A NOTE ON THE GEOMETRICITY OF OPEN HOMOMORPHISMS BETWEEN THE ABSOLUTE GALOIS GROUPS OF $p$-ADIC LOCAL FIELDS

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ABSTRACT. In the present paper, we prove that an open continuous homomorphism between the absolute Galois groups of $p$-adic local fields is geometric [i.e., roughly speaking, arises from an embedding of fields] if and only if the homomorphism is HT-preserving [i.e., roughly speaking, satisfies the condition that the pull-back by the homomorphism of every Hodge-Tate representation is Hodge-Tate].

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INTRODUCTION

Let $p$ be a prime number. Write $\mathbb{Q}_p$ for the $p$-adic completion of the field of rational numbers $\mathbb{Q}$. For $\square \in \{\circ, \bullet\}$, let $k_\square$ be a $p$-adic local field [i.e., a finite extension of $\mathbb{Q}_p$] and $\overline{k}_\square$ an algebraic closure of $k_\square$. Write $G_{k_\square} \overset{\text{def}}{=} \text{Gal}(\overline{k}_\square/k_\square)$.

Let

$$\alpha : G_{k_\circ} \longrightarrow G_{k_\bullet}$$

be an open continuous homomorphism. In [1], [2], S. Mochizuki discussed the geometricity [cf. [2], Definition 3.1, (iv)] of such an $\alpha$. In particular, Mochizuki proved that the following conditions are equivalent [cf. [2], Theorem 3.5, (i)]:

(i) $\alpha$ is geometric, i.e., arises from an isomorphism of fields $\overline{k}_\bullet \cong \overline{k}_\circ$ that determines an embedding $k_\bullet \hookrightarrow k_\circ$.

(ii) $\alpha$ is of CHT-type [cf. [2], Definition 3.1, (iv)], i.e., $\alpha$ is compatible with the respective $p$-adic cyclotomic characters of $G_{k_\circ}$, $G_{k_\bullet}$, and, moreover, there exists an isomorphism of topological modules [but not necessarily the topological fields] $\overline{k}_\circ \cong \overline{k}_\bullet$ — where, for $\square \in \{\circ, \bullet\}$, we write $\overline{k}_\square$ for the $p$-adic completion of $k_\square$ — that is compatible with the respective natural actions of $G_{k_\circ}$, $G_{k_\bullet}$ on $\overline{k}_\circ$, $\overline{k}_\bullet$ [relative to $\alpha$].
(iii) $\alpha$ is of 01-qLT-type [cf. [2], Definition 3.1, (iv)], i.e., for every pair of open subgroups $H_0 \subseteq G_{k_0}, H_0 \subseteq G_{k_0}$ such that $\alpha(H_0) \subseteq H_0$, and every character $\phi: H_0 \to E^\times$ of qLT-type [cf. [2], Definition 3.1, (iii)] — where $E$ is a $p$-adic local field all of whose $\mathbb{Q}_p$-conjugates are contained in the fixed fields $k_0, k_0$ — the composite $H_0 \overset{\alpha}{\to} H_0 \overset{\phi}{\to} E^\times$ is Hodge-Tate, and the set of Hodge-Tate weights of this composite is contained in $\{0, 1\}$.

We shall say that $\alpha$ is HT-preserving [cf. Definition 1.3, (i)] if $\alpha$ preserves the Hodge-Tate-ness of $p$-adic representations, i.e., for every finite dimensional continuous representation $\phi: G_{k_0} \to \text{GL}_n(\mathbb{Q}_p)$ of $G_{k_0}$, if $\phi$ is Hodge-Tate, then the composite $G_{k_0} \overset{\alpha}{\to} G_{k_0} \overset{\phi}{\to} \text{GL}_n(\mathbb{Q}_p)$ is Hodge-Tate. Then it is immediate that if $\alpha$ is of CHT-type, then $\alpha$ is HT-preserving.

Moreover, since a character of qLT-type is Hodge-Tate, and its set of Hodge-Tate weights is contained in $\{0, 1\}$, one verifies easily that if $\alpha$ is not only HT-preserving but also preserves the sets of Hodge-Tate weights of Hodge-Tate representations, then $\alpha$ is of 01-qLT-type.

On the other hand, it does not seem to be clear that the following assertion holds:

If $\alpha$ is HT-preserving, then $\alpha$ is either of CHT-type or of 01-qLT-type.

In particular, the following question may be regarded as a natural question concerning the geometricity of open continuous homomorphisms between the absolute Galois groups of $p$-adic local fields:

Is every HT-preserving open continuous homomorphism between the absolute Galois groups of $p$-adic local fields geometric?

In the present paper, we answer this question in the affirmative by refining the argument of Mochizuki applied in [1], [2]. The main consequence of the present paper is as follows [cf. Corollaries 3.4; 3.5].

**Theorem.** Let $p$ be a prime number. For $\square \in \{\circ, \bullet\}$, let $k_\square$ be a $p$-adic local field and $\overline{k_\square}$ an algebraic closure of $k_\square$. Write $G_{k_\square} \overset{\text{def}}{=} \text{Gal}(\overline{k_\square}/k_\square)$. Let

$$\alpha: G_{k_0} \longrightarrow G_{k_0}$$

be an open continuous homomorphism. Then $\alpha$ is geometric [cf. [2], Definition 3.1, (iv)] if and only if $\alpha$ is HT-preserving [cf. Definition 1.3, (i)]. In particular, if we write

$$\text{Emb}(\overline{k_\circ}/k_\circ, \overline{k_\circ}/k_\circ)$$

for the set of isomorphisms of fields $\overline{k_\circ} \simeq \overline{k_\circ}$ that determine embeddings $k_\circ \hookrightarrow k_\circ$;

$$\text{Emb}(k_\circ, k_\circ)$$

for the set of embeddings of fields $k_\circ \hookrightarrow k_\circ$;

$$\text{Hom}^{\text{open}}_{\text{HT}}(G_{k_0}, G_{k_0})$$

for the set of open continuous homomorphisms between the absolute Galois groups of the $p$-adic local fields $k_0$ and $k_0$. 


for the set of HT-preserving open continuous homomorphisms $G_{k_0} \to G_{k_*}$, then we have a commutative diagram of natural maps

$$
\begin{array}{ccc}
\text{Emb}(\bar{k}_*/k, \bar{k}_0/k_0) & \xrightarrow{\sim} & \text{Hom}_{HT}^\text{open}(G_{k_*}, G_{k_*}) \\
\downarrow & & \downarrow \\
\text{Emb}(k_*, k_0) & \xrightarrow{\sim} & \text{Hom}_{HT}^\text{open}(G_{k_*}, G_{k_*})/\text{Inn}(G_{k_*})
\end{array}
$$

— where the vertical arrows are surjective, and the horizontal arrows are bijective.

**Remark.** The various discussions given in the present paper may be regarded as just slight modifications or improvements of the discussions of [1], [2]. From this point of view, one may consider that some arguments in §2 and the observation that a similar technique of [1], §4, can be available in the situation of the proof of Theorem 3.3 are essentially the only new contributions of the present paper.

### 1. HT-PRESERVING HOMOMORPHISMS

In the present §1, we define the notion of an HT-preserving [i.e., “Hodge-Tate-preserving”] homomorphism [cf. Definition 1.3, (i), below]. Let $p$ be a prime number. Write $\mathbb{Q}_p$ for the $p$-adic completion of the field of rational numbers $\mathbb{Q}$. For $\square \in \{\circ, \bullet, \varnothing\}$, let $k_\square$ be a $p$-adic local field [i.e., a finite extension of $\mathbb{Q}_p$] and $\bar{k}_\square$ an algebraic closure of $k_\square$. Write $\mathfrak{o}_{k_\square}$ for the ring of integers of $k_\square$, $G_{k_\square} \overset{\text{def}}{=} \text{Gal}(\bar{k}_\square/k_\square)$, $I_{k_\square} \subseteq G_{k_\square}$ for the inertia subgroup of $G_{k_\square}$, and $P_{k_\square} \subseteq I_{k_\square}$ for the wild inertia subgroup of $G_{k_\square}$. Now let us recall from local class field theory that we have a natural isomorphism

$$G_{k_*}^{ab} \xrightarrow{\sim} (k^\times)^\wedge$$

— where we write $(k^\times)^\wedge$ for the profinite completion of the topological group $k^\times$ — that determines an isomorphism

$$(G_{k_*}^{ab} \supseteq) \text{ Im}(I_{k_*} \hookrightarrow G_{k_*} \to G_{k_*}^{ab}) \xrightarrow{\sim} \mathfrak{o}_{k_*}^\times \subseteq (k^\times)^\wedge.$$

In the following, let us regard $\mathfrak{o}_{k_*}^\times$ as a closed subgroup of $G_{k_*}^{ab}$ by means of this isomorphism, i.e., $\mathfrak{o}_{k_*}^\times \subseteq G_{k_*}^{ab}$.

**Proposition 1.1.** Let $\alpha : G_{k_*} \to G_{k_*}$ be an open continuous homomorphism. Then $\alpha(I_{k_*})$, $\alpha(P_{k_*}) \subseteq G_{k_*}$ are open subgroups of $I_{k_*}$, $P_{k_*}$, respectively. Moreover, it holds that $\text{Ker}(\alpha) \subseteq P_{k_*}$.

**Proof.** This follows immediately from [2], Proposition 3.4 [cf. also the proof of [2], Proposition 3.4]. \qed

**Definition 1.2.**

(i) Let $A$ be a topological group; $\phi_1, \phi_2 : G_k \to A$ continuous homomorphisms. Then we shall say that $\phi_1$ is inertially equivalent to $\phi_2$ if $\phi_1$ and $\phi_2$ coincide on an open subgroup of $I_k \subseteq G_k$ [cf. the discussion preceding [4], Chapter III, §A.5, Theorem 2].

(ii) Let $E$ be a finite Galois extension of $\mathbb{Q}_p$, that admits an embedding $\sigma : E \hookrightarrow k$. Let $\pi \in \mathfrak{o}_k$ be a uniformizer of $\mathfrak{o}_k$. Then we shall write

$$\chi_{\sigma, \pi}^{LT} : G_k \twoheadrightarrow E^\times$$

for the continuous character obtained by forming the composite

$$G_k \twoheadrightarrow G_k^{ab} \twoheadrightarrow (k^\times)^\wedge \xrightarrow{\sim} \mathfrak{o}_k^\times \times \widehat{\mathbb{Z}} \to \mathfrak{o}_E^\times \to \mathfrak{o}_E^\times \twoheadrightarrow E^\times.$$
— where the first arrow is the natural surjection, the second arrow is the natural isomorphism arising from local class field theory, the third arrow is the isomorphism determined by the uniformizer $\pi \in \mathfrak{o}_k$, the fourth arrow is the first projection, the fifth arrow is the isomorphism induced by the norm map $k^\times \to E^\times$ [with respect to the embedding $\sigma$], the sixth arrow is the isomorphism given by mapping $a$ to $a^{-1}$, and the seventh arrow is the natural inclusion [cf. [4], Chapter III, §A.4]. Since $I_k \subseteq G_k$ surjects onto $\mathfrak{o}_k \times \{1\} \subseteq \mathfrak{o}_k \times \widehat{\mathbb{Z}}$ [cf. the discussion at the beginning of §1], one verifies easily that the inertial equivalence class [cf. (i)] of $\chi^{LT}_{\sigma,\pi}$ does not depend on the choice of $\pi \in \mathfrak{o}_k$. Thus, we shall often write $\chi^{LT}_{\sigma}$ to denote $\chi^{LT}_{\sigma,\pi}$ for some unspecified choice of $\pi \in \mathfrak{o}_k$.

**Definition 1.3.** Let $\alpha : G_{k_\alpha} \to G_{k_\beta}$ be an open continuous homomorphism.

(i) We shall say that $\alpha$ is HT-preserving [i.e., “Hodge-Tate-preserving”] if, for every finite dimensional continuous representation $\phi : G_{k_\beta} \to \text{GL}_n(\mathbb{Q}_p)$ of $G_{k_\beta}$ that is Hodge-Tate, the composite $G_{k_\alpha} \overset{\alpha}{\to} G_{k_\beta} \overset{\phi}{\to} \text{GL}_n(\mathbb{Q}_p)$ is Hodge-Tate.

(ii) We shall say that $\alpha$ is of HT-qLT-type [i.e., “Hodge-Tate-quasi-Lubin-Tate” type] (respectively, of weakly HT-qLT-type [i.e., “weakly Hodge-Tate-quasi-Lubin-Tate” type]) if, for

- every pair of respective finite extensions $k'_\alpha (\subseteq \overline{k}_\alpha), k'_\beta (\subseteq \overline{k}_\beta)$ of $k_\alpha, k_\beta$ such that $\alpha(G_{k'_\alpha}) \subseteq G_{k'_\beta}$,
- every finite Galois extension $E$ of $\mathbb{Q}_p$ that admits a pair of embeddings $\sigma_\alpha : E \hookrightarrow k'_\alpha, \sigma_\beta : E \hookrightarrow k'_\beta$, the composite

$$G_{k'_\alpha} \overset{\alpha|_{G_{k'_\alpha}}}{\to} G_{k'_\beta} \overset{\chi^{LT}_{\sigma_\alpha}}{\to} E^\times$$

[cf. Definition 1.2, (ii)] is Hodge-Tate (respectively, is inertially equivalent [cf. Definition 1.2, (ii)] to a continuous character $G_{k'_\alpha} \to E^\times$ that factors through the natural open injection $G_{k'_\alpha} \hookrightarrow \text{Gal}(\overline{k}_\alpha/E)$ determined by the embeddings $E \overset{\sigma_\alpha}{\subset} k'_\alpha (\subseteq \overline{k}_\alpha)$ [cf. Proposition 1.1]. [Here, we note that, as is well-known — cf., e.g., [4], Chapter III, §A.1, Corollary 2 — the issue of whether or not a finite dimensional continuous representation is Hodge-Tate depends only on the inertial equivalence class of the given representation.]

**Lemma 1.4.** Let $\alpha : G_{k_\alpha} \to G_{k_\beta}$ be an open continuous homomorphism. Consider the following four conditions:

1. $\alpha$ is HT-preserving [cf. Definition 1.3, (i)].

(1') For every pair of respective finite extensions $k'_\alpha (\subseteq \overline{k}_\alpha), k'_\beta (\subseteq \overline{k}_\beta)$ of $k_\alpha, k_\beta$ such that $\alpha(G_{k'_\alpha}) \subseteq G_{k'_\beta}$, the restriction $\alpha|_{G_{k'_\alpha}} : G_{k'_\alpha} \to G_{k'_\beta}$ is HT-preserving.

2. $\alpha$ is of HT-qLT-type [cf. Definition 1.3, (ii)].

3. $\alpha$ is of weakly HT-qLT-type [cf. Definition 1.3, (iii)].

Then we have an equivalence and implications

$$(1) \iff (1') \implies (2) \implies (3).$$

**Proof.** The implication $(1') \implies (1)$ is immediate. Next, let us verify that the implication $(1) \implies (1')$ follows from the following well-known argument: Let $k'_\alpha (\subseteq \overline{k}_\alpha), k'_\beta (\subseteq \overline{k}_\beta)$ be respective finite extensions of $k_\alpha, k_\beta$ such that $\alpha(G_{k'_\alpha}) \subseteq G_{k'_\beta}$; $\phi : G_{k'_\beta} \to \text{GL}_n(\mathbb{Q}_p)$ a finite dimensional continuous representation of $G_{k_\beta}$.
that is Hodge-Tate. Now let us observe [cf., e.g., [4], Chapter III, §A.1, Corollary 2] that, to verify that the composite \( \phi \circ \alpha |_{G_{k'}} \) is Hodge-Tate — by replacing \( k'_p, k'_s \) by suitable finite extensions of \( k_p, k_s \), respectively — we may assume without loss of generality that \( k'_p, k'_s \) are Galois over \( k_p, k_s \), respectively. Write \( \phi_{k_s} \) for the finite dimensional continuous representation of \( G_{k_s} \) obtained by inducing \( \phi \) from \( G_{k'_s} \) to \( G_{k_s} \). Then since \([\text{one verifies easily that}]\) \( \phi_{k_s} |_{G_{k'_s}} \) is isomorphic to the direct product of \([k'_s : k_s] \) copies of \( \phi \), it holds that \( \phi_{k_s} \) is Hodge-Tate. Thus, since \( \alpha \) is HT-preserving, it holds that \( \phi_{k_s} \circ \alpha \), hence also \( (\phi_{k_s} \circ \alpha) |_{G_{k'_s}} \), is Hodge-Tate. On the other hand, one verifies easily that \( \phi \circ \alpha |_{G_{k'_s}} \) is isomorphic to a subrepresentation of \( (\phi_{k_s} \circ \alpha) |_{G_{k'_s}} \). In particular, we conclude that \( \phi \circ \alpha |_{G_{k'_s}} \) is Hodge-Tate. This completes the proof of the implication \((1) \Rightarrow (1')\).

The implication \((1') \Rightarrow (2)\) follows from the fact that \( \chi_{\alpha_{p, \pi}}^{LT} \) defined in Definition 1.2, (ii), is Hodge-Tate [cf. [4], Chapter III, §A.5, Corollary]. Finally, we verify the implication \((2) \Rightarrow (3)\). We shall apply the notational conventions established in Definition 1.3, (ii). Then since \( \alpha \) is of HT-qLT-type, the character \( \chi : G_{k'_s} \to E^\times \) obtained by forming the composite

\[
G_{k'_s} \xrightarrow{\alpha |_{G_{k'_s}}} G_{k_s} \xrightarrow{\chi_{\kappa^T}} E^\times
\]

is Hodge-Tate. Thus, since \( E \) is Galois over \( \mathbb{Q}_p \), it follows immediately from [4], Chapter III, §A.5, Corollary, that \( \chi \) is inertially equivalent [cf. Definition 1.2, (i)] to the character

\[
\prod_{\sigma \in \text{Gal}(E/\mathbb{Q}_p)} (\chi_{\sigma_{\alpha \sigma}}^{LT})^{n_\sigma} : G_{k_s} \to E^\times
\]

for some choices of integers \( n_\sigma \). On the other hand, one verifies easily from local class field theory that this character is inertially equivalent to the restriction to \( G_{k'_s} \subset \text{Gal}(\overline{E}_\sigma / E) \) of the character

\[
\prod_{\sigma \in \text{Gal}(E/\mathbb{Q}_p)} (\chi_{\sigma}^{LT})^{n_\sigma} : \text{Gal}(\overline{E}_\sigma / E) \to E^\times.
\]

This completes the proof of the implication \((2) \Rightarrow (3)\), hence also of Lemma 1.4.

\[\square\]

**Remark 1.4.1.** In the notation of Lemma 1.4, consider the following four conditions:

1. (4) \( \alpha \) is of qLT-type [cf. [2], Definition 3.1, (iv)].
2. (5) \( \alpha \) is of 01-qLT-type [cf. [2], Definition 3.1, (iv)].
3. (6) \( \alpha \) is of CHT-type [cf. [2], Definition 3.1, (iv)].
4. (7) \( \alpha \) is of HT-type [cf. [2], Definition 3.1, (iv)].

Then we have equivalences and implications

\[(7) \iff (4) \iff (5) \iff (6) \iff (1) \iff (1') \iff (2) \iff (3)\].

Indeed, the equivalences \((4) \iff (5) \iff (6)\) follow from [2], Theorem 3.5, (i); the implications \((6) \Rightarrow (1)\) and \((6) \Rightarrow (7)\) are immediate. If, moreover, \( \alpha \) is injective, then we have equivalences and implications

\[(4) \iff (5) \iff (6) \iff (7) \iff (1) \iff (1') \iff (2) \iff (3)\].

Indeed, the implication \((7) \Rightarrow (6)\) follows immediately from [1], Proposition 1.1.
2. INJECTIVITY RESULT

In the present §2, we prove that every open continuous homomorphism of weakly HT-gLT-type is injective [cf. Proposition 2.4 below]. We maintain the notation of the preceding §1.

**Definition 2.1.**

(i) Let $G$ be a profinite group. Then we shall write

\[(G \twoheadrightarrow) \quad G^{\text{p-ab-free}}\]

for the maximal pro-$p$ abelian torsion-free quotient of $G$.

(ii) Let $A$ be an abelian topological group and $\phi: G_k \to A$ a continuous homomorphism. Then we shall write

\[
\text{iner-dim}(\phi) \overset{\text{def}}{=} \dim_{\mathbb{Q}_p}(\phi(I_k)^{\text{p-ab-free}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)
\]

[cf. (i)] and refer to iner-dim$(\phi)$ as the inertial dimension of $\phi$.

**Lemma 2.2.** Let $A$ be an abelian topological group and $\phi: G_k \to A$ a continuous homomorphism. Then the following hold:

(i) It holds that

\[0 \leq \text{iner-dim}(\phi) \leq [k : \mathbb{Q}_p]\]

[cf. Definition 2.1, (ii)].

(ii) Let $H \subseteq I_k$ be a closed subgroup of $I_k$. Suppose that $H$ contains an open subgroup of $P_k$ [e.g., $H$ is an open subgroup of $I_k$ or $P_k$], Then

\[
\text{iner-dim}(\phi) = \dim_{\mathbb{Q}_p}(\phi(H)^{\text{p-ab-free}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)
\]

[cf. Definition 2.1, (i)].

(iii) Let $\phi': G_k \to A$ be a continuous homomorphism that is inertially equivalent to $\phi$ [cf. Definition 1.2, (i)]. Then

\[
\text{iner-dim}(\phi) = \text{iner-dim}(\phi').
\]

(iv) In the notation of Definition 1.2, (ii), it holds that

\[
\text{iner-dim}(\chi^\text{LT}_\sigma) = [E : \mathbb{Q}_p]
\]

[cf. (iii)].

(v) Let $\alpha: G_{k_0} \to G_k$ be an open continuous homomorphism. Then it holds that

\[
\text{iner-dim}(\phi) = \text{iner-dim}(\phi \circ \alpha).
\]

**Proof.** First, I claim that the following assertion holds:

Claim 2.2.A: The natural surjection $I_k \twoheadrightarrow \phi(I_k)^{\text{p-ab-free}}$ factors through the natural surjection $I_k \twoheadrightarrow \sigma^*_k \twoheadrightarrow (\sigma^*_k)^{\text{p-ab-free}}$ [cf. the discussion at the beginning of §1].

Indeed, this follows immediately from our assumption that $A$ is abelian. This completes the proof of Claim 2.2.A.

Assertion (i) follows immediately from Claim 2.2.A, together with the fact that $(\sigma^*_k)^{\text{p-ab-free}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is of dimension $[k : \mathbb{Q}_p]$. Assertion (ii) follows immediately from Claim 2.2.A, together with the [easily verified] fact that the composite $P_k \hookrightarrow I_k \twoheadrightarrow \sigma^*_k$ is open. Assertion (iii) follows immediately from assertion (ii). Assertion (iv) follows immediately from the definition of the character $\chi^\text{LT}_\sigma$, together with the fact that $(\sigma^*_k)^{\text{p-ab-free}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is of dimension $[E : \mathbb{Q}_p]$. Finally, we verify assertion (v). Let us first observe that it follows from Proposition 1.1 that $\alpha$ determines an open homomorphism $P_{k_0} \to P_k$. Thus, assertion (v) follows immediately from assertion (ii). This completes the proof of assertion (v).
Lemma 2.3. Let $N \subseteq G_k$ be a nontrivial normal closed subgroup of $G_k$. Then there exists an open subgroup $H \subseteq G_k$ of $G_k$ such that the image of the composite $N \cap H \twoheadrightarrow H \to H^{\text{p-\ab-free}}$ [cf. Definition 2.1, (i)] is nontrivial.

Proof. Assume that, for every open subgroup $H \subseteq G_k$ of $G_k$, the image of the composite $N \cap H \twoheadrightarrow H \to H^{\text{p-\ab-free}}$ is trivial, i.e., if we write $J_H \subseteq H$ for the kernel of the natural surjection $H \to H^{\text{p-\ab-free}}$, then $N \cap H \subseteq J_H$. Now since $N$ is nontrivial, it is immediate that there exists a normal open subgroup $H \subseteq G_k$ such that the composite $N \twoheadrightarrow G_k \to G_k/H$ is nontrivial. In particular, one verifies easily that, to verify Lemma 2.3, by replacing $G_k$ by the inverse image of the image of $N$ in $G_k/H$ via $G_k \twoheadrightarrow G_k/H$, we may assume without loss of generality that the composite $N \twoheadrightarrow G_k \to G_k/H$ is nontrivial and surjective. Thus, since [we have assumed that] $N \cap H \subseteq J_H$, it follows immediately that the composite $N \twoheadrightarrow G_k \to G_k/J_H$ determines a splitting of the exact sequence of profinite groups

$$1 \twoheadrightarrow H^{\text{p-\ab-free}} \twoheadrightarrow G_k/J_H \twoheadrightarrow G_k/H \twoheadrightarrow 1.$$ 

[Here, we note that since $H \subseteq G_k$ is normal, and $J_H \subseteq H$ is characteristic, one verifies easily that $J_H$ is normal in $G_k$.] In particular, since $N \subseteq G_k$ is normal, the natural action [determined by the above exact sequence] of $G_k/H$ on $H^{\text{p-\ab-free}}$, hence also on $H^{\text{p-\ab-free}} \otimes \mathbb{Z}_p^p$, is trivial. On the other hand, if we write $k'$ (⊆ $\overline{k}$) for the finite Galois extension of $k$ corresponding to $H \subseteq G_k$, then it follows immediately from local class field theory that there exists a $G_k/H$ (= $\text{Gal}(k'/k)$)-equivariant injection of $\mathbb{Q}_p$-vector spaces $k' \hookrightarrow H^{\text{p-\ab-free}} \otimes \mathbb{Z}_p^p$, which contradicts the fact that the action of $G_k/H$ on $H^{\text{p-\ab-free}} \otimes \mathbb{Z}_p^p$ is trivial. This completes the proof of Lemma 2.3.

□

Next, we prove the main result of the present §2. Note that the injectivity result was shown in the proof of the implication (c) ⇒ (d) of [2], Theorem 3.5, (i), for homomorphisms of qLT-type, and that Proposition 2.4 is its improvement for homomorphisms of weakly HT-qLT-type.

Proposition 2.4. Let $\alpha: G_{k_0} \to G_k$ be an open continuous homomorphism. Suppose that $\alpha$ is of weakly HT-qLT-type [cf. Definition 1.3, (ii)]. Then $\alpha$ is injective.

Proof. Assume that the homomorphism $\alpha$ is not injective. Then it follows immediately from Lemma 2.3 that there exists a finite Galois extension $E$ of $\mathbb{Q}_p$ that admits a pair of embeddings $E \hookrightarrow \overline{k}_0$, $E \hookrightarrow \overline{k}_0^\ast$ such that if we write $E_0 \subseteq \overline{k}_0$, $E_0^\ast \subseteq \overline{k}_0^\ast$ for the respective images of these embeddings [so $E_0 \cong E \cong E_0^\ast$, then $k_0 \subseteq E_0$, $k_0 \subseteq E_0^\ast$, and, moreover, the image of the composite $\text{Ker}(\alpha) \cap G_{E_0} \hookrightarrow G_{E_0} \to G_{E_0}^{\text{p-\ab-free}}$ [cf. Definition 2.1, (i)] is nontrivial.

Let $k_0' \subseteq \overline{k}_0$ be a finite extension of $k_0$ such that $E_0 \subseteq k_0'$, and, moreover, $\alpha(G_{k_0'}) \subseteq G_{E_0}$. Write $\chi$ for the composite

$$G_{k_0'} \xrightarrow{\alpha|_{G_{k_0'}}} G_{E_0} \xrightarrow{\chi^{\text{LT}}} E_0^\times \quad \left(\cong E_0^\times \cong E_0^\ast\right)$$

[cf. Definition 1.2, (ii)]. Then since $\alpha|_{G_{k_0'}}$ is open, it follows from Lemma 2.2, (iv), (v), that

$$\text{inert-dim}(\chi) = \text{inert-dim}(\chi^{\text{LT}}) = [E_0^\ast: \mathbb{Q}_p]$$

[cf. Definition 2.1, (ii)]. On the other hand, since $\alpha$ is of weakly HT-qLT-type, the character $\chi$ is inertially equivalent to the continuous character factors as the composite

$$G_{k_0'} \longrightarrow G_{E_0} \xrightarrow{\chi_{E_0}} E_0^\times \quad \left(\cong E_0^\times \cong E_0^\ast\right)$$
of the natural open injection $\gamma_{\ell'_k} \hookrightarrow \gamma_{\ell_k}$ and a continuous character $\chi_{E_\circ} : \gamma_{E_\circ} \to E_\circ^\times$. Thus, it follows from Lemma 2.2, (iii), (v), that

$$([E_\circ : \mathbb{Q}_p] =) \ \text{iner-dim}(\chi) = \text{iner-dim}(\chi_{E_\circ}).$$

Now let us recall from Proposition 1.1 that $\text{Ker}(\alpha) \subseteq P_{k_\circ}$. In particular, it holds that $\text{Ker}(\alpha) = \text{Ker}(\alpha) \cap I_{k_\circ}$, which thus implies that $\text{Ker}(\alpha) \cap I_{k_\circ}$ is open in $\text{Ker}(\alpha)$. On the other hand, it follows from the definition of $\chi$ that $\text{Ker}(\chi) = \text{Ker}(\alpha) \cap I_{k_\circ}$

$$(= \text{Ker}(\alpha) \cap G_{k_\circ}) \subseteq \text{Ker}(\chi).$$

Thus, since $\chi$ is inertially equivalent to $\chi_{E_\circ}|_{G_{k_\circ}}$, we conclude that there exists an open subgroup $J \subseteq \text{Ker}(\alpha)$ of $\text{Ker}(\alpha)$ such that $J \subseteq \text{Ker}(\chi_{E_\circ}) \subseteq \gamma_{E_\circ}$. Now since $J \subseteq \text{Ker}(\alpha)$ is open in $\text{Ker}(\alpha)$, and [we have assumed that] the image of the composite $\text{Ker}(\alpha) \cap \gamma_{E_\circ} \hookrightarrow \gamma_{E_\circ} \to \gamma_{E_\circ}^{pab}$ is nontrivial, it follows that the image of the composite $J \hookrightarrow \gamma_{E_\circ} \to \gamma_{E_\circ}^{pab}$ is nontrivial. Thus, one verifies easily that the image of the homomorphism $J \to \sigma_{E_\circ}^\times (\subseteq \gamma_{E_\circ}^{pab})$ [cf. the discussion at the beginning of §1] determined by the composite $J \hookrightarrow \gamma_{E_\circ} \to \gamma_{E_\circ}^{pab}$ [where we recall that $J \subseteq I_{E_\circ}$] is infinite. In particular, since $J \subseteq \text{Ker}(\chi_{E_\circ})$, we conclude that the kernel of the character $(I_{E_\circ} \to \sigma_{E_\circ}^\times) \to \gamma_{E_\circ}^{pab}$ determined by the restriction of $\chi_{E_\circ}$ to $E_\circ \subseteq \gamma_{E_\circ}^{pab}$ is infinite. Thus, we obtain an inequality

$$([E_\circ : \mathbb{Q}_p] =) \ \text{iner-dim}(\chi_{E_\circ}) < \text{dim}_{\mathbb{Q}_p}(\sigma_{E_\circ}^{pab} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) = [E_\circ : \mathbb{Q}_p],$$

which contradicts the fact that $E_\circ \simeq E \simeq E_\circ$. This completes the proof of Proposition 2.4.

\section{3. The main results}

In the present §3, we prove the main theorem of the present paper [cf. Theorem 3.3 below]. We maintain the notation of §1.

**Definition 3.1.** Let $\alpha : G_{k_\circ} \simeq G_{k_\circ}$ be a continuous isomorphism and $\beta : k_\circ \simeq k_\circ$ an isomorphism of fields. Then we shall say that $\beta$ is inertially compatible with $\alpha$ if the composite

$$\sigma_{k_\circ}^\times \hookrightarrow k_\circ^\times \simeq (k_\circ^\times)^{\wedge}$$

— where the second arrow is the isomorphism determined by $\beta$ — and the composite

$$\sigma_{k_\circ}^\times \hookrightarrow G_{k_\circ}^{ab} \simeq G_{k_\circ}^{ab} \to (k_\circ^\times)^{\wedge}$$

— where the first arrow is the natural inclusion arising from local class field theory [cf. the discussion at the beginning of §1], the second arrow is the isomorphism determined by $\alpha^{-1}$, and the third arrow is the isomorphism arising from local class field theory — coincide on an open subgroup of $\sigma_{k_\circ}^\times$.

**Lemma 3.2.** Let $\alpha : G_{k_\circ} \simeq G_{k_\circ}$ be a continuous isomorphism; $\beta_1, \beta_2 : k_\circ \simeq k_\circ$ isomorphisms of fields. Suppose that $\beta_1, \beta_2$ are inertially compatible with $\alpha$ [cf. Definition 3.1]. Then $\beta_1 \simeq \beta_2$.

**Proof.** Since $\beta_1, \beta_2$ are inertially compatible with $\alpha$, one verifies easily from the various definitions involved that there exists an open subgroup $S_\circ \subseteq \sigma_{k_\circ}^\times$ of $\sigma_{k_\circ}^\times$ such that $\beta_1|_{S_\circ} = \beta_2|_{S_\circ}$. On the other hand, let us recall from [1], Lemma 4.1, that the sub-$\mathbb{Q}_p$-vector space of $k_\circ$ generated by $S_\circ$ coincides with $k_\circ$. Thus, the equality $\beta_1|_{S_\circ} = \beta_2|_{S_\circ}$ implies the equality $\beta_1 = \beta_2$. This completes the proof of Lemma 3.2.

Next, we prove the main theorem of the present paper. Note that the argument given in the proof of Theorem 3.3 is essentially the same as the argument applied in [1] to prove the main theorem of [1].
Theorem 3.3. Let $p$ be a prime number. For $\square \in \{\diamond, \bullet\}$, let $k_\square$ be a $p$-adic local field and $\overline{k}_\square$ an algebraic closure of $k_\square$. Write $G_{k_\square} \overset{\text{def}}{=} \text{Gal}(\overline{k}_\square/k_\square)$. Let

$$\alpha : G_{k_\diamond} \longrightarrow G_{k_\bullet}$$

be an open continuous homomorphism. Suppose that $\alpha$ is of HT-qLT-type [cf. Definition 1.3, (ii)]. Then $\alpha$ is geometric [cf. [2], Definition 3.1, (iv)], i.e., arises from an isomorphism of fields $\overline{k}_\diamond \sim \overline{k}_\bullet$ that determines an embedding $k_\diamond \hookrightarrow k_\bullet$.

Proof. First, let us observe that it follows from Proposition 2.4, together with the implication $(2) \Rightarrow (3)$ of Lemma 1.4, that $\alpha$ is injective. Next, let us observe that, to verify Theorem 3.3, by replacing $G_{k_\bullet}$ by the image of $\alpha$, we may assume without loss of generality that $\alpha$ is an isomorphism.

Now I claim that the following assertion holds:

Claim 3.3.A: Suppose that $k_\diamond$ is Galois over $\mathbb{Q}_p$. Then there exists an open subgroup $\beta$ of $\text{Gal}(\overline{k}_\diamond/k_\diamond)$ that is inertially compatible with $\alpha$ [cf. Definition 3.1].

Indeed, let $E$ be a finite Galois extension of $\mathbb{Q}_p$ that admits embeddings $E \hookrightarrow \overline{k}_\diamond$, $E \hookrightarrow \overline{k}_\bullet$ such that if we write $E_\diamond \subseteq E_\bullet \subseteq \overline{k}_\bullet$ for the respective images of these embeddings [so $E_\diamond \sim E_\bullet \sim E_\diamond$], then $k_\diamond \subseteq E_\diamond$, $k_\bullet \subseteq E_\bullet$. Let $k'_\diamond \subseteq (\overline{k}_\diamond)$ be a finite Galois extension of $k_\diamond$ such that $k'_\diamond$ contains $E_\diamond$, and, moreover, the finite [necessarily Galois] extension $k'_\bullet \subseteq (\overline{k}_\bullet)$ of $k_\bullet$ corresponding to the open subgroup $\alpha(G_{k'_\diamond}) \subseteq G_{k_\bullet}$ contains $E_\bullet$. For $\square \in \{\diamond, \bullet\}$, write $\sigma_\square : E_\square \hookrightarrow k'_\square$ for the natural inclusion. Write $\chi$ for the composite

$$G_{k'_\diamond} \overset{\alpha}{\longrightarrow} G_{k'_\bullet} \overset{\chi_{\text{LT}}}{\longrightarrow} E_\bullet^\times \overset{\sim}{\longrightarrow} E_\diamond^\times \overset{\sim}{\longrightarrow} E_\diamond^\times.$$

Then since $\alpha$ is of HT-qLT-type, it holds that $\chi$ is Hodge-Tate. Thus, since $E_\diamond$ is Galois over $\mathbb{Q}_p$, it follows from [4], Chapter III, §A.5, Corollary, that $\chi$ is inertially equivalent to the character

$$\prod_{\sigma \in \text{Gal}(E_\diamond/\mathbb{Q}_p)} (\chi_{\sigma, \text{cris}})^{n_\sigma} : G_{k'_\diamond} \longrightarrow E_\diamond^\times \overset{\sim}{\longrightarrow} E_\diamond^\times$$

for some choices of integers $n_\sigma$.

For $\square \in \{\diamond, \bullet\}$, write $\text{Ver}_{k'_\square/k_\square} : G_{k'_\square} \longrightarrow G_{k_\square}$ for the Verlagerung map with respect to the finite Galois extension $k'_\square/k_\square$. Then since $\chi$ is inertially equivalent to $\prod_{\sigma \in \text{Gal}(E_\diamond/\mathbb{Q}_p)} (\chi_{\sigma, \text{cris}})^{n_\sigma}$, and [one verifies easily from local class field theory that] $\text{Ver}_{k'_\square/k_\square}$ maps $\sigma^\times_{k_\square} \subseteq G_{k_\square}$ [cf. the discussion at the beginning of §1] to $\sigma^\times_{k_\square} \subseteq G_{k_\square}$, we conclude that there exists an open subgroup $S_\diamond \subseteq \sigma^\times_{k_\diamond} \subseteq G_{k_\diamond}$ of $\sigma^\times_{k_\diamond}$ such that if we write $S_\diamond \subseteq \sigma_{k_\diamond}$ for the image of $S_\diamond \subseteq \sigma^\times_{k_\diamond}$ by the isomorphism

$$(G_{k_\diamond} \overset{\sim}{\longrightarrow} \sigma^\times_{k_\diamond}) \overset{\sim}{\longrightarrow} \sigma_{k_\diamond} \subseteq G_{k_\diamond},$$

induced by $\alpha$ [where let us recall from Proposition 1.1 that $\alpha$ induces an isomorphism $I_{k_\diamond} \sim I_{k_\bullet}$, then the diagram of topological modules

$$
\begin{array}{ccc}
S_\diamond & \longrightarrow & G_{k_\diamond}^\text{ab} \\
\downarrow & & \downarrow \\
S_\bullet & \longrightarrow & G_{k_\bullet}^\text{ab}
\end{array}
$$

— where the left-hand vertical arrow is the isomorphism induced by $\alpha$, and the left-hand horizontal arrows are the natural inclusions — commutes. On the
other hand, it follows immediately from local class field theory, together with
Definition 1.2, (ii), that, for $\square \in \{\circ, \bullet\}$, if we write $\mathrm{Im}(I_{k_{\square}}) \subseteq G_{k_{\square}}^{ab}$ for the image
of the composite $I_{k_{\square}} \hookrightarrow G_{k_{\square}} \twoheadrightarrow G_{k_{\square}}^{ab}$ [i.e., $\omega_{k_{\square}}^{\times} \subseteq G_{k_{\square}}^{ab}$ — cf. the discussion at the
beginning of §1], then we have commutative diagrams of topological modules

$$
\begin{array}{ccc}
\mathrm{Im}(I_{k_{\circ}}) & \xrightarrow{\sigma_{k_{\circ}} / k_{\circ}} & \mathrm{Im}(I_{k_{\bullet}}) \\
\downarrow & & \downarrow \\
\sigma_{k_{\circ}}^\times & \longrightarrow & \sigma_{k_{\bullet}}^\times
\end{array}
\quad
\begin{array}{ccc}
\Pi_{\sigma \in \mathcal{G}(E_{\circ} / Q_p)}(\sigma_{k_{\circ}}) & \xrightarrow{\chi_{k_{\circ}}} & E_{\circ}^\times \\
\Pi_{\sigma \in \mathcal{G}(E_{\bullet} / Q_p)}(\sigma_{k_{\bullet}}) & \xrightarrow{\chi_{k_{\bullet}}} & E_{\bullet}^\times
\end{array}

E_{\circ}^\times \simeq E_{\bullet}^\times $
Claim 3.3.B: For every pair of respective finite extensions $k_0'$ ($\subseteq \overline{k}$), $k_0''$ ($\subseteq \overline{k}$) of $k_0$, there exists $\alpha(n)$ [necessarily unique — cf. Lemma 3.2] $\alpha$-isomorphism of fields $\beta_{k_0',k_0'',k_0'''}: k_0' \sim k_0'''$ that is inertially compatible with the restriction $\alpha|_{G^{\prime}}: G^{\prime} \sim G^{\prime}$. Indeed, let $k_0''$ ($\subseteq \overline{k}$) be a finite extension of $k_0'$ that is Galois over $\mathbb{Q}_p$. Write $k_0'''$ ($\subseteq \overline{k}$) for the finite [necessarily Galois] extension of $k_0'$, corresponding to the open subgroup $\alpha(G^{\prime}k_0'') \subseteq G^{\prime}$. Then it follows from Claim 3.3.A that there exists an isomorphism of fields $\beta_{k_0',k_0''}^{\prime}: k_0'' \sim k_0''$ that is inertially compatible with the restriction $\alpha|_{G^{\prime}}: G^{\prime}k_0'' \sim G^{\prime}k_0''$. Then one verifies easily from Lemma 3.2, together with the fact that $\beta_{k_0',k_0''}^{\prime}$ is inertially compatible with the restriction $\alpha|_{G^{\prime}k_0''}$, that $\beta_{k_0',k_0''}^{\prime}$ is compatible with the respective natural actions of $\text{Gal}(k_0''/k_0')$, $\text{Gal}(k_0''/k_0'')$ on $k_0''$, $k_0'''$ [relative to the isomorphism $\text{Gal}(k_0''/k_0') = G^{\prime}k_0''/G^{\prime}k_0'' \sim G^{\prime}k_0''/G^{\prime}k_0'' = \text{Gal}(k_0''/k_0'')$ induced by $\alpha|_{G^{\prime}}$]. Thus, we conclude that the isomorphism $\beta_{k_0',k_0''}$ determines an isomorphism $\beta_{k_0',k_0''}^{\prime}: k_0' \sim k_0''$. On the other hand, again by Lemma 3.2, together with the fact that $\beta_{k_0',k_0''}$ is inertially compatible with the restriction $\alpha|_{G^{\prime}k_0''}$, it follows immediately from this isomorphism $\beta_{k_0',k_0''}$ is inertially compatible with the restriction $\alpha|_{G^{\prime}k_0''}$. This completes the proof of Claim 3.3.B.

Now, by applying Claim 3.3.B to the various finite extensions of $k_0$, we obtain an isomorphism of fields $\beta_{k_0',k_0''}^{\prime}: k_0' \sim k_0''$, that determines an isomorphism $k_0' \sim k_0$. Moreover, again by applying Claim 3.3.B, one verifies easily that $\alpha$ arises from this isomorphism $\beta_{k_0',k_0''}^{\prime}$. This completes the proof of Theorem 3.3.

Remark 3.3.1. Theorem 3.3 leads naturally to the following observation:

Let $p$ be an odd prime number and $\overline{\mathbb{Q}}_p$ an algebraic closure of the $p$-adic completion $\mathbb{Q}_p$ of the field of rational numbers $\mathbb{Q}$.

Write $G_{\overline{\mathbb{Q}}_p} \overset{\text{def}}{=} \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. Then there exist an automorphism $\alpha$ of $G_{\overline{\mathbb{Q}}_p}$ and a finite dimensional continuous representation $\phi: G_{\overline{\mathbb{Q}}_p} \to \text{GL}_n(\mathbb{Q}_p)$ of $G_{\overline{\mathbb{Q}}_p}$ such that $\phi$ is potentially locally algebraic, i.e., the restriction of $\phi$ to an open subgroup of $G_{\overline{\mathbb{Q}}_p}$ is locally algebraic [cf. [4], Chapter III, §1, Definition] [hence Hodge-Tate], the set of Hodge-Tate weights of $\phi$ is contained in $\{0, 1\}$, but $\phi \circ \alpha$ is not Hodge-Tate.

Indeed, let us first observe that it follows immediately from the discussion given at the final part of [3], Chapter VII, §5, that we have an automorphism $\alpha$ of $G_{\overline{\mathbb{Q}}_p}$ that is not geometric [cf. [2], Definition 3.1, (iv)]. Thus, it follows from Theorem 3.3 that $\alpha$ is not of $HT$-qLT-type [cf. Definition 1.3, (ii)]. In particular, since the character $\chi^{\text{LT}_q}$ defined in Definition 1.2, (ii), is locally algebraic [cf. [4], Chapter III, §1, Example (2)], and the set of Hodge-Tate weights is contained in $\{0, 1\}$ [cf., e.g., [4], Chapter III, §A.5, Theorem 2], it follows from the definition of a homomorphism of $HT$-qLT-type that there exist normal open subgroups $H_1, H_2 \subseteq G_{\overline{\mathbb{Q}}_p}$ and a finite dimensional continuous representation $\phi_{H_2}: H_2 \to \text{GL}_n(\mathbb{Q}_p)$ of $H_2$ such that $\alpha(H_1) \subseteq H_2$, $\phi_{H_2}$ is locally algebraic, the set of Hodge-Tate weights of $\phi_{H_2}$ is contained in $\{0, 1\}$, and, moreover, $\phi_{H_2} \circ \alpha: H_1 \to \text{GL}_n(\mathbb{Q}_p)$ is not Hodge-Tate. Thus, it follows immediately from a similar argument to the argument applied in the proof of the implication (1) $\Rightarrow$ (1’) of Lemma 1.4 that if we write $\phi$ for the finite dimensional continuous
representation of $G_{\Q_p}$ obtained by inducing $\phi_H$, from $H_2$ to $G_{\Q_p}$, then $\phi$ is potentially locally algebraic [cf. also [4], Chapter III, §A.7, Theorem 3], the set of Hodge-Tate weights of $\phi$ is contained in $\{0, 1\}$, but $\phi \circ \alpha$ is not Hodge-Tate.

**Corollary 3.4.** In the notation of Theorem 3.3, consider the following nine conditions:

1. $\alpha$ is HT-preserving [cf. Definition 1.3, (i)].
2. $\alpha$ is of HT-qLT-type [cf. Definition 1.3, (ii)].
3. $\alpha$ is geometric [cf. [2], Definition 3.1, (iv)].
4. $\alpha$ is of qLT-type [cf. [2], Definition 3.1, (iv)].
5. $\alpha$ is of 01-qLT-type [cf. [2], Definition 3.1, (iv)].
6. $\alpha$ is of CHT-type [cf. [2], Definition 3.1, (i)].
7. $\alpha$ is HT-type [cf. [2], Definition 3.1, (iv)].
8. $\alpha$ is [an isomorphism and] RF-preserving [cf. [2], Definition 3.6, (iii)].
9. $\alpha$ is [an isomorphism and] uniformly toral [cf. [2], Definition 3.6, (iii)].

Then we have equivalences and implications

$$ (8) \iff (9) \implies (1) \iff (2) \iff (3) \iff (4) \iff (5) \iff (6) \implies (7). $$

If, moreover, $\alpha$ is an isomorphism, then the above nine conditions are equivalent.

**Proof.** Let us recall from Remark 1.4.1 that we have implications

$$ (4) \implies (5) \implies (6) \implies (1) \implies (2) $$

and

$$ (6) \implies (7). $$

The implication $(2) \implies (3)$ follows from Theorem 3.3. The implication $(3) \implies (4)$ follows from [2], Theorem 3.5, (i). The equivalence $(8) \iff (9)$ and the implication $(8) \implies (3)$ follow from [2], Corollary 3.7. Finally, the implication $(7) \implies (6)$ (respectively, $(3) \implies (8)$) in the case where $\alpha$ is an isomorphism follows immediately from [1], Proposition 1.1 (respectively, [2], Corollary 3.7). This completes the proof of Corollary 3.4.

**Corollary 3.5.** Let $p$ be a prime number. For $\square \in \{\circ, \bullet\}$, let $k_\square$ be a $p$-adic local field and $\bar{k}_\square$ an algebraic closure of $k_\square$. Write $G_{k_\square} \defeq \Gal(\bar{k}_\square/k_\square)$;

$$ \Emb(\bar{k}_\bullet/k_\bullet, \bar{k}_\circ/k_\circ) $$

for the set of isomorphisms of fields $\bar{k}_\bullet \cong \bar{k}_\circ$ that determine embeddings $k_\bullet \hookrightarrow k_\circ$;

$$ \Emb(k_\bullet, k_\circ) $$

for the set of embeddings of fields $k_\bullet \hookrightarrow k_\circ$;

$$ \Hom_{\text{HT}}^{\text{open}}(G_{k_\bullet}, G_{k_\circ}) $$

for the set of open continuous homomorphisms $\alpha: G_{k_\circ} \to G_{k_\bullet}$ that are HT-preserving [cf. Definition 1.3, (i)], i.e., for every finite dimensional continuous representation $\phi: G_{k_\bullet} \to \GL_n(\Q_p)$ of $G_{k_\bullet}$, if $\phi$ is Hodge-Tate, then $\phi \circ \alpha$ is Hodge-Tate. Then we have a commutative diagram of natural maps

$$ \begin{array}{ccc}
\Emb(\bar{k}_\bullet/k_\bullet, \bar{k}_\circ/k_\circ) & \longrightarrow & \Hom_{\text{HT}}^{\text{open}}(G_{k_\bullet}, G_{k_\circ}) \\
\downarrow & & \downarrow \\
\Emb(k_\bullet, k_\circ) & \longrightarrow & \Hom_{\text{HT}}^{\text{open}}(G_{k_\bullet}, G_{k_\circ})/\text{Inn}(G_{k_\bullet})
\end{array} $$

where the vertical arrows are surjective, and the horizontal arrows are bijective.
Proof. The injectivity of the horizontal arrows follow immediately from the injectivity portion of [1], Theorem 4.2 [cf. also the proof of [1], Theorem 4.2]. The surjectivity of the horizontal arrows follow immediately from Theorem 3.3, together with the implication (1) \( \Rightarrow (2) \) of Lemma 1.4. This completes the proof of Corollary 3.5.

References


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