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Continuous wavelet transforms and non-commutative Fourier analysis

By

Hideyuki Ishi*

Abstract

We discuss continuous wavelet transforms for the semidirect product group of a unimodular (not necessarily commutative) normal subgroup $N$ with a closed subgroup $H$ of Aut($N$), which is a generalization of the wavelet theory for an affine transformation group on a vector space. The operator-valued Fourier transform for $N$ plays a substantial role in the arguments.

§ 1. Introduction

Let $G$ be a locally compact group, and $(\pi, \mathcal{H})$ an irreducible unitary representation of $G$. The representation $\pi$ is said to be \textit{square-integrable} if there exists a vector $\phi \in \mathcal{H}$ for which $\int_G |(\pi(g)\phi|\phi)|^2 \, dg < +\infty$, where $dg$ is a left Haar measure on $G$. Such $\phi$ is called an \textit{admissible vector} of $\pi$. If $(\pi, \mathcal{H})$ is square-integrable, there exists a unique positive self-adjoint operator (the \textit{Duflo-Moore operator}) $C_\pi$ with the following two properties ([7], [13]):

(C1) $\phi$ is admissible $\iff \phi \in \text{dom}(C_\pi)$,

(C2) For $f_1, f_2 \in \mathcal{H}$ and $\phi_1, \phi_2 \in \text{dom}(C_\pi)$, one has

$$\int_G (f_1|\pi(g)\phi_1|\mathcal{H})(\pi(g)\phi_2|f_2)\mathcal{H} \, dg = (f_1|f_2)\mathcal{H}(C_\pi\phi_1|C_\pi\phi_2)\mathcal{H}.$$  

When $G$ is unimodular, $C_\pi$ is a scalar operator. Furthermore, if $G$ is a compact group, then $C_\pi = (\dim \mathcal{H})^{-1/2} \text{Id}$. Thus, $C_\pi^{-2}$ is called the \textit{formal degree} of $\pi$ in general.
Regarding the equality in (C2) as an identity for \( f_2 \in \mathcal{H} \), we obtain

\[
(1.1) \quad f_1 = \frac{1}{(C_\pi \phi_1 | C_\pi \phi_2)_{\mathcal{H}}} \int_G (f_1 | \pi(g) \phi_1)_{\mathcal{H}} \pi(g) \phi_2 \, dg,
\]

where the integral is taken in the weak sense. For an admissible vector \( \phi \in \text{dom}(C_\pi) \), the continuous wavelet transform \( W_\phi \) is defined as a linear map from the Hilbert space \( \mathcal{H} \) into the space \( C(G) \) of continuous functions on the group \( G \) defined by

\[
W_\phi f(g) := (f | \pi(g) \phi)_{\mathcal{H}} \quad (f \in \mathcal{H}, \ g \in G).
\]

Then (1.1) tells us that the inverse formula of \( W_\phi \) is given by

\[
(1.2) \quad f = \frac{1}{c_\phi} \int_G W_\phi f(g) \pi(g) \phi \, dg,
\]

where \( c_\phi = \|C_\pi \phi\|^2_{\mathcal{H}} \).

For instance, let us consider the affine transformation group \( G_{\text{aff}} \) on the real line \( \mathbb{R} \) consisting of the maps \( g_{b,a} : \mathbb{R} \ni x \mapsto ax + b \in \mathbb{R} \) with \( a \in \mathbb{R} \setminus \{0\} \) and \( b \in \mathbb{R} \). The unitary representation \( L \) of \( G_{\text{aff}} \) is defined on the Hilbert space \( L^2(\mathbb{R}) \) by \( L(g_{b,a})f(x) := |a|^{-1/2} f(g_{b,a}^{-1}x) \) \( (x \in \mathbb{R}) \). This representation is square-integrable, and the Duflo-Moore operator is given by

\[
(C_L \phi)^\wedge(\xi) = \sqrt{\frac{2\pi}{|\xi|}} \hat{\phi}(\xi) \quad (\xi \in \mathbb{R}),
\]

where \( \wedge \) stands for the Fourier transform given by \( \hat{\phi}(\xi) := \int_{\mathbb{R}} e^{ix\xi} \phi(x) \, dx \) \( (\xi \in \mathbb{R}) \). Then \( c_\phi = \|C_L \phi\|^2 \) equals \( \int_{\mathbb{R}\setminus\{0\}} |\hat{\phi}(\xi)|^2 |\xi|^{-1} \, d\xi \), and if \( c_\phi < +\infty \), the equality (1.2) yields the Calderón formula

\[
(1.3) \quad f = \frac{1}{c_\phi} \int_{\mathbb{R}} \int_{\mathbb{R}^2} W_\phi f(b,a) L(g_{b,a}) \phi \frac{db \, da}{|a|^2} \quad (f \in L^2(\mathbb{R})),
\]

where \( W_\phi f(b,a) = W_\phi f(g_{b,a}) = |a|^{-1/2} \int_{\mathbb{R}} f(x) \hat{\phi}\left(\frac{x-b}{a}\right) \, dx \).

The results about the wavelet transform for \( G_{\text{aff}} \) have been generalized to a multi-dimensional affine group \( G = H \times \mathbb{R}^n \), where \( H \) is a closed subgroup of \( GL(\mathbb{R}^n) \) (see [4], [8] and [10] for example). In this article, we consider a further generalization to the case that \( G \) is a semidirect product group \( N \rtimes H \), where \( N \) is a unimodular (not necessarily commutative) group, and \( H \) is a closed subgroup of \( \text{Aut}(N) \) satisfying certain conditions. Although the situation becomes complicated if \( N \) is not commutative, the argument goes in parallel with the commutative case, where the operator-valued Fourier transform for \( N \) plays a substantial role instead of the ordinary Fourier transform.

The content of Sections 2–4 is essentially a summary of [16], while Section 5 is devoted to discuss a concrete example that \( N \) is the Heisenberg group. Such Heisenberg
case is already studied by He-Liu [14], whereas we shall present a new example of admissible vector \( \phi_{\pm, \alpha} \in L^2(N) \). In this article, we write \( \mathbb{T} \) for the set of complex numbers with absolute value 1. For a Hilbert space \( \mathcal{H} \), we denote by \( \mathcal{B}(\mathcal{H}) \), (resp. \( \mathcal{B}_{HS}(\mathcal{H}), \mathcal{B}_{Tr}(\mathcal{H}), U(\mathcal{H}) \)) the space of bounded (resp. Hilbert-Schmidt, trace class, unitary) operators on \( \mathcal{H} \).

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§ 2. Preliminaries

Let \( N \) be a separable locally compact unimodular group of type I. We denote by \( \hat{N} \) the unitary dual of \( N \), that is, the set of equivalence classes of irreducible unitary representations of \( N \). For each \( \lambda \in \hat{N} \), we take a unitary representation \( (\pi_\lambda, \mathcal{H}_\lambda) \) for which \( \lambda \) equals the equivalence class \([\pi_\lambda]\) of \( \pi_\lambda \). Let us fix a Haar measure \( \nu \) on \( N \). For \( f \in L^1(N) \), we define the bounded operator \( \pi_\lambda(f) \in \mathcal{B}(\mathcal{H}_\lambda) \) by \( \pi_\lambda(f) := \int_N f(n) \pi_\lambda(n) \, d\nu(n) \). It is known that the Plancherel measure \( \mu \) on \( \hat{N} \) is uniquely determined by the abstract Plancherel formula [6]:

\[
\int_N |f(n)|^2 d\nu(n) = \int_{\hat{N}} \|\pi_\lambda(f)\|_{HS}^2 d\mu(\lambda) \quad (f \in L^1(N) \cap L^2(N)),
\]

where \( \| \cdot \|_{HS} \) stands for the Hilbert-Schmidt norm. We define the operator-valued Fourier transform \( \mathbf{F} : L^2(N) \to \int_{\hat{N}} \mathcal{B}_{HS}(\mathcal{H}_\lambda) \, d\mu(\lambda) \) as the unitary isomorphism which is the extension of the map \( L^1(N) \cap L^2(N) \ni f \mapsto (\pi_\lambda(f))_{\lambda \in \hat{N}} \in \int_{\hat{N}} \mathcal{B}_{HS}(\mathcal{H}_\lambda) \, d\mu(\lambda) \).

The inverse formula of \( \mathbf{F} \) is given as follows:

**Proposition 2.1** ([11, Theorem 4.15]). Let \( (A(\lambda))_{\lambda \in \hat{N}} \) be an element of the direct integral \( \int_{\hat{N}} \mathcal{B}_{Tr}(\mathcal{H}_\lambda) \, d\mu(\lambda) \) of the Banach space \( \mathcal{B}_{Tr}(\mathcal{H}_\lambda) \). Define a function \( f \) on \( N \) by

\[
f(n) := \int_{\hat{N}} \text{tr} \, A(\lambda) \pi_\lambda(n)^* \, d\mu(\lambda) \quad (n \in N).
\]

Then \( f \) belongs to \( L^2(N) \) if and only if \( (A(\lambda))_{\lambda \in \hat{N}} \in \int_{\hat{N}} \mathcal{B}_{HS}(\mathcal{H}_\lambda) \, d\mu(\lambda) \). In that case, one has \( \mathbf{F}f(\lambda) = A(\lambda) \) (a.a. \( \lambda \in \hat{N} \)).

Let \( H \) be a closed subgroup of the automorphism group \( \text{Aut}(N) \) on \( N \), and \( G \) the semidirect product group \( N \rtimes H \). We write the action of \( h \) to \( n \in N \) as \( h \cdot n \). For \( h \in H \), we have a positive number \( \delta(h) \) for which \( d\nu(h \cdot n) = \delta(h) \, d\nu(n) \) \( (n \in N) \). Clearly, \( \delta : H \to \mathbb{R}_+ \) is a representation of \( H \). Define the unitary representation \( L \) of \( G \) on \( L^2(N) \) by

\[
(2.2) \quad L(h)f(n_0) := \delta(h)^{-1/2}f(h^{-1} \cdot n_0),
\]

\[
L(n)f(n_0) := f(n^{-1}n_0) \quad (f \in L^2(N), \, h \in H, \, n, n_0 \in N).
\]
It is easy to see that the representation $L$ is equivalent to the induced representation $\text{Ind}_H^G 1$, where $1$ is the trivial representation of $H$.

We define the action of $H$ on the unitary dual $\hat{N}$ by $h \cdot \lambda := [\pi_\lambda \circ h^{-1}]$ ($h \in H, \lambda \in \hat{N}$). Then we have a unitary operator $C(h, \lambda) : \mathcal{H}_\lambda \to \mathcal{H}_{h \cdot \lambda}$ with the property

$$C(h, \lambda)\pi_\lambda(h^{-1} \cdot n) = \pi_{h \cdot \lambda}(n)C(h, \lambda) \quad (n \in N).$$

The operator $C(h, \lambda)$ is unique up to multiple by elements of $T$ by Schur’s lemma. Moreover, for $h, h' \in H$ and $\lambda \in \hat{N}$, we have

$$(2.3) \quad C(hh', \lambda) = s_{h, h', \lambda}C(h, h' \cdot \lambda)C(h', \lambda)$$

with $s_{h, h', \lambda} \in T$. Thus we have a uniquely determined operator $D(h, \lambda) : B(\mathcal{H}_\lambda) \ni T \mapsto C(h, \lambda)TC(h, \lambda)^* \in B(\mathcal{H}_{h \cdot \lambda})$, which satisfies the chain rule

$$D(hh', \lambda) = D(h, h' \cdot \lambda)D(h', \lambda).$$

Using the operator-valued Fourier transform $F$, we describe the representation $(L, L^2(N))$ of $G$ as follows:

**Proposition 2.2.** For $f \in L^2(N)$, $h \in H$ and $n \in N$, one has

$$(2.4) \quad F[L(h)f](\lambda) = \delta(h)^{1/2}D(h, h^{-1} \cdot \lambda)Ff(h^{-1} \cdot \lambda),$$

$$(2.5) \quad F[L(n)f](\lambda) = \pi_\lambda(n)Ff(\lambda)$$

for almost all $\lambda \in \hat{N}$ with respect to the Plancherel measure $\mu$.

§ 3. Decomposition of $L^2(N)$

For $\lambda \in \hat{N}$, let $\mathcal{O}_\lambda^* \subset \hat{N}$ be the $H$-orbit through $\lambda$. Now let us assume that there exist elements $\lambda_1, \ldots, \lambda_K \in \hat{N}$ satisfying the following conditions:

(1) $\mu(\mathcal{O}_{\lambda_k}^*) > 0$ ($k = 1, \ldots, K$),

(2) The stabilizer $H_k := \{ h \in H ; h \cdot \lambda_k = \lambda_k \}$ is compact for all $k = 1, \ldots, K$.

(3) The map $H/H_k \ni hH_k \mapsto h \cdot \lambda_k \in \mathcal{O}_{\lambda_k}^*$ is a homeomorphism for $k = 1, \ldots, K$, where the topology on $\mathcal{O}_{\lambda_k}^*$ is induced from the Fell topology on $\hat{N}$.

(4) $\mathcal{O}_{\lambda_k}^* \cap \mathcal{O}_{\lambda_l}^* = \emptyset$ ($k \neq l$) and $\mu(\hat{N} \setminus \bigsqcup_{k=1}^K \mathcal{O}_{\lambda_k}^*) = 0$.

Under this assumption, we shall see in Sections 3 and 4 that the unitary representation $(L, L^2(N))$ of $G$ is decomposed into the direct sum of countably many irreducible subrepresentations, and such subrepresentations are all square-integrable. Their Duflo-Moore operators are described by using the operator-valued Fourier transform $F$.

Thanks to (2.3), we have a projective unitary representation $\tau_k : H_k \ni h \mapsto C(h, \lambda_k) \in U(\mathcal{H}_{\lambda_k})$ of the group $H_k$ for $k = 1, \ldots, K$. Since $H_k$ is compact, we have an
irreducible decomposition $\mathcal{H}_{\lambda k} = \bigoplus_{\alpha \in A_k} \mathcal{H}_{\lambda k, \alpha}$, where $A_k$ is an at most countable index set. The subspaces $\mathcal{H}_{\lambda k, \alpha}$ are finite dimensional. For $\lambda = \tilde{h} \cdot \lambda_k \in \mathcal{O}_{\lambda_k}^*$ with $\tilde{h} \in H$, we put $\mathcal{H}_{\lambda, \alpha} := C(\tilde{h}, \lambda_k)\mathcal{H}_{\lambda k, \alpha} \ (\alpha \in A_k)$, where the right-hand side is independent of the choice of $\tilde{h}$. Moreover we have $\mathcal{H}_\lambda = \bigoplus_{\alpha \in A_k} \mathcal{H}_{\lambda, \alpha}$, which gives an irreducible decomposition of the projective representation $\tau_\lambda : H \ni h \mapsto C(h, \lambda) \in U(\mathcal{H}_\lambda)$ of the compact group $H_\lambda := \{ h \in H ; h \cdot \lambda = \lambda \}$. On the other hand, the relation

$$C(h, \lambda)\mathcal{H}_{\lambda, \alpha} = \mathcal{H}_{h \cdot \lambda, \alpha}$$

holds for $h \in H$, $\lambda \in \mathcal{O}_{\lambda_k}^*$ and $\alpha \in A_k$ in general. Using the orthogonal projection $P_{\lambda, \alpha} : \mathcal{H}_\lambda \to \mathcal{H}_{\lambda, \alpha}$, we define

$$B_{\lambda, \alpha} := \left\{ T \in \mathcal{B}_{\mathcal{H}_\lambda} ; TP_{\lambda, \alpha} = T \right\}.$$  

If we identify $\mathcal{B}_{\mathcal{H}_\lambda}$ with the tensor product $\mathcal{H}_\lambda \otimes \overline{\mathcal{H}}_{\lambda}$, the space $B_{\lambda, \alpha}$ is nothing but $\mathcal{H}_\lambda \otimes \overline{\mathcal{H}}_{\lambda, \alpha}$. Thus we see that

$$B_{\mathcal{H}_\lambda} = \bigoplus_{\alpha \in A_k} B_{\lambda, \alpha},$$

while (3.1) yields

$$D(h, \lambda)B_{\lambda, \alpha} = B_{h \cdot \lambda, \alpha}.$$  

Now we set

$$L_{k, \alpha}(N) := \mathbf{F}^{-1} \left( \int_{\mathcal{O}_{\lambda_k}^*} B_{\lambda, \alpha} \, d\mu(\lambda) \right) $$

$$= \left\{ f \in L^2(N) : \begin{array}{ll} \mathbf{F}f(\lambda) = \mathbf{F}f(\lambda)P_{\lambda, \alpha} & (\text{a.a. } \lambda \in \mathcal{O}_{\lambda_k}^*) \\ \mathbf{F}f(\lambda) = 0 & \text{(otherwise)} \end{array} \right\}$$

for $k = 1, \ldots, K$ and $\alpha \in A_k$. By (3.3) and (H4), we have

$$L^2(N) = \bigoplus_{1 \leq k \leq K} \bigoplus_{\alpha \in A_k} L_{k, \alpha}(N),$$

and each $L_{k, \alpha}(N)$ is $G$-invariant thanks to Proposition 2.2 and (3.4).

§ 4. Main results

Fixing a left Haar measure $d_H$ on $H$, we define a left Haar measure $d_G$ on $G$ by $d_G(nh) := \delta(h)^{-1} d\nu(n) d_H(h) \ (n \in N, h \in H)$. Let us consider the square-integrability
of the matrix coefficients \((f|L(g)\phi) \quad (f, \phi \in L_{k,\alpha}(N), \ g \in G)\) of the representation \((L, L_{k,\alpha}(N))\) of \(G\). By (2.1) and (2.5), we have

\[
(f|L(n)L(h)\phi) = \int_{\mathcal{S}_k} \text{tr } Ff(\lambda)F[L(n)L(h)\phi](\lambda)^* \, d\mu(\lambda)
\]

\[
= \int_{\mathcal{S}_k} \text{tr } (Ff(\lambda)F[L(h)\phi](\lambda)^*)\pi_\lambda(n)^* \, d\mu(\lambda).
\]

(4.1)

We note that the operator \(Ff(\lambda)F[L(h)\phi](\lambda)^*\) is of trace class because both \(Ff(\lambda)\) and \(F[L(h)\phi](\lambda)\) are Hilbert-Schmidt operators. Furthermore, \((Ff(\lambda)F[L(h)\phi](\lambda)^*\lambda \in \mathcal{S}_k\) belongs to \(\int_{\mathcal{S}_k} B_{\text{Tr}}(\mathcal{H}_\lambda) \, d\mu(\lambda)\) because

\[
\int_{\mathcal{S}_k} \|Ff(\lambda)F[L(h)\phi](\lambda)^*\|_{\text{Tr}} \, d\mu(\lambda)
\]

\[
\leq \int_{\mathcal{S}_k} \|Ff(\lambda)\|_{\text{HS}} \|F[L(h)\phi](\lambda)\|_{\text{HS}} \, d\mu(\lambda)
\]

\[
\leq \left\{ \int_{\mathcal{S}_k} \|Ff(\lambda)\|_{\text{HS}}^2 \, d\mu(\lambda) \right\}^{1/2} \left\{ \int_{\mathcal{S}_k} \|F[L(h)\phi](\lambda)\|_{\text{HS}}^2 \, d\mu(\lambda) \right\}^{1/2}
\]

\[
= \|f\| \|L(h)\phi\| = \|f\| \|\phi\| < +\infty,
\]

where \(\| \cdot \|_{\text{Tr}}\) denotes the Trace norm. Now we assume that

\[
\int_G |(f|L(g)\phi)|^2 \, dG(g) = \int_H \int_N |(f|L(n)L(h)\phi)|^2 \delta(h)^{-1} \, d\nu(n) \, dH(h) < +\infty.
\]

Then \(\int_N |(f|L(n)L(h)\phi)|^2 \, d\nu(n)\) is finite for almost all \(h \in H\). Thus, applying Proposition 2.1, we see from (4.1) and (2.1) that

\[
\int_N |(f|L(n)L(h)\phi)|^2 \, d\nu(n) = \int_{\mathcal{S}_k} \|Ff(\lambda)F[L(h)\phi](\lambda)^*\|_{\text{HS}}^2 \, d\mu(\lambda) \quad (\text{a.a. } h \in H).
\]

(4.3)

Therefore the integral \(\int_G |(f|L(g)\phi)|^2 \, dG(g)\) is equal to

\[
\int_H \int_{\mathcal{S}_k} \|Ff(\lambda)F[L(h)\phi](\lambda)^*\|_{\text{HS}}^2 \delta(h)^{-1} \, d\mu(\lambda) \, dH(h).
\]

(4.4)

Now we observe that

\[
\int_H \|Ff(\lambda)F[L(h)\phi](\lambda)^*\|_{\text{HS}}^2 \delta(h)^{-1} \, dH(h)
\]

\[
= \int_H \int_{H_\lambda} \left( \text{tr } Ff(\lambda)F[L(h_1h)\phi](\lambda)^*F[L(h_1h)\phi](\lambda)Ff(\lambda)^* \right) \delta(h_1h)^{-1} \, dh_1 \, dH(h),
\]

(4.5)
where \( dh_1 \) is the normalized Haar measure on the compact group \( H_\lambda \). Since \( L(h_1h)\phi \) belongs to \( L_{k,\alpha}(N) \), we have by (3.5)

\[
P_{\lambda,\alpha}F[L(h_1h)\phi](\lambda)^*F[L(h_1h)\phi](\lambda)P_{\lambda,\alpha} = F[L(h_1h)\phi](\lambda)^*F[L(h_1h)\phi](\lambda),
\]

which means that we can regard \( F[L(h_1h)\phi](\lambda)^*F[L(h_1h)\phi](\lambda) \) as a linear operator on the finite dimensional vector space \( \mathcal{H}_{\lambda,\alpha} \). Furthermore, applying Schur’s lemma to the representation \((\tau_\lambda, \mathcal{H}_{\lambda,\alpha})\) of \( H_\lambda \), we get

\[
\int_{H_\lambda} \delta(h_1h)^{-1}F[L(h_1h)\phi](\lambda)^*F[L(h_1h)\phi](\lambda) \, dh_1 = (\dim \mathcal{H}_{\lambda,\alpha})^{-1}\delta(h)^{-1}\|F[L(h)\phi](\lambda)\|^2_{HS}P_{\lambda,\alpha} \in \text{End}(\mathcal{H}_{\lambda,\alpha}),
\]

see [16, pp. 43-44] for the detail. Note that \( \dim \mathcal{H}_{\lambda,\alpha} = \dim \mathcal{H}_{\lambda,\alpha} \), which we denote by \( n_{k,\alpha} \) in what follows. By (4.5) and (3.5), the integral (4.4) equals

\[
\frac{1}{n_{k,\alpha}} \int_{\mathcal{O}_{\lambda k}^*} \int_H (\text{tr} \, Ff(\lambda)P_{\lambda,\alpha}Ff(\lambda)^*) \|F[L(h)\phi](\lambda)\|^2_{HS} \delta(h)^{-1} \, dH(h) \, d\mu(\lambda)
\]

\[
= \frac{1}{n_{k,\alpha}} \int_{\mathcal{O}_{\lambda k}^*} \int_H \|Ff(\lambda)\|^2_{HS} \|F[L(h)\phi](\lambda)\|^2_{HS} \delta(h)^{-1} \, dH(h) \, d\mu(\lambda),
\]

and the right-hand side is rewritten as

\[
\frac{1}{n_{k,\alpha}} \int_{\mathcal{O}_{\lambda k}^*} \|Ff(\lambda)\|^2_{HS} \left( \int_H \|F\phi(h^{-1} \cdot \lambda)\|^2_{HS} \, dH(h) \right) \, d\mu(\lambda)
\]

by (2.4). Thanks to [16, Proposition 3], the integral \( \int_H \|F\phi(h^{-1} \cdot \lambda)\|^2_{HS} \, dH(h) \) does not depend on \( \lambda \in \mathcal{O}_{\lambda k}^* \), and it is equal to \( \int_{\mathcal{O}_{\lambda k}^*} \|F\phi(\lambda)\|^2_{HS} D_k(\lambda) \, d\mu(\lambda) \) with a certain \( H \)-relatively invariant function \( D_k \) on \( \mathcal{O}_{\lambda k}^* \). Therefore we have

\[
+\infty > \int_G |(f|L(g)\phi)|^2 \, dG(g) = \int_{\mathcal{O}_{\lambda k}^*} \|Ff(\lambda)\|^2_{HS} \, d\mu(\lambda) \cdot \frac{1}{n_{k,\alpha}} \int_H \|F\phi(h^{-1} \cdot \lambda)\|^2_{HS} \, dH(h)
\]

\[
= \|f\|^2 \cdot \frac{1}{n_{k,\alpha}} \int_{\mathcal{O}_{\lambda k}^*} \|F\phi(\lambda)\|^2_{HS} D_k(\lambda) \, d\mu(\lambda).
\]

In particular, if \( f \neq 0 \), then

\[
c(\phi) := \frac{1}{n_{k,\alpha}} \int_{\mathcal{O}_{\lambda k}^*} \|F\phi(\lambda)\|^2_{HS} D_k(\lambda) \, d\mu(\lambda) < +\infty
\]

and

\[
\int_G |(f|L(g)\phi)|^2 \, dG(g) = c(\phi) \|f\|^2.
\]
Conversely, if \( \phi \) satisfies the condition (4.6), the integral (4.4) converges for any \( f \in L_{k,\alpha}(N) \), so that the right-hand side of (4.3) converges for almost all \( h \in H \). Therefore Proposition 2.1 implies (4.3), so that we get (4.7) again and thus, \( \int_G |(f|L(g)\phi)|^2 dG(g) < +\infty \).

**Theorem 4.1.** The unitary representation \((L, L_{k,\alpha}(N))\) of \( G \) is irreducible.

**Proof.** Let \( \mathcal{L} \) be a nonzero invariant subspace of \( L_{k,\alpha}(N) \). The orthogonal complement \( \mathcal{L}^\perp \subset L_{k,\alpha}(N) \) is also invariant. We take \( \phi \in \mathcal{L} \setminus \{0\} \) and \( f \in \mathcal{L}^\perp \). By (4.7) we have

\[
0 = \int_G |(f|L(g)\phi)|^2 dG(g) = c(\phi)\|f\|^2,
\]

which implies \( f = 0 \). Actually, the argument is valid even if \( c(\phi) \) is not finite. Therefore \( \mathcal{L}^\perp = \{0\} \) and Theorem 4.1 is proved. \( \square \)

Furthermore, we deduce from (4.6) and (4.7) the following result.

**Theorem 4.2.** The representation \((L, L_{k,\alpha}(N))\) is square-integrable, whose Duflo-Moore operator \( C_{k,\alpha} \) is given by

\[
F[C_{k,\alpha}\phi](\lambda) = \sqrt{\frac{D_k(\lambda)}{n_{k,\alpha}}} F\phi(\lambda) \quad (\phi \in L_{k,\alpha}(N), \lambda \in O_{\lambda_k}^*)
\]

A classification of the representations \((L, L_{k,\alpha}(N))\) is also given in [16]. The result is as follows:

**Theorem 4.3** ([16, Theorem 4]). The unitary representations \((L, L_{k,\alpha}(N))\) and \((L, L_{k',\alpha'}(N))\) of \( G \) are equivalent if and only if \( k = k' \) and the projective representations \((\tau_k, \mathcal{H}_{k,\alpha})\) and \((\tau_k, \mathcal{H}_{k,\alpha'})\) of \( H_k \) are equivalent, that is, there exists an isometry \( A : \mathcal{H}_{k,\alpha} \rightarrow \mathcal{H}_{k,\alpha'} \) such that \( \tau_k(h) \circ A = A \circ \tau_k(h) \) for all \( h \in H_k \).

Keeping the decomposition (3.6) in mind, and applying (1.2) to each representation \((L, L_{k,\alpha}(N))\), we obtain

**Theorem 4.4.** For each \( k = 1, \ldots, K \) and \( \alpha \in A_k \), take admissible vectors \( \phi_{k,\alpha} \in \text{dom}(C_{k,\alpha}) \subset L_{k,\alpha}(N) \). Then for all \( f \in L^2(N) \) one has

\[
f = \sum_{k=1}^{K} \sum_{\alpha \in A_k} \frac{1}{\|C_{k,\alpha}\phi_{k,\alpha}\|^2} \int_G W_{\phi_{k,\alpha}} f(g) L(g)\phi_{k,\alpha} dG(g),
\]

where \( W_{\phi_{k,\alpha}} f(g) = (f|L(g)\phi_{k,\alpha}) \).
§ 5. Example

As an illustrative example, we shall consider the case that the unimodular group \( N \) is the Heisenberg group of \((2\ell + 1)\)-dimension and \( H \) is isomorphic to \( \mathbb{R}_+ \times U(\ell) \). The continuous wavelet transform for this case is first considered by He-Liu [14].

Let \( N \) be the Lie group consisting of elements \( n(z, c) \ (z \in \mathbb{C}^\ell, c \in \mathbb{R}) \) with multiplication rule

\[
n(z, c)n(z', c') := (z + z', c + c' + \Im(z|z')) \quad (z, z' \in \mathbb{C}^\ell, c, c' \in \mathbb{R}),
\]

where \( (z|z') := \sum_{k=1}^\ell z_k z'_k \). Define a Haar measure \( d\nu \) on \( N \) by \( d\nu(n(z, c)) := dm(z)dc \), where \( dm \) is the standard Euclidean measure on \( \mathbb{C}^\ell \).

For \( \zeta \in \mathbb{C}^\ell \), we define the unitary character \( \chi_\zeta \) by \( \chi_\zeta(n(z, c)) := e^{i\zeta(z)} \). For \( \lambda > 0 \), we define

\[
\mathcal{H}_\lambda := \left\{ \varphi : \mathbb{C}^\ell \rightarrow \mathbb{C} \text{ (holomorphic) ; } ||\varphi||^2 = \frac{\lambda^\ell}{\pi^\ell} \int_{\mathbb{C}^\ell} |\varphi(w)|^2 e^{-\lambda |w|^2} \, dm(w) < +\infty \right\},
\]

and \( \mathcal{H}_{-\lambda} := \{ \overline{\varphi} ; \varphi \in \mathcal{H}_\lambda \} \). These \( \mathcal{H}_{\pm \lambda} \) are Hilbert spaces, on which we define the irreducible unitary representations \( \pi_{\pm \lambda} \) of \( N \) by

\[
\pi_\lambda(n(z, c))\varphi(w) := e^{-i\lambda c + \lambda(z|w) - \lambda|z|^2/2}\varphi(w - z) \quad (\varphi \in \mathcal{H}_\lambda),
\]

\[
\pi_{-\lambda}(n(z, c))\varphi(w) := e^{i\lambda c + \lambda(z|w) - \lambda|z|^2/2}\varphi(w - z) \quad (\varphi \in \mathcal{H}_{-\lambda}).
\]

The Stone-von Neumann theorem states that every irreducible unitary representation of \( N \) is equivalent to one of \( \chi_\zeta \ (\zeta \in \mathbb{C}^\ell) \) and \( \pi_\lambda \ (\lambda \in \mathbb{R} \setminus \{0\}) \), so that \( \hat{N} \) can be identified with \( \mathbb{C}^\ell \sqcup (\mathbb{R} \setminus \{0\}) \). For \( f_1, f_2 \in L^1(N) \cap L^2(N) \), we have

\[
(f_1|f_2) = \frac{2^{\ell-1}}{\pi^{\ell+1}} \int_{\mathbb{R} \setminus \{0\}} (\pi_\lambda(f_1)|\pi_\lambda(f_2))_{\text{HS}} |\lambda|^\ell d\lambda
\]

by [9, Chapter I, section 5], which implies that the Plancherel measure \( \mu \) on \( \hat{N} \) is given by \( \mu(\mathcal{H}_\ell) = 0 \) and \( d\mu(\lambda) = 2^{\ell-1} \pi^{-\ell-1} |\lambda|^\ell d\lambda \ (\lambda \in \mathbb{R} \setminus \{0\}) \).

For \( a > 0 \) and \( u \in U(\ell) \), define \( h(a, u) \in \text{Aut}(N) \) by \( h(a, u) \cdot n(z, c) := n(auz, a^2c) \). Then we have

\[
h(a, u) h(a', u') = h(aa', uu') \quad (a, a' > 0, \ u, u' \in U(\ell)),
\]

so that \( H := \{ h(a, u) ; a > 0, u \in U(\ell) \} \) is a subgroup of \( \text{Aut}(N) \). For \( h = h(a, u) \in H \), we have \( \delta(h) = a^{-2\ell + 2} \), and we define the representation \( (L, L^2(N)) \) of \( G = N \rtimes H \) by (2.2). The action of \( H \) on \( \hat{N} \) is described as

\[
h(a, u) \cdot \zeta = a^{-1} u \zeta \quad (\zeta \in \mathbb{C}^\ell),
\]

\[
h(a, u) \cdot \lambda = a^{-2}\lambda \quad (\lambda \in \mathbb{R} \setminus \{0\}).
\]
Thus, setting which give the irreducible decomposition of the representation \((\lambda, \alpha)\): 
\begin{equation}
C(h, \lambda)\varphi(w) := \varphi(a^{-1}u^{-1}w) \quad (\varphi \in \mathcal{H}_\lambda).
\end{equation}

We denote by \(\mathcal{O}_\pm^*\) the \(H\)-orbit through \(\pm 1 \in \mathbb{R} \setminus \{0\} \subset \hat{N}\). Then \(\mathcal{O}_\pm^* = \{ \lambda; \pm \lambda > 0 \}\), and the conditions \((\text{H}1)-(\text{H}4)\) are satisfied for the two orbits. In particular, the stabilizer \(H_\lambda\) at any \(\lambda \in \mathcal{O}_\pm^* \cup \mathcal{O}_\pm^*\) equals the compact group \(\{ h(1, u); u \in U(\ell) \} \simeq U(\ell)\). For a non-negative integer \(\alpha \in \mathbb{Z}_{\geq 0}\), let \(\mathcal{P}_\alpha(\mathbb{C}^\ell)\) be the space of holomorphic polynomials of degree \(\alpha\) on \(\mathbb{C}^\ell\), and \(\overline{\mathcal{P}}_\alpha(\mathbb{C}^\ell)\) the space \(\{ \overline{\varphi}; \varphi \in \mathcal{P}_\alpha(\mathbb{C}^\ell) \}\). Then we have for \(\lambda > 0\)
\[\mathcal{H}_\lambda = \bigoplus_{\alpha = 0}^{\infty} \mathcal{P}_\alpha(\mathbb{C}^\ell), \quad \mathcal{H}_{-\lambda} = \bigoplus_{\alpha = 0}^{\infty} \overline{\mathcal{P}}_\alpha(\mathbb{C}^\ell),\]
which give the irreducible decomposition of the representation \((\tau_{\pm \lambda}, H_{\pm \lambda})\) of \(H_{\pm \lambda}\) respectively. Let \(P_{\lambda, \alpha} : \mathcal{H}_\lambda \to \mathcal{P}_\alpha(\mathbb{C}^\ell)\) and \(P_{-\lambda, \alpha} : \mathcal{H}_{-\lambda} \to \overline{\mathcal{P}}_\alpha(\mathbb{C}^\ell)\) be the orthogonal projections, and set
\[L_{\pm, \alpha}(N) := \left\{ f \in L^2(N); \begin{array}{ll} \mathbf{F}f(\lambda) = \mathbf{F}f(\lambda) P_{\lambda, \alpha} & (\text{a.a. } \lambda \in \mathcal{O}_\pm^*) \\ \mathbf{F}f(\lambda) = 0 & (\text{otherwise}) \end{array} \right\}.
\]
Then an irreducible decomposition of the unitary representation \((L, L^2(N))\) of \(G\) is given by
\begin{equation}
L^2(N) = \bigoplus_{\alpha = 0}^{\infty} L_{+, \alpha}(N) \oplus \bigoplus_{\alpha = 0}^{\infty} L_{-, \alpha}(N).
\end{equation}

The decomposition is multiplicity-free thanks to Theorem 4.3.

We define a Haar measure \(d_H\) on \(H\) by \(d_H(h(a, u)) := a^{-1} da du\), where \(du\) is the normalized Haar measure on \(U(\ell)\). For \(\lambda_0 \in \mathcal{O}_\pm^*\) and a measurable function \(p : \mathcal{O}_\pm^* \to \mathbb{R}\), we observe
\[\int_H p(h^{-1} \lambda_0)d_H(h) = \int_0^{+\infty} p(a^2 \lambda_0) \frac{da}{a} = \int_{\mathcal{O}_\pm^*} p(\lambda) \frac{d\lambda}{2|\lambda|^\ell} = \int_{\mathcal{O}_\pm^*} p(\lambda) \cdot 2 \left( \frac{\pi}{2|\lambda|^\ell} \right)^{\ell+1} \cdot \frac{d\lambda}{\lambda}.\]

Thus, setting \(D_\pm(\lambda) := 2 \left( \frac{\pi}{2|\lambda|^\ell} \right)^{\ell+1} (\lambda \in \mathcal{O}_\pm^*)\), the Duflo-Moore operator \(C_{\pm, \alpha}\) of the representation \((L, L_{\pm, \alpha}(N))\) is given by \(\mathbf{F}[C_{\pm, \alpha} \phi](\lambda) = \sqrt{\frac{D_\pm(\lambda)}{n_\alpha}} \mathbf{F}\phi(\lambda) (\phi \in L_{+, \alpha}(N), \lambda \in \mathcal{O}_\pm^*)\), where \(n_\alpha = \dim_{\mathbb{C}} \mathcal{P}_\alpha(\mathbb{C}^\ell) = \left( \frac{\alpha + \ell - 1}{\alpha} \right)^\ell\).

Finally, we give an example of admissible vector in \(L_{\pm, \alpha}(N)\). We set
\[\phi_{\pm, \alpha}(n(z, c)) := \int_{\mathcal{O}_\pm^*} |\lambda| e^{-|\lambda|} \cdot \text{tr} P_{\lambda, \alpha} \pi_\lambda(n(z, c))^* \, d\mu(\lambda).\]
Therefore \(|\lambda|e^{-|\lambda|}P_{\lambda,\alpha}\) belongs to both of the direct integrals \(\int_{\mathcal{O}_+^\times} \mathcal{B}_{\text{Tr}}(\mathcal{H}_\lambda) \, d\mu(\lambda)\) and \(\int_{\mathcal{O}_-^\times} \mathcal{B}_{\text{HS}}(\mathcal{H}_\lambda) \, d\mu(\lambda)\). Applying Proposition 2.1, we see that \(\phi_{\pm,\alpha} \in L^2(N)\) and that

\[
F\phi_{\pm,\alpha}(\lambda) = \begin{cases} |\lambda|e^{-|\lambda|}P_{\lambda,\alpha} & (\text{a.a. } \lambda \in \mathcal{O}_\pm^\times), \\ 0 & (\text{otherwise}). \end{cases}
\]

Thus \(\phi_{\pm,\alpha} \in L_{\pm,\alpha}(N)\). Furthermore we have

\[
\|C_{\pm,\alpha}\phi_{\pm,\alpha}\|^2 = n_{\alpha}^{-1} \int_{\mathcal{O}_-^\times} \|F\phi_{\pm,\alpha}(\lambda)\|^2_{\text{HS}} \frac{d\lambda}{2|\lambda|} = n_{\alpha}^{-1} \int_{\mathcal{O}_+^\times} n_{\alpha}|\lambda|^2 e^{-2|\lambda|} \frac{d\lambda}{2|\lambda|}
\]

\[
= \frac{1}{2} \int_0^{+\infty} e^{-2\lambda} \lambda \, d\lambda = \frac{1}{8},
\]

so that \(\phi_{\pm,\alpha}\) is admissible.

The function \(\phi_{\pm,\alpha}\) can be calculated explicitly. By definition, we have \(\phi_{-,\alpha} = \overline{\phi}_{+,\alpha}\), so that we shall consider only \(\phi_{+,\alpha}\). For non-negative integers \(k\) and \(\alpha\), the Laguerre polynomial \(L_{\alpha}^k(s)\) is defined by \(L_{\alpha}^k(s) := e^{s/\alpha}(d/ds)^\alpha e^{-s\alpha} [e^{-s\alpha} / s^\alpha]^k\). By [3, Proposition 6.2], we have

\[
\text{tr} \, P_{\lambda,\alpha} \pi_{\lambda}(n(z,c))^* = e^{(ic-|z|^2/2)\lambda} L_{\alpha}^{\ell-1}(\lambda|z|^2).
\]

Thus

\[
\phi_{+,\alpha}(n(z,c)) = \frac{2\ell-1}{\pi^{\ell+1}} \int_0^{+\infty} e^{-(1-ic+|z|^2/2)\lambda} L_{\alpha}^{\ell-1}(\lambda|z|^2) \lambda^{\ell+1} \, d\lambda.
\]

On the other hand, for a parameter \(|r| < 1\), we have \(\sum_{\alpha=0}^{\infty} r^\alpha L_{\alpha}^{\ell-1}(s) = (1-r)^{-\ell} e^{-r^2/r}\). Therefore

\[
\sum_{\alpha=0}^{\infty} r^\alpha \phi_{+,\alpha}(n(z,c)) = \frac{2\ell-1}{\pi^{\ell+1}} \int_0^{+\infty} e^{-(1-ic+|z|^2/2)\lambda} (1-r)^{-\ell} e^{-r^2/r} \lambda^{\ell+1} \, d\lambda
\]

\[
= \frac{2\ell-1}{\pi^{\ell+1}} (1-r)^{-\ell} \cdot (\ell + 1)! \left\{ (1-ic+|z|^2/2) + \frac{r|z|^2}{1-r} \right\}^{-(\ell+2)}
\]

\[
= \frac{2\ell-1}{\pi^{\ell+1}} \left(1-ic+|z|^2/2\right)^{-(\ell+2)} \times (1-r)^2 \left(1 + \frac{1+ic+|z|^2/2}{1-ic+|z|^2/2} \cdot r\right)^{-(\ell+2)}.
\]

Putting \(\theta := \frac{-1+ic+|z|^2/2}{1-ic+|z|^2/2}\), we have by the binomial theorem

\[
(1-r)^2(1+\theta r)^{-(\ell+2)} = \sum_{\alpha=0}^{\infty} \frac{(-1)^\alpha (\alpha + \ell - 1)!}{(\ell + 1)! \alpha!} \{\alpha(\alpha-1)\theta^{\alpha-2} + 2\alpha(\alpha+\ell)\theta^{\alpha-1} + (\alpha+\ell)(\alpha+\ell+1)\theta^\alpha\} r^\alpha.
\]
Hence
\[
\phi_{+\alpha}(n(z,c)) = \frac{2^\ell-1}{\pi^{\ell+1}} (1 - ic + |z|^2/2)^{-(\ell+2)} \times \left( \frac{-1}{\alpha!} \right)^{\ell+1} \left\{ \alpha(\alpha - 1)\theta^{\alpha-2} + 2\alpha(\alpha + \ell)\theta^{\alpha-1} + (\alpha + \ell)(\alpha + \ell + 1)\theta^\alpha \right\}.
\]

References
