvature bounds was extended to Finsler manifolds in [Oh3, Theorem 1.2]. Precisely, given an $n$-dimensional Finsler manifold $(M, F)$ with a positive $C^\infty$-measure $m$, we consider the weighted Ricci curvature $\text{Ric}_N$ for $N \in [n, \infty]$ given by

$$\text{Ric}_N(v) := \text{Ric}(v) + (\psi \circ \eta)^n(0) - \frac{(\psi \circ \eta)^n(0)^2}{N - n} \text{ for } v \in T_x M,$$  

where $\text{Ric}$ is the Finsler-Ricci curvature, $\eta$ is the geodesic with $\eta(0) = v$, and $\psi$ is determined by $m = e^{-\psi} \text{vol}_q$ around $x$ with the volume measure $\text{vol}_q$ of the Riemannian metric $g_q$ induced from $\eta$ (see [Oh3] for the precise definition). Then $(M, F, m)$ satisfies $\text{CD}(K, N)$ if and only if $\text{Ric}_N(v) \geq K F(v)^2$ for all $v \in TM$. By combining this equivalence with the stability of $\text{CD}(K, N)$ under convergence of metric measure spaces, all normed spaces $(\mathbb{R}^n, |\cdot|)$ with the Lebesgue measure $m_L$ satisfy $\text{CD}(0, n)$ (see also [Vi, Theorem in page 908]). In particular, the $\ell_\infty$-space $(\mathbb{R}^n, |\cdot|_\infty, m_L)$ satisfies $\text{CD}(0, n)$.

Geodesics in $(\mathbb{R}^n, |\cdot|_\infty)$ can branch in the sense that there exist two minimal geodesics $\eta_1, \eta_2: [0, 1] \to \mathbb{R}^n$ with respect to $|\cdot|_\infty$ such that $\eta_1(t) = \eta_2(t)$ on $[0, \varepsilon]$ for some $\varepsilon \in (0, 1)$ and that $\eta_1(t) \neq \eta_2(t)$ at some $t \in (\varepsilon, 1)$. The existence of branching geodesics causes difficulties in the study of spaces satisfying the curvature-dimension condition. Although the non-branching assumption has been recently removed in some cases (see [Ra1], [Ra2]), there still remain important results for those we need to assume that geodesics do not branch. See, for instances, [St1, Proposition 4.16] and [DS] for products of CD-spaces, and [Oh2, §5] for a maximal diameter rigidity. If geodesics do not branch (then $(X, d)$ is called a non-branching space), then there is essentially only one minimal geodesic between
absolutely continuous measures $\mu, \nu \ll m$ (see [St2, Lemma 4.1]), so that the entropy is convex along every minimal geodesic. This enables us to localize $\text{CD}(K, N)$ to a certain inequality (concavity of Jacobian) along almost all geodesics in $X$ describing the minimal geodesic between $\mu$ and $\nu$ (see [St2, Proposition 4.2]). Such an infinitesimal inequality is sometimes necessary for sharper estimates.

As far as the author knows, only known examples of spaces satisfying $\text{CD}(K, N)$ and containing branching geodesics are normed spaces. We shall present two more examples of branching spaces (built as singular Finsler manifolds) in this note, whereas they are still modifications of normed spaces in a sense. We also discuss open problems related to these examples.

2 An essentially non-branching example

2.1 Construction

Given a bounded convex domain $D \subset \mathbb{R}^n$, Hilbert [Hi] introduced the distance function

$$d_H(x, y) := \frac{1}{2} \log \left( \frac{|x' - y| \cdot |x - y'|}{|x' - x| \cdot |y - y'|} \right)$$

for $x, y \in D, x \neq y,$ where $| \cdot |$ is the Euclidean norm, and $x' = x + s(y - x)$ and $y' = x + t(y - x)$ are the intersections of the boundary $\partial D$ and the line passing through $x$ and $y$ with $s < 0 < t$ (see Figure 1). This is a generalization of the Klein model of a hyperbolic space which corresponds to the case where $D$ is the unit ball. It is easily seen that, for any $x, y \in D$, the line segment $\eta(t) := x + t(y - x)$ ($t \in [0, 1]$) gives a minimal geodesic from $x$ to $y$ with respect to $d_H$. If $D$ is strictly convex (i.e., $x, y \in \partial D$ implies $(x + y)/2 \in D$ unless $x = y$), then $\eta$ is the unique minimal geodesic between $x$ and $y$. Otherwise, minimal geodesics may not be unique. For instance, consider the square $D = (-1, 1)^2 \subset \mathbb{R}^2$. Since $d_H((0, 0), (a, b)) = d_H((0, 0), (a, b'))$ for any $a \in (0, 1)$ and $b, b' \in [-a, a]$, we can construct uncountably many minimal geodesics between $(0, 0)$ and $(a, 0)$ with $a > 0$ (e.g., thick broken segments in Figure 2).
If $\partial D$ is $C^\infty$ and positively curved (the latter will be called the strong convexity of $D$), then $d_H$ is realized by the $C^\infty$-Finsler metric
\[ F_H(x, v) := \frac{|v|}{2} \left( \frac{1}{|x-a|} + \frac{1}{|x-b|} \right) \quad \text{for } v \in T_x D = \mathbb{R}^n, \]
where $a = x + sv$ and $b = x + tv$ are the intersections of $\partial D$ and the line passing through $x$ in the direction $v$ with $s < 0 < t$ (see Figure 1). It is well-known that the Finsler manifold $(D, F_H)$ has the constant flag curvature $-1$ (see [Sh, §12.2]). Recently, the author showed that $(D, F_H)$ equipped with the Lebesgue measure $m_L$ on $D$ has the bounded weighted Ricci curvature:
\[ \text{Ric}_\infty(v) \in (- (n-1), 2], \quad \text{Ric}_N(v) \in \left( - (n-1) - \frac{(n+1)^2}{N-n}, 2 \right) \]
for any unit vector $v \in TD$ and $N \in (n, \infty)$ ([Oh4, Theorem 1.2]). Thus, for any bounded convex domain $D \subset \mathbb{R}^n$, $(D, d_H, m_L)$ satisfies $\text{CD}(K, N)$ with
\[ K = -(n-1) \quad \text{for } N = \infty, \quad K = -(n-1) - \frac{(n+1)^2}{N-n} \quad \text{for } N \in (n, \infty) \]
by approximating $D$ by strongly convex domains with $C^\infty$-boundaries.

Now, let us consider a domain $D \subset \mathbb{R}^3$ such that $P := D \cap (\mathbb{R}^2 \times \{0\}) = (-1, 1)^2 \times \{0\}$ and that $\partial D \setminus (\mathbb{R}^2 \times \{0\})$ is $C^\infty$ and positively curved. Then $P$ contains branching geodesics, while geodesics not included in $P$ do not branch. Indeed, for any $x, y \in D$ with $x \notin P$ or $y \notin P$, the line segment is a unique minimal geodesic between them. Hence $(D, d_H, m_L)$ is essentially non-branching in the sense of Theorem 2.2 below. A concrete example of such a domain is
\[ D_{\square} := \{(a, b, c) \in \mathbb{R}^3 \mid (a, b) \in (-1, 1)^2, \ c^2 < (1 - a^2)(1 - b^2)\}. \quad (2.1) \]

**Remark 2.1** In [Oh4], we also gave a lower bound of the weighted Ricci curvature for Funk’s distance [Fu]:
\[ d_F(x, y) = \log \left( \frac{|x - y|}{|y - y'|} \right) \quad \text{for } x, y \in D, \ x \neq y, \]
which can be regarded as a ‘non-symmetrization’ of $d_H$. In $(D_{\square}, d_F)$, however, geodesics in a plane containing only one edge of $\partial P$ (e.g., $\{1\} \times [-1, 1] \times \{0\}$) can branch. Therefore there are many more branching geodesics than $(D_{\square}, d_H)$.

### 2.2 Open problems

The above example is inspired by the following recent work of Rajala and Sturm. We say that $(X, d, m)$ satisfies the strong curvature-dimension condition $s\text{CD}(K, \infty)$ if (1.1) holds along any minimal geodesic in $\mathcal{P}^2(X)$.

**Theorem 2.2** ([RS, Corollary 1.2]) Let $(X, d)$ be a complete separable metric space with a locally finite measure $m$ on $X$, and suppose that $(X, d, m)$ satisfies $s\text{CD}(K, \infty)$. Then any minimal geodesic between absolutely continuous measures $\mu_0, \mu_1 \in \mathcal{P}^2(X)$ is concentrated on a set of non-branching geodesics.
See [RS] for the precise statement. Our example $(D_\square, d_\mathcal{H}, m_\mathcal{L})$ in (2.1) satisfies the hypothesis of Theorem 2.2, thus it shows that geodesics still can branch in this kind of essentially non-branching spaces.

An important class of spaces covered by Theorem 2.2 is the one of metric measure spaces enjoying the Riemannian curvature-dimension condition $\text{RCD}(K, \infty)$ introduced by Ambrosio, Gigli and Savaré [AGS]. Roughly speaking, $\text{RCD}(K, \infty)$ is defined by the combination of $\text{sCD}(K, 1)$ and the linearity of the heat flow. The heat flow on a Finsler manifold $(M, F, m)$ is linear if and only if $F$ comes from a Riemannian metric (see [OS]), so that $(D_\square, d_\mathcal{H}, m_\mathcal{L})$ does not satisfy $\text{RCD}(K, 1)$ for any $K$.

**Problem A** Is there a metric measure space satisfying $\text{RCD}(K, \infty)$ for some $K \geq 0$ and containing branching geodesics?

The author does not have any idea how to construct such a space. If the answer is NO, then we can remove the non-branching assumption from a number of theorems on RCD-spaces. Furthermore, since all limit spaces of Riemannian manifolds with $\text{Ric} \geq K$ satisfy $\text{RCD}(K, \infty)$, the negative answer to Problem A contributes to the development of the study of the local structure of these limit spaces. After seminal work of Cheeger and Colding [CC], it is the important open question whether such limit spaces admit branching geodesics or not.

**Problem B** In the case where the answer to Problem A is YES, can one construct a branching space satisfying $\text{RCD}(K, \infty)$ as a limit space of Riemannian manifolds with $\text{Ric} \geq K$?

This is related to another deep question: Can any metric measure space satisfying $\text{RCD}(K, \infty)$ be realized as the limit of a sequence of Riemannian manifolds with $\text{Ric} \geq K$ (or weighted Riemannian manifolds with $\text{Ric}_\infty \geq K$)?

### 3 A positively curved example

#### 3.1 Construction

We next construct a branching space satisfying $\text{CD}(K, N)$ with $K > 0$ and $N < \infty$. Fix $N > 2$, $R > 0$ and $a \in (0, (N - 2)R^{-2})$. Consider the metric measure space $(X_R, d_\infty, m_a)$ given by

$$X_R := \{x \in \mathbb{R}^2 \mid |x|_2 < R\}, \quad d_\infty(x, y) := |x - y|_\infty, \quad m_a := \exp\left(-\frac{a|x|_2^2}{2}\right) m_\mathcal{L}(dx).$$

Clearly geodesics can branch in $(X_R, d_\infty)$. Note that, along any line segment $\eta$ in $X_R$,

$$\left(\frac{a|\eta|_2^2}{2}\right) \leq a|\eta|_2 \cdot |\dot{\eta}|_2 \leq aR|\dot{\eta}|_2, \quad \left(\frac{a|\eta|_2^2}{2}\right)^\prime = a|\dot{\eta}|_2^2.$$

Since $|\dot{\eta}|_2 \geq |\dot{\eta}|_\infty$, by approximating $|\cdot|_\infty$ by $C^\infty$ and strongly convex norms to be precise, we see that $(X_R, d_\infty, m_a)$ satisfies $\text{CD}(K, N)$ with (recall (1.2))

$$K = a - \frac{(aR)^2}{N - 2}.$$
Note that $K > 0$ by the choice of $a$. We similarly find that $(\mathbb{R}^2, d_\infty, m_a)$ satisfies $CD(a, \infty)$ for any $a > 0$.

### 3.2 Open problems

The above example is related to a rigidity result for positively curved spaces. By the Bonnet–Myers type theorem, the diameter of a metric measure space satisfying $CD(K, N)$ with $K > 0$ and $N < \infty$ is at most $\pi \sqrt{(N - 1)/K}$.

In [Oh2, §5], under the weaker assumption of the measure contraction property $MCP(K, N)$ (see [Oh1], [St2, §5]), we showed that a non-branching space with the maximal diameter $\pi \sqrt{(N - 1)/K}$ is homeomorphic to the spherical suspension of a topological measure space. The non-branching property was necessary for proving the continuity of the homeomorphism.

In our example $(X, d_1)$, the diameter is $2R$ and is smaller than the maximal one:

$$\pi \sqrt{\frac{N - 1}{K}} = \pi \sqrt{\frac{N - 1}{a - (aR)^2/(N - 2)}} \geq \pi \sqrt{(N - 1) \frac{4R^2}{N - 2}} > 2\pi R.$$

**Problem C** Is there a metric measure space satisfying $CD(K, N)$ with $K > 0$ and $N < \infty$, attaining the maximal diameter $\pi \sqrt{(N - 1)/K}$, and containing branching geodesics? If YES, can such a space be realized as the spherical suspension of some space?

One can weaken the condition $CD(K, N)$ to $MCP(K, N)$, or strengthen it to the combination of $CD(K, N)$ and $RCD(K, \infty)$. Related to the latter, we can also ask the following.

**Problem D** Is there a metric measure space which satisfies $CD(K, N)$ with $K > 0$ (or $K = 0$), contains branching geodesics, but is essentially non-branching in the sense of Theorem 2.2?

This would be a more accessible one than other problems. Recall that the space $(D_\square, d_\mathcal{H}, m_L)$ in the previous section admits only a negative curvature bound.

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**References**


