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Kyoto University
Dispersive estimates for Schrödinger equations in dimension one

By

Haruya Mizutani

Abstract

We study the time decay of scattering solutions to one dimensional Schrödinger equations and prove a weighted dispersive estimate with stronger time decay than the case of unweighted estimates. Furthermore an asymptotic expansion in time of scattering solutions is given.

§ 1. Introduction

This report is concerned with the time decay of scattering solutions $e^{-itH} P_{ac} u$ to Schrödinger equations

$$i\partial_t u = Hu,$$

where

$$H = -\frac{d^2}{dx^2} + V(x), \ x \in \mathbb{R}$$

is a one dimensional Schrödinger operator and $P_{ac}$ is the projection onto the absolutely continuous subspace for $H$. We assume that $V(x)$ is real valued and $V \in L^1$ at least, where $L^p_\gamma$ is the weighted $L^p$ space:

$$L^p_\gamma := \{ f \mid \langle x \rangle^\gamma f \in L^p(\mathbb{R}) \}, \ |||f|||_{L^p_\gamma} := ||\langle x \rangle^\gamma f||_{L^p},$$

$$\langle x \rangle := \sqrt{1 + |x|^2}, \ 1 \leq p \leq \infty, \ \gamma \in \mathbb{R}.$$ 

Under the above conditions, $H$ is self-adjoint on $L^2(\mathbb{R})$ with form domain $H^1(\mathbb{R})$ and the absolutely continuous spectrum of $H$ is the half line $[0, \infty)$, the singular continuous spectrum of $H$ is absent, and the eigenvalues of $H$ are strictly negative.
Let $H_0 = -\frac{d^2}{dx^2}$. It is well known that the propagator $e^{-itH_0}$ has the following asymptotic expansion in $B(L^2_s, L^2_{-s})$ for sufficiently large $s > 0$:

$$e^{-itH_0} = t^{-\frac{1}{2}}C_{-1} + t^{-\frac{3}{2}}C_0 + \cdots, \; t \to \infty,$$

where $C_{j-1}$ are given by

$$C_{j-1}(x) := \frac{1}{\sqrt{4\pi i j^2}} \int_{\mathbb{R}} (i|y|)^{2j} u(y) dy,$$

and $B(X, Y)$ denotes the Banach spaces of bounded operators form $X$ to $Y$. The asymptotic expansion of $e^{-itH}$ as $t \to \infty$ in $B(L^2_s, L^2_{-s})$ was proved by Murata [14], under the assumption that $|V(x)| \leq C \langle x \rangle^{-\sigma}$ for sufficiently large $\sigma > 7$. In higher dimension, such expansions were proved by [11, 10, 14]. In this paper we prove an asymptotic expansion of $e^{-itH_{Pac}}$ in $B(L^1_s, L^\infty_{-s})$ as $t \to \infty$.

In order to state our results, we introduce a few notations. The Jost functions $f_{\pm}(\lambda, x)$ are the solutions to the equation

$$-f''(\lambda, x) + V(x)f(\lambda, x) = \lambda^2 f(\lambda, x), \quad \lambda, x \in \mathbb{R}$$

satisfying following asymptotic conditions

$$|f_{\pm}(\lambda, x) - e^{\pm i\lambda x}| \to 0 \text{ as } x \to \pm \infty.$$

It is well known (see [3]) that if $V \in L^1_1$, then the Jost functions are uniquely defined for all $\lambda, x \in \mathbb{R}$. We denote by $W(\lambda)$ their Wronskian

$$W(\lambda) := f_+(\lambda, x) \cdot \partial_x f_-(\lambda, x) - \partial_x f_+(\lambda, x) \cdot f_-(\lambda, x).$$

$W(\lambda)$ is independent of $x$ and does not vanish for $\lambda \neq 0$.

**Definition 1.1.** We say that the potential $V$ is of generic type if $W(0) \neq 0$ and is of exceptional type if $W(0) = 0$. We also say that zero is a resonance of $H$ if the potential $V$ is of exceptional type.

**Theorem 1.2.** Let $m$ be a positive integer. Suppose that $V \in L^1_{2m}$ and $V$ is of generic type, or $V \in L^1_{2m+2}$ and $V$ is of exceptional type. Let

$$s = \begin{cases} 
2m - 1 & \text{if } V \text{ is of generic type,} \\
2m & \text{if } V \text{ is of exceptional type.}
\end{cases}$$

Then

$$||\langle x \rangle^{-s} (e^{-itH_{Pac}} - P_{m-1}) u||_{L^\infty} \leq C t^{-\frac{1}{2} - m} ||\langle x \rangle^s u||_{L^1}. $$
for all \( t > 0 \), where \( P_{m-1} \) is given by

\[
P_{m-1} = \sum_{j=0}^{m-1} t^{-\frac{1}{2} - j} C_{j-1}.
\]

Furthermore, the coefficients \( C_{j-1} \) satisfy the following:

1. If \( V \) is of generic type, then \( C_{-1} \equiv 0 \), \( \text{rank} C_j \leq 2j \) and
   \[
   \|\langle x \rangle^{-2j+1} C_{j-1} u \|_{L^\infty} \leq C \|\langle x \rangle^{2j-1} u \|_{L^1}, \quad j = 1, 2, \ldots, m - 1.
   \]

2. If \( V \) is of exceptional type, then \( \text{rank} C_j \leq 2j + 1 \) and
   \[
   \|\langle x \rangle^{-2j} C_{j-1} u \|_{L^\infty} \leq C \|\langle x \rangle^{2j} u \|_{L^1}, \quad j = 0, 1, \ldots, m - 1.
   \]

Remark. In exceptional case, we can compute \( C_{-1} \) explicitly:

\[
C_{-1} u = \frac{1}{\sqrt{4\pi i}} \langle u, f_0 \rangle f_0,
\]

where \( f_0 \) is a non trivial bounded solution to the equation \( Hf = 0 \) normalized as

\[
\lim_{x \to +\infty} \frac{1}{2} (|f_0(x)|^2 + |f_0(-x)|^2) = 1.
\]

Dispersive estimates for Schrödinger equations have been studied by many authors. \( L^1 - L^\infty \) estimates:

\[
(1.2) \quad \|e^{-itH} P_{ac} u \|_{L^\infty} \leq C |t|^{-\frac{d}{2}} \|u\|_{L^1},
\]

was proved by [12] under the suitable decay and regularity assumptions for \( V \). Later (1.2) has been proved by [6, 7, 9, 15, 17, 18, 19, 21, 22] under various assumptions on the potential \( V \) and the assumption that zero is neither an eigenvalue nor a resonance of \( H \). When zero is either an eigenvalue or a resonance of \( H \), similar estimates were studied by [4, 23]. Such estimates are very important since (1.2) implies Strichartz estimates which can be applied to prove well-posedness for nonlinear Schrödinger equations. The time decay \( t^{-\frac{1}{2}} \) in \( d = 1 \) is not integrable at infinity and is unsuitable for applying to NLS. We hence are interested in a dispersive estimate whose time decay is integrable at infinity. Goldberg [8] also proved (1.1) with \( m = 1 \) under the assumptions that \( V \in L^1_3 \) and \( V \) is of generic type, or \( V \in L^1_3 \) and \( V \) is of exceptional type. Compared to his results, our assumptions on the potential \( V(x) \), which are used in Theorem 1.2, are weaker.

\section{Sketch of the proof}

To prove Theorem 1.2, we use the representation

\[
\langle e^{-itH} P_{ac} u, v \rangle = \frac{1}{\pi i} \int_{\mathbb{R}} e^{-it\lambda^2} \lambda \langle R(\lambda^2 + i0)u, v \rangle d\lambda,
\]

where
where we denote an extended resolvent \( \lim_{\varepsilon \to +0} (H - (\lambda + i\varepsilon)^2)^{-1} \) by \( R(\lambda^2 + i0) \). We split the propagator into high and low energy parts. For the high energy part, the following proposition holds.

**Proposition 2.1.** Suppose \( V \in L^1_N, N \in \mathbb{N} \) and set \( \lambda_0 := ||V||_{L^1_N} \). Let \( \chi \) be an even smooth cut-off function such that \( \chi(\lambda) = 1 \) for \( |\lambda| \leq \lambda_0 \) and \( \chi(\lambda) = 0 \) for \( |\lambda| \geq 2\lambda_0 \). Then

\[
||\langle x \rangle^{-N} e^{-itH}(1 - \chi(\sqrt{H}))u||_{L^\infty} \leq Ct^{-\frac{1}{2} - N}||\langle x \rangle^N u||_{L^1}, \quad u \in L^2 \cap L^1_N
\]

for all \( t > 0 \).

**Proof.** Set \( \tilde{\chi}(\lambda) := 1 - \chi(\lambda) \). Let \( \eta \) be an even smooth function on \( \mathbb{R} \) such that \( \eta(\lambda) = 1 \) if \( |\lambda| \leq 1 \), \( \eta(\lambda) = 0 \) if \( |\lambda| \geq 2 \) and let \( \tilde{\chi}_L(\lambda) := \eta(\lambda/L)\tilde{\chi}(\lambda) \) for \( L \geq 1 \). Using the Born series expansion of \( R(\lambda^2 + i0) \), we have

\[
\langle e^{itH} \tilde{\chi}_L(\sqrt{H})u, v \rangle = \frac{1}{\pi i} \sum_{n=0}^{\infty} \frac{1}{(-2i)^{n+1}} \int_{\mathbb{R}^{n+3}} e^{-it\lambda^2 + i\lambda \sum_{j=0}^n |x_{j+1} - x_j|} \frac{\tilde{\chi}_L(\lambda)}{\lambda^n} \times u(x_0) \prod_{j=1}^n V(x_j)v(x_{n+1})d\lambda dx_0...dx_{n+1}.
\]

Let us consider the oscillatory integral

\[
\Phi(t, a) = \int_{\mathbb{R}} e^{-it\lambda^2 + ia\lambda} \frac{\tilde{\chi}_L(\lambda)}{\lambda^n} d\lambda, \quad a \in \mathbb{R}.
\]

Using integration by parts and Fourier inversion formula, we have

\[
|\Phi(t, a)| \leq Ct^{-\frac{1}{2} - N}||\mathcal{F}P_\lambda^N(e^{ia\lambda} \tilde{\chi}_L(\lambda)\lambda^{-n})||_{L^1}
\]

\[
\leq Ct^{-\frac{1}{2} - N} \sum_{k=0}^N |a|^{N-k}||\mathcal{F}\partial_\lambda^k(\tilde{\chi}_L(\lambda)\lambda^{-n-N})||_{L^1},
\]

where \( P_\lambda = \frac{\partial}{\partial \lambda} \frac{1}{\lambda} \) and \( \mathcal{F} \) is the Fourier transform with respect to \( \lambda \). A direct computation yields

\[
\sup_{L \geq 1} \sup_{0 \leq k \leq N} ||\mathcal{F}\partial_\lambda^k(\tilde{\chi}_L(\lambda)\lambda^{-n-N})||_{L^1} \leq C(n + 2N)^N \lambda_0^{-n-N}
\]

for \( n \geq 0 \). Since

\[
\sum_{j=0}^n |x_{j+1} - x_j| \leq \prod_{j=1}^n (1 + |x_j|),
\]
we conclude that
\[
\sup_{L \geq 1} \left| \langle e^{itH} \chi_L(\sqrt{H})u, v \rangle \right| \\
\leq Ct^{-\frac{1}{2} - N} \sum_{n=0}^{\infty} 2^{-n} n^N \lambda_0^{-n-N} ||(1 + |x|)^N V||_{L^1} ||(1 + |x|)^N u||_{L^1} ||(1 + |x|)^N v||_{L^1}
\]
\[
\leq Ct^{-\frac{1}{2} - N} ||\langle x \rangle^N u||_{L^1} ||\langle x \rangle^N v||_{L^1}
\]
for all \( t > 0 \).

\[\Box\]

\section{Jost Functions}

Using the Jost functions and their Wronskian, the integral kernel of the resolvent \( R(\lambda^2 \pm i0) \) is given by
\[
f_\pm(\pm \lambda, y)W(\pm \lambda)\chi_{\{x<y\}} + f_\pm(\pm \lambda, x)W(\pm \lambda)\chi_{\{x>y\}}.
\]
In this subsection, we collect results on the Jost functions \( f_\pm(\pm \lambda, x) \) needed later. \( f_+(\lambda, x) \) and \( f_+(-\lambda, x) \) are independent for \( \lambda \neq 0 \) since their Wronskian
\[
W[f_+(\lambda, \cdot), f_+(-\lambda, \cdot)] := f_+(\lambda, x) \cdot \partial_x f_+(-\lambda, x) - \partial_x f_+(\lambda, x) \cdot f_+(-\lambda, x)
\]
\[
= \lim_{x \to +\infty} [e^{i\lambda x}(-i\lambda)e^{-i\lambda x} - i\lambda e^{i\lambda x}e^{-i\lambda x}]
\]
\[
= -2i\lambda \neq 0.
\]
Similarly \( W[f_-(\lambda, \cdot), f_-(-\lambda, \cdot)] = 2i\lambda \). These imply the relations
\begin{align}
T(\lambda) f_-(\lambda, x) &= R_1(\lambda) f_+(\lambda, x) + f_+(\lambda, x), \\
T(\lambda) f_+(\lambda, x) &= R_2(\lambda) f_-(\lambda, x) + f_-(\lambda, x),
\end{align}
where \( T(\lambda), R_1(\lambda) \) and \( R_2(\lambda) \) are called the transmission and reflection coefficients, respectively. It is well known that
\[
|T(\lambda)|^2 + |R_j(\lambda)|^2 = 1, \quad j = 1, 2.
\]
Furthermore the following holds (see [3], [13] and [1]).

\textbf{Lemma 2.2.} \quad (1) Suppose that \( V \) is of generic type and \( V \in L^1_N, N \geq 1 \). Then \( T, R_1 \) and \( R_2 \in C^{N-1}(\mathbb{R}) \) and for \( 1 \leq k \leq N-1 \),
\begin{align}
|\partial_x^k T(\lambda)| + |\partial_x^k R_1(\lambda)| + |\partial_x^k R_2(\lambda)| \leq C(\lambda)^{-1}, \lambda \in \mathbb{R}.
\end{align}
Furthermore, we have
\[
T(\lambda) = \alpha \lambda + o(\lambda), \quad \alpha \neq 0, \quad \lambda \to 0,
\]
\[
R_1(0) = R_2(0) = -1.
\]
Suppose that $V$ is of exceptional type and $V \in L^1_N$, $N \geq 2$. Then $T$, $R_1$ and $R_2 \in C^{N-2}(\mathbb{R})$ and (2.2) holds for $1 \leq k \leq N-2$. Furthermore, as $\lambda \to 0$, we have

$$T(\lambda) = \frac{2a}{1+a^2} + o(1),$$

$$R_1(\lambda) = \frac{1-a^2}{1+a^2} + o(1),$$

$$R_2(\lambda) = \frac{a^2-1}{1+a^2} + o(1),$$

with $a := \lim_{x \to -\infty} f_+(0, x) \neq 0$.

The following inequality was proved by Artbazar and Yajima [1]:

$$|\partial^k_\lambda f_\pm(\lambda, x)| \leq C\langle x \rangle^k (1 + \max(\mp x, 0)), \quad (\lambda, x) \in \mathbb{R}^2.$$  

(2.3)

for $0 \leq k \leq N-1$. Using Lemma 2.2, we can improve the above estimates.

**Lemma 2.3.**  
(1) Suppose that $V$ is of generic type and $V \in L^1_N$, $N \geq 1$. Then

$$|\partial^k_\lambda (T(\lambda)f_\pm(\lambda, x))| \leq C\langle x \rangle^k, \quad \lambda \neq 0, \quad x \in \mathbb{R},$$

for $0 \leq k \leq N-1$. If in addition $N \geq 2$, then

$$|\partial^k_\lambda f_\pm(0, x)| \leq C\langle x \rangle^k, \quad x \in \mathbb{R},$$

for $1 \leq k \leq N-1$ and $k$ odd.

(2) Suppose that $V$ is of exceptional type and $V \in L^1_N$, $N \geq 2$, then

$$|\partial^k_\lambda f_\pm(\lambda, x)| \leq C\langle x \rangle^k, \quad (\lambda, x) \in \mathbb{R}^2.$$  

for $0 \leq k \leq N-2$.

Lemma 2.3 follows from (2.1), (2.3) and Lemma 2.2, and we omit the proof.

We next study Fourier properties of the Jost functions. Set

$$B_\pm(\xi, x) := \int_{\mathbb{R}} e^{2i\lambda \xi} (m_\pm(\lambda, x) - 1) \, d\lambda,$$

where $m_\pm(\lambda, x) := e^{\mp i\lambda x} f_\pm(\lambda, x)$ are the modified Jost functions. Then the function $B_+(\xi, x)$ satisfies the Marchenko equation:

$$B_+(\xi, x) = \int_{x+\xi}^{\infty} V(\sigma) \, d\sigma + \int_{0}^{\xi} d\zeta \int_{x+\xi-\zeta}^{\infty} V(\sigma) B_+(\zeta, \sigma) \, d\sigma$$

and $B_-(\xi, x)$ also satisfies a corresponding equation. It is well known (see [3]) that if
$V \in L^1_1$, then the function $B_+(\xi, x)$ is well defined for $\xi \geq 0$, $x \in \mathbb{R}$ and satisfies the following estimates

\[ |B_+(\xi, x)| \leq e^{\gamma(x)} \eta(x + \xi), \quad \xi \geq 0, \quad x \in \mathbb{R}, \]

where $\eta(x) = \int_x^{\infty} |V(\sigma)|d\sigma$, $\gamma(x) = \int_x^{\infty} (\sigma - x)|V(\sigma)|d\sigma$. $B_-(\xi, x)$ also satisfies a similar inequality. Since $\gamma(x)$ is dominated by $||V||_{L^1_1}$ for all $x \geq 0$, (2.5) implies $||B_+(\cdot, x)||_{L^1_1}$ is bounded for $x \geq 0$ with the bound depending on $||V||_{L^1_1}$. Similarly $||B_-(\cdot, x)||_{L^1_1}$ is bounded for $x \leq 0$. Iterating Marchenko equations, we can prove the following (see [2]).

**Lemma 2.4.** Let $N \in \mathbb{N}$, $N \geq 1$ and suppose $V \in L^1_N$. Then the functions $B_{\pm}(\xi, x)$ satisfy the estimates

\[ ||B_{\pm}(\cdot, x)||_{L^1_{N-1}} \leq C(1 + \max(\mp x, 0))^N, \quad x \in \mathbb{R}, \]

where $C$ depends on $||V||_{L^1_N}$.

The following Lemma follows from Lemma 2.4 and the representation

W($\lambda$) = $-2i\lambda + \int_{\mathbb{R}} V(\sigma)m_+(\lambda, \sigma)d\sigma$.

**Lemma 2.5.** Let $\chi \in C_0^\infty(\mathbb{R})$.

1. Let $N \in \mathbb{N}$, $N \geq 1$ and assume that $V \in L^1_N$ and $V$ is of generic type, then

$\mathcal{F}(\frac{\chi}{W}) \in L^1_{N-1}$.

2. Let $N \in \mathbb{N}$, $N \geq 2$ and assume that $V \in L^1_N$ and $V$ is of exceptional type, then

$\mathcal{F}(\frac{\chi}{W}) \in L^1_{N-2}$.

Here $W(\lambda)$ is the Wronskian of the Jost functions.

Lemma 2.4, Lemma 2.5 and (2.1) imply the following.

**Lemma 2.6.** Let $\chi \in C_0^\infty(\mathbb{R})$. Suppose that $V \in L^1_N$, $N \geq 1$. Then

\[ ||\mathcal{F}(\chi(\cdot)f_{\pm}(\cdot, x))||_{L^1_{N-1}} \leq C(1 + \max(\mp x, 0))^N, \quad x \in \mathbb{R}. \]

Furthermore,

1. If $V$ is of generic type, then

\[ ||\mathcal{F}(\chi(\cdot)T(\cdot)f_{\pm}(\cdot, x))||_{L^1_{N-1}} \leq C(1 + \max(\mp x, 0))^N, \quad x \in \mathbb{R}. \]

2. If $V$ is of exceptional type and $V \in L^1_N$, $N \geq 2$, then

\[ ||\mathcal{F}(\chi(\cdot)f_{\pm}(\cdot, x))||_{L^1_{N-2}} \leq C(1 + \max(\mp x, 0))^N, \quad x \in \mathbb{R}. \]
§ 2.2. The Low Energy Estimates

In this subsection, we prove the following Proposition to complete the proof of Theorem 1.2.

Proposition 2.7. Let $m \in \mathbb{N}$ and let $\chi$ be an even smooth cut-off function such that $\chi(\lambda) = 1$ close to zero. Suppose that $V \in L^1_{2m}$ and $V$ is of generic type, or $V \in L^1_{2m+2}$ and $V$ is of exceptional type. Let $P_{m-1}$ as in Theorem 1.2. Let

$$s = \begin{cases} 
2m - 1 & \text{if } V \text{ is of generic type}, \\
2m & \text{if } V \text{ is of exceptional type}.
\end{cases}$$

Then

$$||\langle x \rangle^{-s} (e^{-itH} \chi(\sqrt{H}) P_{ac} - P_{m-1}) u ||_{L^\infty} \leq C t^{-\frac{1}{2} - m} ||\langle x \rangle^s u ||_{L^1},$$

for all $t > 0$.

Proof. We consider the generic case. The proof of the exceptional case is similar and we omit the proof. Set

$$K(\lambda, x, y) := T(\lambda)f_+ (\lambda, y)f_-(\lambda, x), \quad G(\lambda, x, y) := \frac{K(\lambda, x, y)}{\lambda}.$$ 

We start from the representation

$$\langle e^{-itH} \chi(\sqrt{H}) P_{ac} u, v \rangle = \frac{1}{\pi i} \int_{\mathbb{R}} e^{-it\lambda^2} \lambda \chi(\lambda) \langle R(\lambda^2 + i0) u, v \rangle d\lambda
\begin{aligned}
= & \frac{1}{2\pi} \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}} e^{-it\lambda^2} \lambda \chi(\lambda) \tilde{G}(\lambda, x, y) d\lambda \right) u(y)v(x) dy dx,
\end{aligned}$$

where $\tilde{G}(\lambda, x, y)$ denotes the kernel of $-2iR(\lambda^2 + i0)$ and is given by

$$\tilde{G}(\lambda, x, y) = \begin{cases} 
G(\lambda, x, y) & \text{for } x < y, \\
G(\lambda, y, x) & \text{for } x > y.
\end{cases}$$

Consider the integral

$$I(t, G) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{-it\lambda^2} \lambda \chi(\lambda) G(\lambda, x, y) d\lambda,$$ 

for $x < y$. The proof for the case $x > y$ is analogous. Integrating by parts (2.9), we have

$$I(t, G) = \frac{1}{4\pi it} \int_{\mathbb{R}} e^{-it\lambda^2} \partial_\lambda (\chi(\lambda) G(\lambda, x, y)) d\lambda.$$
The case $m = 1$: it suffice to prove that

$$|I(t, G)| \leq Ct^{-\frac{3}{2}}\langle x \rangle \langle y \rangle, \ x < y.$$  

Using the Fourier inversion formula, we obtain

$$|I(t, G)| \leq Ct^{-\frac{3}{2}}\|\mathcal{F}\partial_\lambda \chi(\cdot, x, y)\|_{L^1}$$

for all $t > 0$ and $x < y$, where $\mathcal{F}$ is the Fourier transform with respect to $\lambda$. By Young’s inequality, Lemma 2.5 (1) and Lemma 2.6, we have

$$\|(\mathcal{F}\partial_\lambda \chi G)(\cdot, x, y)\|_{L^1} \leq C\langle x \rangle \langle y \rangle, \ x < y$$

and this implies

$$|I(t, G)| \leq Ct^{-\frac{3}{2}}\langle x \rangle \langle y \rangle$$

for $x < y$.

The case $m \geq 2$: Applying the stationary phase theorem to the integral (2.10), we have

$$I(t, G) = \frac{1}{\sqrt{\pi i}} \sum_{j=1}^{m-1} \frac{t^{-\frac{1}{2}-j}}{(j-1)!(4i)^j} (\partial^{2j-1}_\lambda G)(0, x, y) + t^{-\frac{1}{2}-m} S_{m-1}(t, G)$$

with

$$|S_{m-1}(t, G)| \leq C\|(\mathcal{F}\partial^2_{\lambda} \chi - 1 G)(\cdot, x, y)\|_{L^1} \leq C\langle x \rangle^{2m-1} \langle y \rangle^{2m-1}, \ x < y.$$  

For the last inequality, we used Lemma 2.5 (1) and Lemma 2.6. We now define the coefficients $C_{j-1}$: since $T(0) = 0$, we have

$$(\partial^{2j-1}_\lambda G)(0, x, y) = \frac{1}{2j} (\partial^{2j}_\lambda K)(0, x, y).$$

Considering the fact that

$$(\partial^{2j}_\lambda K)(0, x, y) = (\partial^{2j}_\lambda K)(0, y, x), \ x, y \in \mathbb{R}, \ j = 1, 2, \ldots, m,$$

we define $C_{j-1}$ and $P_{m-1}$ by

$$C_{j-1}u(x) := \frac{1}{\sqrt{4\pi ij!(4i)^j}} \int_{\mathbb{R}} (\partial^{2j}_\lambda K)(0, x, y)u(y)dy, \ x \in \mathbb{R},$$

$$P_{m-1} := \sum_{j=0}^{m-1} t^{-\frac{1}{2}-j} C_{j-1}.$$
Then we have
\[
\| \langle x \rangle^{-2m+1} (e^{-itH} \chi(\sqrt{H}) P_{ac} - P_{m-1}) u \|_{L^\infty} \leq C t^{-\frac{1}{2} - m} \| \langle x \rangle^{2m-1} u \|_{L^1}.
\]
By the definition of \( C_{j-1} \) and Corollary 2.3 (1), we can see that
\[
\text{rank} \, C_{j-1} \leq 2j,
\]
and there exists \( C > 0 \) such that
\[
\| \langle x \rangle^{-2j+1} C_{j-1} u \|_{L^\infty} \leq C \| \langle x \rangle^{2j-1} u \|_{L^1}.
\]
In particular,
\[ C_{-1} \equiv 0. \]
These complete the proof.

References


