$T_X$-Approaches to Multiflows and Metrics (Combinatorial Optimization and Discrete Algorithms)

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$T_X$-Approaches to Multiflows and Metrics

By

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Abstract

This paper is an exposition of a unified approach to multiflow problems using certain polyhedral objects called tight spans or $T_X$-spaces. The tight span was introduced by Isbell and Dress, independently of the multiflow research. In the middle of 90’s, Karzanov and Chepoi explored the significance of tight spans in the multiflow theory. We explain how the tight span derives min-max relations to multiflow problems and how its geometry affects discreteness issues of flows and potentials.

§ 1. Introduction

This paper is an exposition of a unified approach to multiflow problems using certain polyhedral objects called tight spans or $T_X$-spaces. The tight span was introduced by Isbell [22] and Dress [9] for a metric, independently of the multiflow research. By a metric $\mu$ on a set $S$ we mean a function defined on $S \times S$ satisfying $\mu(s, t) = \mu(t, s) \geq \mu(s, s) = 0$ and the triangle inequality $\mu(s, t) + \mu(t, u) \geq \mu(s, u)$ for $s, t, u \in S$. The tight span $T_\mu$ for $\mu$ is defined to be the set of (pointwise) minimal elements of the unbounded polyhedron

$$P_\mu = \{ p \in \mathbb{R}^S \mid p(s) + p(t) \geq \mu(s, t) \ (s, t \in S) \}.$$ 

Although a duality relationship between multiflows and metrics was known in 70’s [21, 33], it was the middle of 90’s when the significance of tight spans in the multiflow theory was revealed by Karzanov [27, 28] and Chepoi [4]. Recently, the author [12] considered the tight span of a possibly nonmetric distance $\mu$, where by a distance $\mu$ we mean a function on $S \times S$ satisfying only $\mu(s, t) = \mu(t, s) \geq \mu(s, s) = 0$ for $s, t \in S$. The
subsequent paper [13] showed that nonmetric tight spans provide general combinatorial
duality relations for weighted maximum multiflow problems, unifying previously known
results for 0-1-weighted and metric-weighted maximum multiflow problems [24, 28]. In
this paper, following [13] (and [14]), we explain how the tight span \( T_\mu \) derives such a
combinatorial min-max relation in multiflow problems and how geometry of \( T_\mu \) affects
discreteness issues of flows and potentials.

We begin by introducing basic notions. Let \( G \) be an undirected graph with nonneg-
ative edge capacity \( c : EG \to \mathbb{R}_+ \). Let \( S \subseteq VG \) be the set of terminals. A multiflow is a
pair \((P, \lambda)\) of a set \( P \) of paths connecting distinct terminals in \( S \) and a flow-value func-
tion \( \lambda : P \to \mathbb{R}_+ \) satisfying the capacity constraint \( \sum_{P \in P : e \in P} \lambda(P) \leq c(e) \) \( (e \in EG) \).
This paper mainly deals with the following multiflow maximization problems. For a
distance \( \mu \) on terminal set \( S \), the \( \mu \)-weighted maximum multi
ow problem \((\mu \text{-problem})\) is formulated as follows:

\[
(1.1) \quad \text{Maximize } \sum_{P \in P} \mu(s_P, t_P) \lambda(P) \text{ over all multiflows } (P, \lambda) \text{ for } (G, c),
\]

where \( s_P, t_P \) are the ends of \( P \). \( \mu \)-problems contain several basic multiflow problems. In
particular, the 0-1 case is of particular combinatorial interest. In this case, 0-1 distance
\( \mu \) is regarded as the commodity graph, and \( \mu \)-problem is the problem of maximizing
the total sum of multiflows connecting pairs of terminals specified by \( \mu(s, t) = 1 \). For
example, the case where \( S \) is a 2-set \( \{s, t\} \) with \( \mu(s, t) = 1 \) corresponds to the single-
commodity flow problem. The case where \( S \) is a 4-set \( \{s, t, s', t'\} \) and \( \mu(s, t) = \mu(s', t') = 1 \) and others are zero corresponds to the two-commodity flow problem. The case of
\( \mu(s, t) = 1 \) for all distinct \( s, t \in S \) is called the free multiflow problem. In the three cases
above, there are combinatorial duality relations: Ford-Fulkerson’s max-flow min-cut
theorem [10], Hu’s max-biflow min-cut theorem [19], and Lovász-Cherkassky’s duality
theorem [6, 31].

A \( TX \)-approach we would like to describe here gives a unified derivation of such
duality relations. The core of this approach is to consider the following continuous location problem on the tight span \( T_\mu \) endowed with the \( l_\infty \)-distance:

\[
(1.2) \quad \text{Minimize } \sum_{xy \in EG} c(xy) ||\rho(x) - \rho(y)||_\infty
\]

subject to \( \rho : VG \to T_\mu, \)

\[\rho(s) \in T_{\mu,s} \quad (s \in S),\]

where the sets \( T_\mu, T_{\mu,s} \subseteq \mathbb{R}^S \) are defined by

\( P_\mu = \{p \in \mathbb{R}^S \mid p(s) + p(t) \geq \mu(s, t) \ (s, t \in S)\}, \)

\( T_\mu = \text{the set of minimal elements of } P_\mu, \)

\( T_{\mu,s} = \{p \in T_\mu \mid p(s) = 0\} \quad (s \in S). \)
We call the problem (1.2) the tight-span-dual (T-dual) to μ-problem. In analogy to the network flow theory, we call ρ in T-dual (1.2) a potential. T-dual is indeed a dual of μ-problem as follows.

**Theorem 1.1 ([13]).** The maximum value of μ-problem is equal to the minimum value of T-dual.

Furthermore, if dim $T_\mu \leq 2$, then T-dual can be discretized as follows.

**Theorem 1.2 ([13]).** If $\mu$ is a rational distance with dim $T_\mu \leq 2$, then there exists a finite set $Z$ in $T_\mu$ such that for every graph $(G, c)$ with $S \subseteq VG$ we can take an optimal potential $\rho$ in T-dual with $\rho(VG) \subseteq Z$.

Namely, if dim $T_\mu \leq 2$, then T-dual is reduced to the following discrete location problem:

$$\text{Minimize } \sum_{xy \in EG} c(xy)\|\rho(x) - \rho(y)\|_\infty$$

subject to $\rho : VG \to T_\mu \cap Z$,

$$\rho(s) \in T_{\mu,s} \cap Z (s \in S).$$

This might be regarded as an analogous phenomenon of the discreteness of potential in the network flow theory, and gives a general combinatorial duality relation for μ-problems with dim $T_\mu \leq 2$, which includes max-flow min-cut theorem, max-biflow min-cut theorem, and so on. This duality relation was recognized in the case of metrics $\mu$ [27, 28]. The main contribution of [13] is to extend it to general distances $\mu$.

This paper is organized as follows. In Section 2, we prove Theorem 1.1. In Section 3, we illustrate several examples of T-dual to explain how T-dual derives combinatorial duality relations. In Section 4, we give a sketch of the proof of Theorem 1.2 with emphasis on a geometric intuition. In Section 5, we explain an application of tight spans to metric packing problems, which are dual to multiflow feasibility problems. In Section 6, we describe related issues, future research directions, and open questions.

**Notation.** $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{Z}$ denote the sets of reals, nonnegative reals, and integers, respectively. For a set $S$, the characteristic vector $\chi_S$ is defined as: $\chi_S(s) = 1$ for $s \in S$ and $\chi_S(t) = 0$ for $t \notin S$. We simply denote $\chi_{\{s\}}$ by $\chi_s$. For a graph $G$, $VG$ and $EG$ denote the sets of vertices and edges of $G$, respectively. For a graph $G$, $\text{dist}_G$ is the graph metric on $VG$ induced by $G$. A subgraph $G'$ of $G$ is called isometric if $\text{dist}_{G'}(x, y) = \text{dist}_G(x, y)$ for $x, y \in VG'$. We use the basic terminology in the polytope theory, such as faces, extreme points, polyhedral subdivisions; see [38]. We call a $k$-dimensional face a $k$-face.
§ 2. T-dual to maximum multiflow problems

In this section, we prove Theorem 1.1. It is known that the LP-dual of $\mu$-problem is given by the following:

\begin{equation}
\begin{aligned}
\text{Minimize} & \quad \sum_{xy \in EG} c(xy)d(x, y) \\
\text{subject to} & \quad d: \text{metric on } VG, \\
& \quad d(s, t) \geq \mu(s, t) \quad (s, t \in S).
\end{aligned}
\end{equation}

This is a variant of the so-called Japanese theorem due to Onaga and Kakusho [33] and Iri [21]. We show that (2.1) is reduced to $T$-dual (1.2). The proof consists of two lemmas. The first lemma states that LP-dual (2.1) is reduced to a location problem on $P_{\mu}$.

**Lemma 2.1.** The optimal value of $\mu$-problem is equal to the minimum value of the following problem:

\begin{equation}
\begin{aligned}
\text{Minimize} & \quad \sum_{xy \in EG} c(xy)\|\rho(x) - \rho(y)\|_\infty \\
\text{subject to} & \quad \rho: VG \to P_{\mu}, \\
& \quad \rho(s) \in P_{\mu,s} \quad (s \in S),
\end{aligned}
\end{equation}

where the subset $P_{\mu,s} \subseteq P_{\mu}$ for $s \in S$ is defined by

\begin{equation}
P_{\mu,s} = \{ p \in P_{\mu} \mid p(s) = 0 \}.
\end{equation}

**Proof.** For $\rho: VG \to P_{\mu}$ with $\rho(s) \in P_{\mu,s} \quad (s \in S)$, define a metric $d^\rho$ on $VG$ by

$$d^\rho(x, y) = \|\rho(x) - \rho(y)\|_\infty \quad (x, y \in VG).$$

By definition, we have $d^\rho(s, t) = \|\rho(s) - \rho(t)\|_\infty \geq (\rho(s))(t) - (\rho(t))(t) = (\rho(s))(t) + (\rho(s))(s) - \mu(s, t)$, where we use $\rho(s)(s) = 0$ by $\rho(s) \in P_{\mu,s}$ and $(\rho(s))(t) + (\rho(s))(s) \geq \mu(s, t)$ by $\rho(s) \in P_{\mu}$; see also Lemma 2.3 (2). Thus $d^\rho$ is feasible to (2.1). Conversely, take a metric $d$ feasible to (2.1). Define a map $\rho^d: VG \to \mathbb{R}^S$ by

$$\rho^d(x)(s) = d(s, x) \quad (s \in S, x \in VG).$$

By definition of $\rho^d(x)$ and the triangle inequality, $\rho^d$ is feasible to (2.2). Furthermore, the triangle inequality $d(x, y) \geq |d(x, s) - d(s, y)|$ implies $d(x, y) \geq \|\rho^d(x) - \rho^d(y)\|_\infty$. The nonnegativity of $c$ implies

$$\sum_{xy \in EG} c(xy)d(x, y) \geq \sum_{xy \in EG} c(xy)\|\rho^d(x) - \rho^d(y)\|_\infty.$$
Thus we can always take an optimal solution of (2.1) as $d^\rho$ for some potential $\rho$ in (1.2).

The second lemma, due to Dress, states the existence of a nonexpansive retraction from $P_\mu$ to $T_\mu$.

**Lemma 2.2** ([9, (1.9)]). There is a map $\phi : P_\mu \to T_\mu$ such that

1. $\|\phi(p) - \phi(q)\|_\infty \leq \|p - q\|_\infty$ for $p, q \in P_\mu$, and
2. $\phi(p) \leq p$ for $p \in P_\mu$, and thus $\phi$ is identical on $T_\mu$.

**Proof.** For $s \in S$, define $\phi_s : P_\mu \to P_\mu$ by

$$\phi_s(p) := p - \chi_s \max\{\epsilon \geq 0 \mid p - \epsilon \chi_s \in P_\mu\}.$$ 

Then, one can show that $\phi_s$ satisfies (1-2). Indeed, (2) is obvious by definition. For (1), we have

$$\phi_s(p)(s) = \max_{u \in S \setminus s} \{0, \mu(u, s) - p(u)\} = \max_{u \in S \setminus s} \{0, \mu(u, s) - q(u) + q(u) - p(u)\}$$

$$\leq \max_{u \in S \setminus s} \{0, \mu(u, s) - q(u)\} + \max_{u \in S \setminus s} \{0, q(u) - p(u)\}$$

$$\leq \phi_s(q)(s) + \|p - q\|_\infty.$$ 

Thus we have $|\phi_s(p)(s) - \phi_s(q)(s)| \leq \|p - q\|_\infty$ and (1). Let $S = \{s_1, s_2, \ldots, s_m\}$. Then the map $\phi_{s_m} \circ \phi_{s_{m-1}} \circ \cdots \circ \phi_{s_1}$ is a desired one. 

Since $c$ is nonnegative, by Lemma 2.2, we can always take an optimal solution of (2.2) from potentials in $T$-dual (1.2). Thus we obtain Theorem 1.1. By the proof of Lemma 2.1, the relationship between LP-dual (2.2) and $T$-dual (1.2) is given as follows:

- For a metric $d$ *minimal* in the feasible region of LP-dual (2.2), the map $\rho^d$ defined by

$$\rho^d(x)(s) = d(s, x) \quad (x \in VG, s \in S)$$

is a potential in $T$-dual (1.2).

- For a potential $\rho$ in $T$-dual (1.2), a metric $d^\rho$ defined by

$$(2.4) \quad d^\rho(x, y) = \|\rho(x) - \rho(y)\|_\infty \quad (x, y \in VG)$$

is feasible to LP-dual (2.2).

We end this section with listing basic properties of $T_\mu$. 

Lemma 2.3.

1. \((T_\mu, l_\infty)\) is geodesic, i.e., for \(p, q \in T_\mu\) there is a path in \(T_\mu\) of length \(\|p - q\|_\infty\).

2. \(\mu(s, t) = \inf\{\|p - q\|_\infty \mid p \in T_{\mu, s}, q \in T_{\mu, t}\}\) for \(s, t \in S\).

3. If \(\mu\) is a metric, then \(T_{\mu, s}\) is a single point \(\mu_s\) defined by
   \[\mu_s(t) = \mu(s, t) \quad (t \in S),\]
   i.e., \(\mu_s\) is the \(s\)-th row vector of \(\mu\).

4. \(p(s) = \inf\{\|p - q\|_\infty \mid q \in T_{\mu, s}\}\) for \(p \in T_\mu\) and \(s \in S\).

5. \(|p(s) - p(t)| \leq \mu(s, t)\) for \(p \in T_\mu\) and \(s, t \in S\).

(1) is an easy corollary of Lemma 2.2. (2-3) were shown in [12] and (4) was shown in [13]. (4) follows from (1) and (3). They are extensions of [9, Theorem 2 (i), (ii), and (iv)]. The properties (2-3) mean that \(\mu\) is isometrically embedded into \(T_\mu\) as the \(l_\infty\)-distance among subsets \(T_{\mu, s}\).

§ 3. Some examples

In this section, we explain how \(T\)-duals derive combinatorial duality relations in multiflow maximization problems. The message of this section is:

The shape of \(T_\mu\) is a min-max formula of \(\mu\)-problem.

§ 3.1. Single commodity flows

The first example is well-known single commodity flow problem. In this case, the terminal \(S\) is a 2-set \(\{s, t\}\), and \(\mu(s, t) = 1\). Therefore, \(P_\mu\) is an unbounded polyhedron in the plane, \(T_\mu\) is the segment, and \(T_{\mu, s} = \{\chi_t\}\) and \(T_{\mu, t} = \{\chi_s\}\) are the endpoints of the segment; see Figure 1 (a). Therefore, \(T\)-dual is equivalent to the following problem:

\[
\text{(3.1) } \begin{cases} \min & \sum_{xy \in EG} c(xy)|\rho(x) - \rho(y)| \\ \text{subject to } & \rho : V G \to [0, 1], \rho(s) = 0, \rho(t) = 1. \end{cases}
\]

Namely, \(\rho\) is an ordinary (scalar) potential. This problem can be discretized into the minimum cut problem as follows. We can easily see that for any map \(\rho : V G \to [0, 1]\), the corresponding metric \(d^\rho\) can be represented as a convex combination of \(\{d^\rho_i\}_{i \in I}\) for maps \(\rho_i : V \to \{0, 1\}\). Therefore, (3.1) is discretized into

\[
\text{Minimize } \sum_{xy \in EG} c(xy)|\rho(x) - \rho(y)| \\
\text{subject to } \rho : V G \to \{0, 1\}, \rho(s) = 0, \rho(t) = 1.
\]
This is nothing but the minimum cut problem. Then we obtain Ford-Fulkerson’s max-flow min-cut theorem (without integrality of optimal flows).

§ 3.2. Free multiflows

The second example we consider is the free multiflow problem. The corresponding distance \( \mu \) is all-one distance, that is, \( \mu(s, t) = 1 \) for distinct \( s, t \in S \). Then \( T_\mu \) is a star having the center \( 1/2 \chi_S \), \#S leaves \( T_{\mu, s} = \{\chi_{S \setminus s}\} \ (s \in S) \), and edge length \( 1/2 \); see Figure 1 (b) for the three terminal case. By the argument similar to the previous single flow example, \( T \)-dual is discretized into the discrete location problem on the star as follows. Let \( \Gamma \) be the graph of 1-skeleton of \( T_\mu \), which is the star with the center \( p^O = 1/2 \chi_S \) and the leaves \( p^s = \chi_{S \setminus s} \ (s \in S) \). Then \( T \)-dual is reduced to:

\[
\text{Minimize } \frac{1}{2} \sum_{xy \in EG} c(xy) \text{dist}_\Gamma(\rho(x), \rho(y))
\]

subject to \( \rho : VG \to V \Gamma, \rho(s) = p^s \ (s \in S) \).

From this, we obtain Lovász-Cherkassky duality relation (without half-integrality of optimal flows):

\[
\text{the optimal multiflow value} = \frac{1}{2} \sum_{t \in S \setminus t \text{ minimum cut value}}.
\]

§ 3.3. Two-commodity flows

The third example is the two-commodity flow maximization problem. This case corresponds to: \( S \) is a 4-set \( \{s, s', t, t'\} \), and \( \mu(s, t) = \mu(s', t') = 1 \) and others are

\[
\begin{array}{c|ccc}
\mu & s & t & u \\
\hline
s & 0 & 1 & 1 \\
t & 1 & 0 & 1 \\
u & 1 & 1 & 0 \\
\end{array}
\]
zero. Then $P_\mu$ is a 4-dimensional polyhedron, and thus we cannot draw it. Its minimal element $T_\mu$, however, is 2-dimensional. Indeed, an easy calculation shows

$$T_\mu = \frac{1}{2} \chi_S + \{a(\chi_s - \chi_t) + b(\chi_{s'} - \chi_{t'}) \mid -1/2 \leq a, b \leq 1/2\}.$$ 

Therefore $T_\mu$ is isomorphic to the unit square in the $l_\infty$-plane ($\mathbb{R}^2, l_\infty$) by projection to $\mathbb{R}^{(s,s')}$, and $T_{\mu,s}$ ($s \in S$) are its four edges; see Figure 2 (a). We show that $T$-dual in this case is also discretized. Recall the well-known fact that the $l_\infty$-plane ($\mathbb{R}^2, l_\infty$) is isomorphic to the $l_1$-plane ($\mathbb{R}^2, l_1$) by 45 degree rotation

$$(x_1, x_2) \mapsto \left(\frac{x_1 + x_2}{2}, \frac{x_1 - x_2}{2}\right).$$

By the map, $T_\mu$ is isomorphic to the square in the $l_1$-plane; see Figure 2 (b). We can subdivide $T_\mu$ into four isosceles right triangles with its shorter edges parallel to $l_1$-axes as in Figure 2 (c), where $l_1$-axes mean vectors $(1, 1)$ and $(1, -1)$ in ($\mathbb{R}^2, l_\infty$) or $(0, 1)$ and $(1, 0)$ in ($\mathbb{R}^2, l_1$). This subdivision is denoted by $\Delta$. Let $\Gamma$ be the graph formed by the shorter edges of these four triangles. The graph $\Gamma$ is a star with the center $p^O$ and four leaves $p^{ss'}, p^{st'}, p^{ts'}, p^{tt'}$ and edge lengths are $1/2$. Then $T$-dual is again reduced to the discrete location problem on $\Gamma$:

\begin{equation}
\begin{align*}
\text{(3.2)} \\
\quad \text{Minimize} \quad & \frac{1}{2} \sum_{xy \in EG} c(xy) \text{dist}_\Gamma(\rho(x), \rho(y)) \\
\text{subject to} \quad & \rho : VG \rightarrow V\Gamma, \\
& \rho(s) \in \{p^{ss'}, p^{st'}\}, \quad \rho(s') \in \{p^{ss'}, p^{ts'}\}, \\
& \rho(t) \in \{p^{ts'}, p^{tt'}\}, \quad \rho(t') \in \{p^{st'}, p^{tt'}\}.
\end{align*}
\end{equation}

Indeed, for any (rational) potential $\rho : VG \rightarrow T_\mu$, the corresponding metric $d^\rho$ defined by (2.4) can be represented as a convex combination of $d^{\rho_i}$ ($i \in I$) for potentials $\rho_i$ satisfying the constraints of (3.2). We give an intuitive proof of this fact by using
illustrations. We may assume that the image of $\rho$ is rational. Then we can further subdivide $T_\mu$ into $1/k$-smaller squares and isosceles right triangles so that the image of $\rho$ lies on the vertices of this subdivision as in Figure 3. This subdivision is denoted by $\Delta^k$. We choose a set $O$ of edges, called an orbit, of this subdivision by the following way. Take an arbitrary edge $e$ of this subdivision, and set $O = \{e\}$. If there is a square having $e' \in O$ and $e'' \notin O$ as a parallel pair of edges, then set $O \leftarrow O \cup \{e''\}$. If there is a triangle having $e' \in O$ and $e'' \notin O$, then set $O \leftarrow O \cup \{e''\}$. Then all edges are partitioned into $k$ orbits.

Contract all edges in $O$. Then we obtain $(k-1)/kT_\mu$. Expand it in factor $k/(k-1)$. From this, we obtain a feasible potential $\rho' : VG \to T_\mu$ whose the image lies on the vertices of the subdivision $\Delta^{k-1}$. We construct one more potential. Contract all edges not in $O$. Then we obtain $1/kT_\mu$. Expand it in factor $k$. From this, we obtain a potential $\rho' : VG \to T_\mu$ whose the image lies on the vertices of $\Delta$. Then one can easily see that

$$d'' = \frac{k-1}{k} d' + \frac{1}{k} d'''. $$

Repeat this process to $\rho'$. We obtain a desired convex combination.

Moreover, we can take an optimum $\rho$ satisfying $(\rho(s), \rho(t)) = (\rho(s'), \rho(t')) = (p^{st'}, p^{ts'})$ or $(\rho(s), \rho(t)) = (\rho(t'), \rho(s')) = (p^{st'}, p^{ts'})$. From this, we obtain Hu’s max-biflow min-cut theorem [19] (without half-integrality of optimal flows):

the optimal multiflow value = min\{ss'-tt' mincut value, st'-ts' mincut value\}.

§ 4. Geometry of $T_\mu$ and a general combinatorial min-max formula

In this section, we explain the constriction of $Z$ in Theorem 1.2. The essential idea has already been described in the two-commodity example in Section 3.3. Namely, subdivide $T_\mu$ into squares and isosceles right triangles. Then we can take $Z$ to be vertices
Figure 4. (a) gluing 2-faces and (b) an $l_1$-grid

of this subdivision. The following two propositions guarantee that this approach indeed works. The first one concerns the shape of 2-faces of $T_\mu$.

**Proposition 4.1.** Let $F$ be a 2-face of $T_\mu$. Then the metric space $(F, l_\infty)$ is isomorphic to the polygon $Q$ in the $l_\infty$-plane represented as

$$Q = \left\{ (x_1, x_2) \in \mathbb{R}^2 \left| \begin{array}{l} a_1 \leq x_1 \leq a'_1, b \leq x_1 + x_2 \leq b', \\ a_2 \leq x_2 \leq a'_2, c \leq x_1 - x_2 \leq c' \end{array} \right. \right\}$$

for some $a_1, a'_1, a_2, a'_2, b, b', c, c' \in \mathbb{R}$. Moreover, the isometry is given by the projection $\mathbb{R}^S \to \mathbb{R}^{\{s,t\}}$ for some $s, t \in S$.

A polygon represented as (4.1) is exactly a convex polygon each of whose edges is parallel to one of the four vectors $(1,0), (0,1), (1,1), (1,-1)$. Recall that the $l_\infty$-plane is the $l_1$-plane. By the map $(x_1, x_2) \mapsto ((x_1 + x_2)/2, (x_1 - x_2)/2)$, we again obtain a convex polygon in the $l_1$-plane each of whose edges is parallel to one of the four vectors $(1,0), (0,1), (1,1), (1,-1)$. If we draw the $l_1/l_\infty$-coordinate on a 2-face $F$, then we observe that there are two types of edges of $F$: edges parallel to an $l_1$-axis and edges parallel to an $l_\infty$-axis. Here an $l_1$-axis means a vector $(0,1)$ or $(1,0)$, and an $l_\infty$-axis means a vector $(1,1)$ or $(1,-1)$ by the isometric projection to $(\mathbb{R}^2, l_\infty)$ in Proposition 4.1.

The second one says that if $\dim T_\mu \leq 2$, the metric space $(T_\mu, l_\infty)$ is constructed by gluing such polygons along the same type of edges; see Figure 4 (a).

**Proposition 4.2.** Suppose $\dim T_\mu \leq 2$. Let $F, F'$ be 2-faces of $T_\mu$ sharing an edge $e$. The edge $e$ is parallel to an $l_1$-axis on $F$ if and only if $e$ is parallel to an $l_1$-axis on $F'$. 
In the following, we prove the first proposition (Proposition 4.1) by explaining a basic method to investigate \( T_\mu \), which will be often used in the subsequent arguments. For a point \( p \in P_\mu \), we define an undirected graph \( K(p) \) on \( S = V K(p) \) by

\[
\text{st} \in EK(p) \overset{\text{def}}{=} p(s) + p(t) = \mu(s, t) \quad (s, t \in S).
\]

Note that a loop appears at \( s \in S \) exactly when \( p(s) = 0 \). The graph \( K(p) \) expresses the information of faces of \( P_\mu \) which contain \( p \).

Take \( p^* \) in the relative interior of a face \( F \). Suppose that \( K(p^*) \) has \( m \) bipartite components with bipartitions \( \{A_1, B_1\}, \{A_2, B_2\}, \ldots, \{A_m, B_m\} \). Then it is easy to see that the set of vectors \( \{\chi_{A_i} - \chi_{B_i}\}_{i=1}^m \) is a basis of the vector space \( \{p \in \mathbb{R}^S \mid p(s) + p(t) = 0 \ (st \in EK(p^*))\} \). Then every point \( p \) in \( F \) is uniquely represented as

\[
(4.2) \quad p = p^* + \sum_{i=1}^m x_i(\chi_{A_i} - \chi_{B_i})
\]

for \( (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m \). Therefore we have the following.

**Proposition 4.3** ([9]). For \( p \in T_\mu \), let \( F(p) \) be the face containing \( p \) as its relative interior. Then we have

\[
\dim F(p) = \text{the number of bipartite components of } K(p),
\]

where loops are regarded as odd cycles.

In the expression (4.2), the map \( p \mapsto (x_1, x_2, \ldots, x_m) \) is an injective isometry from \( (F, l_\infty) \) to \( (\mathbb{R}^m, l_\infty) \) since each \( \chi_{A_i} - \chi_{B_i} \) is a 0-1 vector. From this fact, we easily obtain Proposition 4.1. Indeed, By substituting (4.2) with \( k = 2 \) to linear inequalities \( p(s) + p(t) \geq \mu(s, t) \ (s, t \in S) \), we obtain the linear inequality representation (4.1). In particular, the isometry is given by the projection \( \mathbb{R}^S \rightarrow \mathbb{R}^{\{s, t\}} \) for \( s \in A_1 \cup B_1, t \in A_2 \cup B_2 \).

### § 4.1. \( l_1 \)-grids

Suppose that \( \mu \) is a rational distance with \( \dim T_\mu \leq 2 \). By Propositions 4.1 and 4.2, there exists a polyhedral subdivision \( \Delta \) of \( T_\mu \) satisfying the following conditions:

\( (*1) \) 2-faces of \( \Delta \) consist of squares and isosceles right triangles.

\( (*2) \) for each square \( F \in \Delta \), its edges are parallel to \( l_1 \)-axes of the 2-face containing \( F \), and for each triangle \( F \in \Delta \), its shorter edges are parallel to \( l_1 \)-axes of the 2-face containing \( F \).
We call a polyhedral subdivision $\Delta$ of $T_\mu$ with (*1-2) an $l_1$-grid; see Figure 4 (b). An edge of $\Delta$ is called an $l_1$-edge if it is not the longer edge of a triangle in $\Delta$. If all $l_1$-edges of $\Delta$ have the same length $\alpha$, then we call $\Delta$ $\alpha$-uniform. A uniform $l_1$-grid always exists if $\mu$ is rational. The graph of $l_1$-edges realizes the $l_\infty$-distance on $T_\mu$ as follows.

**Proposition 4.4.** Let $\Delta$ be an $l_1$-grid of $T_\mu$. For two vertices $p, q$ in $\Delta$, there is a path of length $\|p - q\|_\infty$ between $p$ and $q$ consisting of $l_1$-edges of $\Delta$.

The finite set $Z$ in Theorem 1.2 can be taken to be the vertices of an $l_1$-grid satisfying a certain orientability condition. An $l_1$-grid $\Delta$ is orientable if edges of $\Delta$ are oriented so that

(o1) a parallel pair of edges of each square has the same direction, and

(o2) an acute angle of each triangle is a sink or a source.

See Figure 5. By subdivision, we can always make a nonorientable $l_1$-grid orientable; see below. Let $\Delta$ be an orientable $\alpha$-uniform $l_1$-grid $\Delta$ for a rational $\alpha$. Let $\Gamma$ be the graph of $l_1$-edges of $\Delta$. Then we have the following.

**Proposition 4.5.** There exists an optimal potential $\rho$ in $T$-dual with $\rho(VG) \subseteq \gamma T$ for any capacitated-graph $(G, c)$ with $S \subseteq VG$

Let $V_s = \gamma T \cap T_{\mu,s}$ be the subset of vertices contained in $T_{\mu,s}$. Then $T$-dual is discretized into the following. This is a general combinatorial dual that yields previously known multiflow duality relations.

$$\begin{align*}
\text{Minimize} & \quad \alpha \sum_{xy \in EG} c(xy) \text{dist}_T(\rho(x), \rho(y)) \\
\text{subject to} & \quad \rho : VG \to \gamma T, \\
& \quad \rho(s) \in V_s \quad (s \in S).
\end{align*}$$

The proof of Proposition 4.5 is essentially the same as the proof of two-commodity example in Section 3.3. We do not repeat it here. Instead, we explain why the orientability condition is needed.
condition is required. Consider the 0-1 distance $\mu$ of commodity graph $C_5$ (five cycle). Then $T_\mu$ is a pentagon obtained by gluing five isosceles right triangles along their right angle. Therefore $T_\mu$ has the 1/2-uniform $l_1$-grid $\Delta$ consisting of these five triangles. Then $\Delta$ is not orientable. If we apply the method used in two-commodity example, then an orbit of the subdivided $l_1$-grid $\Delta^k$ goes around the original $l_1$-grid $\Delta$ twice and the image of $\rho''$ does not lies on the vertices of $\Delta$. In this case, $\Delta^2$ is a desired orientable 1/4-uniform $l_1$-grid. This method making $\Delta$ orientable by subdivision is called an orbit splitting [13]; it is a slight modification of the original definition given in [28]. In particular, if an $l_1$-grid exists, then an orientable $l_1$-grid always exists.

§ 4.2. Half-integrality, lattice, and the folder decomposition

If there is an orientable 1/k-uniform $l_1$-grid for a positive integer $k$, then we can take an 1/k-integral optimal solution in (2.1) by Proposition 4.4 and the correspondence $\rho \mapsto d^\rho$ in (2.4).

**Theorem 4.6** ([13]). **If $\mu$ is an integral distance with $\dim T_\mu \leq 2$, then there exists an orientable 1/4-uniform $l_1$-grid, and thus there exists a 1/4-integral optimal solution in LP-dual (2.1) of $\mu$-problem.**

To show the existence of the 1/4-uniform $l_1$-grid is not difficult. This immediately follows from the fact that the polyhedron $P_\mu$ is half-integral if $\mu$ is integral. To show the orientability of this 1/4-uniform $l_1$-grid is not so easy. Here we give a sketch of this fact for the case of metric $\mu$. We state it in a shaper form. An integral metric $\mu$ is called a cyclically even if $\mu(x, y) + \mu(y, z) + \mu(z, x)$ is even for $x, y, z \in S$. Clearly $2\mu$ is cyclically even for every integral metric $\mu$.

**Theorem 4.7.** **If $\mu$ is a cyclically even metric, then there exists an orientable 1/2-uniform $l_1$-grid.**

This theorem is essentially due to Karzanov [28]. His approach is graph-theoretical. Here we describe a different approach using a lattice (a discrete subgroup) in $\mathbb{R}^S$. Let
$L$ be a lattice in $\mathbb{R}^S$ defined by

$$L = \{ p \in \mathbb{R}^S \mid p(s) + p(t) \in \mathbb{Z} \ (s, t \in S) \}.$$ 

This lattice is known as the *weight lattice of type B* in the representation theory of semisimple Lie algebras. Note that $L \subseteq (1/2)\mathbb{Z}^S$ and all extreme points of $T_\mu$ lie on $L$. By cyclically evenness, a simple calculation shows

$$\mu_s - \mu_t \in 2L \quad (s, t \in S),$$

where $\mu_s$ is the $s$-th row vector of $\mu$ defined in Lemma 2.3 (3). So we can define the affine lattice $A_\mu$ by

$$A_\mu = \mu_s + 2L.$$ 

Consider the graph $\Gamma_0$ of $A_\mu$ by connecting pairs of points in $A_\mu$ having unit $l_\infty$-distance. Namely, $V\Gamma_0 = A_\mu$ and $pq \in E\Gamma_0$ if $\|p - q\|_\infty = 1$, or equivalently, $p - q \in \{1, -1\}^S$. Let $\Gamma$ be the subgraph of $\Gamma_0$ induced by $T_\mu \cap A_\mu$. For 2-face $F$ of $T_\mu$, the projection of $F \cap A_\mu$ to $\mathbb{R}^2$ coincides with the intersection of a polygon and a translation of the lattice

$$\{(x_1, x_2) \in \mathbb{Z}^2 \mid x_1 + x_2 \in 2\mathbb{Z}\}.$$ 

This immediately follows from (4.2). Figure 7 illustrates $A_\mu$ and $L$ with a 2-face. If all extreme points of $T_\mu$ belong to $A_\mu$, then $\Gamma$ coincides with the graph of the integral uniform $l_1$-grid of $T_\mu$. However there may exist an extreme point of $T_\mu$ not in $A_\mu$ as indicated by the arrow in Figure 7.

Delete edges $E\Gamma$ from $T_\mu$ and consider (the closure of) the connected components. Then the connected components are classified into the following:

1. a square formed by a 4-cycle in $\Gamma$ lying on some 2-face.
(2) a set containing a part of an edge of $T_\mu$ parallel to an $l_\infty$-axis of some 2-face, which is obtained by gluing $m(\geq 3)$ isosceles right triangles along the common longer edge and is the interior of a subgraph $K_{2,m}$ of $\Gamma$.

(3) a set containing an extreme point $p$ not belonging to $A_\mu$, which is obtained by taking join of $p$ and a complete bipartite subgraph $K_{n,m}$ ($n, m \geq 3$) of $\Gamma$.

They are called folders. A folder of (1), (2), and (3) is called a square, a $K_{2,m}$-folder, and a $K_{n,m}$-folder, respectively. See Figure 8 for three types of folders. The decomposition of $T_\mu$ into these folders is called the folder decomposition.

By subdividing each folder and orienting its graph as in Figure 9, we obtain the 1/2-uniform $l_1$-grid $\Delta$ with orientation. Therefore the 1/2-uniform $l_1$-grid $\Delta$ is orientable. Thus we have Theorem 4.7. In particular, the vertices of this 1/2-uniform $l_1$-grid $\Delta$ is given explicitly by the intersection of $T_\mu$ and the lattice $L$. Then the discrete version of
T-dual (4.3) is also represented as a discrete convex optimization on the lattice $L$:

Minimize $\sum_{xy \in EG} c(xy)\|\rho(x) - \rho(y)\|_{\infty}$

subject to $\rho : VG \to T_\mu \cap L$, $\rho(s) \in T_\mu,s \cap L$ ($s \in S$).

This problem resembles an $L$-convex function minimization in Murota’s theory of discrete convex analysis [32]. It would be interesting to explore such an analogy and develop discrete convexity theory for multiflows.

§ 4.3. Unbounded fractionality

In Section 4.1, we saw that 2-dimensionality of $T_\mu$ brings a combinatorial duality relation to $\mu$-problem. On the other hand, we cannot expect such a combinatorial duality relation for the case $\dim T_\mu \geq 3$. Here we explain this fact. Recall that LP-dual (2.1) to $\mu$-problem is a linear optimization over the polyhedron

$$\mathcal{P}_{\mu,V} = \{ d : \text{metric on } V(\supseteq S) \mid d(s,t) \geq \mu(s,t) \ (s,t \in S) \} + \mathbb{R}_{+}^{V \times V}.$$

**Theorem 4.8 ([13]).** For an integral distance $\mu$ on $S$, if $\dim T_\mu \geq 3$, then there is no positive integer $k$ such that $\mathcal{P}_{\mu,V}$ is $1/k$-integral for every $V$ with $V \supseteq S$.

This follows from the fact that there exists an infinite increasing series of finite subsets $P_1 \subseteq P_2 \subseteq \cdots$ in $(\mathbb{R}^3, l_\infty)$ such that the corresponding metrics are extreme in the metric cones. The standard TDI argument shows the following corollary.

**Corollary 4.9.** If $\dim T_\mu \geq 3$, then there is no positive integer $k$ such that $\mu$-problem has a $1/k$-integral optimal multflow for every integer-capacitated graph $(G,c)$ with $S \subseteq VG$.

§ 5. An application to metric packing

In this section, we describe an application of the folder decomposition of 2-dimensional tight spans, introduced in Section 4.2, to metric packing problems. The basic idea of such an approach is due to Chepoi [4]. Extending his approach, we solve Karzanov’s conjecture concerning metric packing problems for the case where the commodity graph is vertex-disjoint sum of two triangles.

Let us introduce metric packing problems. Let $G,H$ be undirected graphs with $VH \subseteq VG$. $H$ is called a commodity graph. We assume that $G$ is connected. A set of
metrics $\{\mu_i\}_{i=1}^m$ on $VG$ is called an $H$-packing if it satisfies
\[
\sum_{i=1}^k \mu_i(x, y) \leq \text{dist}_G(x, y) \quad (x, y \in VG),
\]
\[
\sum_{i=1}^k \mu_i(s, t) = \text{dist}_G(s, t) \quad (st \in EH).
\]

The existence of an $H$-packing by special combinatorial metrics is of central interest. The most simplest metric is a cut metric. A metric $\mu$ on $V$ is called a cut metric if there is $X \subseteq V$ such that
\[
\mu(x, y) = \begin{cases} 1 & \text{if } x \neq y, \#(\{x, y\} \cap X) = 1, \\ 0 & \text{otherwise} \end{cases} \quad (x, y \in V).
\]

A classical theorem in the network flow theory, often called the max-potential min-work theorem, says:

**Theorem 5.1** ([35]). If $H = K_2$, then there exists an $H$-packing by cut metrics.

This is a polar theorem of Ford-Fulkerson’s max-flow min-cut theorem. In fact, a metric packing problem is known to be polar to a multiflow feasibility problem; see [36, Section 70.12].

Karzanov extended Theorem 5.1 to the following multiterminal version, which generalizes Seymour’s two-commodity cut packing theorem [37], and also strengthens Papernov’s characterization of commodity graphs with the property that the cut condition is sufficient for the multiflow feasibility [34].

**Theorem 5.2** ([23]). If $G$ is bipartite and $H$ is $K_4$, $C_5$ or the union of two stars, then there exists an $H$-packing by cut metrics.

If $H$ is none of those graphs in this theorem, then an $H$-packing by cut metrics does not exist in general. However, by using some class of metrics beyond cut metrics, one can expect further combinatorial metric packing results. To describe it, we need some notation. For a graph $\Gamma$, a metric $\mu$ on $V$ is called a $\Gamma$-metric if there is a map $\phi : V \rightarrow V\Gamma$ such that
\[
\mu(x, y) = \text{dist}_\Gamma(\phi(x), \phi(y)) \quad (x, y \in V).
\]

In particular, a cut metric is just a $K_2$-metric. For a set $\mathcal{G}$ of graphs, a $\mathcal{G}$-metric is a $\Gamma$-metric for some $\Gamma \in \mathcal{G}$. Extending Theorem 5.2, Karzanov showed the following.

**Theorem 5.3** ([25]). If $G$ is bipartite and $H$ is $K_5$ or the union of a star and a triangle, then there exists an $H$-packing by $\{K_2, K_{2, 3}\}$-metrics.
It is known that if $H$ has a matching of size 3, there is no finite set $G$ of graphs with the property that every graph $G$ with $VH \subseteq VG$ admits $H$-packing by proportions of $G$-metrics [25, Section 3]. The graphs without matching of size 3 are classified into the following:

1. $K_4$, $C_5$, and the union of two stars.
2. $K_5$ and the union of a star and a triangle.
3. The vertex-disjoint union of two triangles.

Theorems 5.2 and 5.3 solve the cases (1) and (2), respectively. For the case (3), Karzanov conjectured that there is an $H$-packing by \{$K_2$, $K_{2,3}$, $1/2\Gamma_{3,3}$\}-metrics, where $\Gamma_{3,3}$ is the graph obtained by subdividing $K_{3,3}$ and connecting edges between all subdivided points and newly added one point [25, Section 3]. Namely, $\Gamma_{3,3}$ is the graph of the subdivision of the $K_{3,3}$-folder in Figure 9. Recently, [14] solved this conjecture in a stronger form.

**Theorem 5.4** ([14]). If $G$ is bipartite and $H$ is the vertex-disjoint union of two triangles, then there exists an $H$-packing by \{$K_2$, $K_{2,3}$, $K_{3,3}$, $\Gamma_{3,3}$\}-metrics.

Note that a $K_{3,3}$-metric is a submetric of a $1/2\Gamma_{3,3}$-metric.

§ 5.1. Extremal graphs and geometry of $T_\mu$

Here we sketch a proof of Theorem 5.4 given in [14]. For a metric $\mu$ on $V$, an extremal graph $H$ of $\mu$ is a graph with $VH \subseteq V$ satisfying the following property:

(*) for any distinct $x, y \in V$, there exists $st \in EH$ such that

$$\mu(s,t) = \mu(s,x) + \mu(x,y) + \mu(y,t).$$

This means that every pair $x, y \in V$ is a part of a shortest path of some $s, t \in VH$. Metric packing problems in bipartite graphs with commodity graph $H$ are reduced to a problem of decomposing cyclically even metrics having $H$ as its extremal graph. Recall that an integral metric $\mu$ is called cyclically even if $\mu(x,y) + \mu(y,z) + \mu(z,x)$ is even for any $x,y,z$.

**Lemma 5.5.** Let $H$ be a graph. Let $G$ be a finite set of graphs. If any cyclically even metric $\mu$ having $H$ as an extremal graph is decomposed into an integral sum of $G$-metrics, then every bipartite graph $G$ with commodity graph $H$ admits an $H$-packing by $G$-metrics.

For a proof of this fact, see [25, pp. 476–477]. Therefore, it suffices to consider the decomposition property of cyclically even metrics having $H$ as its extremal graph. The following proposition connects extremal graph $H$ and geometry of $T_\mu$. 

Proposition 5.6. If an extremal graph $H$ of a metric $\mu$ has no matching of size $k$, then the dimension of $T_\mu$ is at most $k - 1$.

Proof. Suppose that $\dim T_\mu \geq k$. By Proposition 4.3, there is $p \in T_\mu$ such that $K(p)$ has $k$ bipartite components. We show that each bipartite component has at least one edge of $H$. Take an edge $xy$ from a bipartite component of $K(p)$. Then, by definition of $K(p)$, we have $p(x) + p(y) = \mu(x, y)$. For some $s, t \in VH$, we have

$$\mu(s, t) \leq p(s) + p(t) = p(s) - p(x) + p(x) + p(y) - p(y) + p(t) \leq \mu(s, x) + \mu(x, y) + \mu(y, t) = \mu(s, t),$$

where we use Lemma 2.3 (5) in the second inequality. This implies $st \in EK(p)$, and similarly $sy, xt \in EK(p)$. Therefore, if $H$ has no matching of size 3, then $T_\mu$ is 2-dimensional, and thus we can apply the folder decomposition of 2-dimensional tight spans, introduced in Section 4.2. Let $\Gamma$ be the graph of $T_\mu \cap A_\mu$. By Lemma 2.3 and (a slight modification of) Proposition 4.4, $\mu$ is a submetric of $\text{dist}_\Gamma$. We decompose $\text{dist}_\Gamma$ by using orbits as in Section 3.3. A pair of edges $e, e'$ of $\Gamma$ is called projective if there is a sequence of edges $e = e_1, e_2, \ldots, e_m = e'$ such that $e_i$ and $e_{i+1}$ are edges of some folder of type (2) or (3), or are parallel edges of a folder of type (1). The projectivity is an equivalence relation on $E\Gamma$. An equivalence class is called an orbit. Let $O$ be the set of all orbits of $\Gamma$. For an orbit $o \in O$, the orbit graph $\Gamma^o$ is the graph obtained from $\Gamma$ by contracting edges $E\Gamma \setminus o$ and deleting parallel edges appeared. This construction naturally gives a map $\phi^o : V\Gamma \to V\Gamma^o$ by defining $\phi^o(p)$ to be the contracted point. Then the following formula holds:

$$(5.1) \quad \text{dist}_\Gamma(p, q) = \sum_{o \in O} \text{dist}_{\Gamma^o}(\phi^o(p), \phi^o(q)) \quad (p, q \in V\Gamma).$$

This is a special case of the decomposition of a modular graph by Bandelt [1] or the canonical metric representation of a bipartite graph by Lomonosov and Sebő [30]; also see [8, Section 20.1]. Therefore, it suffices to determine orbit graphs of $\Gamma$. By analyzing $T_\mu$, one can show the following.

Lemma 5.7 ([14]). If an extremal graph $H$ of a cyclically even metric $\mu$ is the vertex-disjoint sum of two triangles, then an orbit graph of $\Gamma$ is $K_2$, $K_{2,3}$, $K_{3,3}$, or an isometric subgraph of $\Gamma_{3,3}$.

Thus we obtain Theorem 5.4.
§ 6. Concluding remarks

In this paper, we explained a unified approach to multiflow problems by using tight spans. We think that the potential of such a $T_X$-approach has not yet been fully exploited. Finally, we explain further related topics, future research directions, and open questions.

Minimum 0-extensions and minimizable graphs. Here we explain a relationship between tight spans and minimum 0-extension problems discovered by Karzanov [27]. For an undirected graph $G$ with nonnegative capacity $c \in \mathbb{R}^E_{\geq}$ and an undirected graph $\Gamma$ with $V\Gamma \subseteq V$, the minimum 0-extension problem is:

\[
\begin{align*}
\text{Minimize} & \quad \sum_{xy \in EG} c(xy) \text{dist}_\Gamma(\rho(x), \rho(y)) \\
\text{subject to} & \quad \rho : VG \to V\Gamma, \\
& \quad \rho(s) = s \ (s \in V\Gamma).
\end{align*}
\]

This problem is NP-hard since it contains the 3-terminal cut problem for $\Gamma = K_3$. Karzanov considered the following relaxation problem:

\[
\begin{align*}
\text{Minimize} & \quad \sum_{xy \in EG} c(xy) d(x, y) \\
\text{subject to} & \quad d : \text{metric on } VG, \\
& \quad d(s, t) = \text{dist}_\Gamma(s, t) \ (s, t \in V\Gamma).
\end{align*}
\]

$\Gamma$ is said to be minimizable if (6.1) and (6.2) have the same optimal value for every capacitiated graph $(G, c)$ with $V\Gamma \subseteq VG$. Karzanov gave an elegant characterization of minimizable graphs as follows.

Theorem 6.1 ([27]). $\Gamma$ is minimizable if and only if $\Gamma$ is bipartite, has no isometric cycles of length $k \geq 6$, and orientable.

Here a graph $\Gamma$ is orientable if $\Gamma$ can be oriented so that the orientations of edges $pq$ and $rs$ in every 4-cycle $(p, q, r, s)$ are opposite along the cycle as in Figure 5 (a). A bipartite graph without isometric cycles of length $k \geq 6$ is just a hereditary modular graph [2]. Orbits and related concepts that we used for $l_1$-grids were originally introduced for a class of modular graphs [27, 28].

A relation to our approach using $l_1$-grids is explained as follows. The relaxation (6.2) is the LP-dual to the $\mu$-problem for $\mu = \text{dist}_\Gamma$; the equality of the constraint of (2.1) is attained since $\mu = \text{dist}_\Gamma$ is a metric. Then $\Gamma$ necessarily coincides with the graph of an orientable $l_1$-grid of $T_\mu$, and (4.3) coincides with (6.1). Thus, $T_{\text{dist}_\Gamma}$ is obtained by
filling $l_1$-space to each 4-cycle in $\Gamma$ as in Figure 8 [27, 28]. Conversely, the graph $\Gamma$ of an orientable $l_1$-grid of $T_\mu$ for a (possibly nonmetric) distance $\mu$ is necessarily minimizable. See [5, 3, 16] for further developments in this direction.

**Fractionality of optimal multiflows.** In Section 4, we gave a general combinatorial min-max relation to $\mu$-problems for a distance $\mu$ with $\dim T_\mu \leq 2$. However, it is not a fully combinatorial min-max relation, since it says nothing about the existence of integral, half-integral, quarter-integral, or $1/k$-integral optimal multiflows for a fixed positive integer $k$. As seen in Section 4.3, we cannot expect that $\mu$-problems for a distance $\mu$ with $\dim T_\mu \geq 3$ have such a combinatorial min-max relation.

For 0-1 distance case, Karzanov conjectured:

**Conjecture 6.2 ([26]).** If commodity graph $H$ having no isolated vertices satisfies the following condition:

(P) for every intersecting triple $A, B, C$ of maximal stable sets, we have $A \cap B = B \cap C = C \cap A$,

then there exists a positive integer $k$ such that the maximum multiflow problem with respect to $H$ for any integer-capacitated graph $(G, c)$ with $V_H \subseteq V_G$ has a $1/k$-integral optimal multiflow.

Karzanov [24] showed that the condition (P) is a necessary condition for the existence of such a positive integer $k$, and gave a combinatorial min-max relation in this case. In fact, (P) is equivalent to the condition of the 2-dimensionality of the tight span of the 0-1 distance corresponding to $H$. A detailed description of $T_\mu$ for a 0-1 distance with the property (P) is given by [13, Section 7], and a combinatorial min-max relation from (4.3) coincides with Karzanov’s one. Therefore, the following conjecture extending the previous one might be reasonable.

**Conjecture 6.3.** For a distance $\mu$ on $S$, if $\dim T_\mu \leq 2$, then there exists a positive integer $k$ such that $\mu$-problem for any integer-capacitated graph $(G, c)$ with $S \subseteq V_G$ has a $1/k$-integral optimal multiflow.

Quite recently, there was a significant development in this research direction. [15] devised a new method for finding an optimal multiflow with bounded denominator. Based on this method together with the framework developed in [16], finally [17] solves this conjecture affirmatively for $k = 24$. However it is not known whether this constant $k = 24$ is tight.

**Directed multiflows and tropical polytopes.** It is natural to ask whether a geometric dual similar to $T$-dual exists for $\mu$-problems on digraphs. The forthcoming
paper [18] joint with S. Koichi answers this question; a part of this work appeared in his Ph.D. thesis [29]. For not necessarily symmetric distance $\mu : S \times S \to \mathbb{R}_+$, consider the following polyhedral sets:

$$Q_\mu = \{(p, q) \in \mathbb{R}^{S \times S} \mid p(s) + q(t) \geq \mu(s, t) \ (s, t \in S)\},$$

$$T_\mu = \text{the set of minimal elements of } (Q_\mu \cap \mathbb{R}^{S \times S}_+).$$

Then $T_\mu$ plays the same role of tight spans. Furthermore, when we restrict $\mu$-problems on Eulerian digraphs, the following subset $R_\mu$ of $T_\mu$

$$R_\mu = \mathbb{R}_+^{S \times S} \cap \{\text{the set of minimal elements of } Q_\mu\}$$

gives sharper duality relations including Frank’s directed version of free multiflow theorem [11] and Ibaraki-Karzanov-Nagamochi’s directed version of the multiflow locking theorem [20]. Interestingly, $R_\mu$ coincides with the intersection of the nonnegative orthant and a tropical polytope introduced by Sturmfels and Develin [7]. Then the dual of $\mu$-problem for an Eulerian digraph is reduced to a certain location problem on a tropical polytope.

References


