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Self-similar blow-up for a chemotaxis system in higher dimensional domains

By

Yüki Naito * and Takasi Senba **

§ 1. Introduction and statement of main results

We consider solutions of a parabolic-elliptic system

\begin{equation}
\begin{cases}
  u_t = \nabla \cdot (\nabla u - u \nabla v) & \text{in } \Omega \times (0, T) \\
  0 = \Delta v + u & \text{in } \Omega \times (0, T),
\end{cases}
\end{equation}

where either $\Omega = \{x \in \mathbb{R}^N : |x| < L\}$ or $\Omega = \mathbb{R}^N$ with $N \geq 3$. In the former case, we assume $\partial u/\partial \nu - u \partial v/\partial \nu = 0$ and $v = 0$ on $\partial \Omega$, where $\nu$ denotes the outer unit normal vector. This system arises in the study of the motion of bacteria by chemotaxis as a simplification of the Keller-Segel model (see [16], [22]). Here, $u$ and $v$ represent the density of the bacteria and the concentration of the chemo-attractant, respectively. This system also has been used as a model for the evolution of self-attracting clusters (see [27], [28], [2]).

In this note we consider the blow-up rate of solutions to the system

\begin{equation}
\begin{cases}
  u_t = \nabla \cdot (\nabla u - u \nabla v) & \text{in } \Omega \times (0, T) \\
  0 = \Delta v + u & \text{in } \Omega \times (0, T) \\
  \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} = 0 \text{ and } v = 0 & \text{on } \partial \Omega \times (0, T) \\
  u(x, 0) = u_0(|x|) & \text{in } \Omega,
\end{cases}
\end{equation}
where $\Omega = \{ x \in \mathbb{R}^N : |x| < L \}$ with $N \geq 3$ and $0 < L < \infty$, and $u_0(r)$ is a nonnegative continuous function on $[0, L]$. We restrict ourselves to the study of radially symmetric solutions. It is known by [2] that the system (1.2) has a unique local classical solution $(u, v)$. It is easy to see that $u$ and $v$ are positive for $0 < t < T$, and that the conservation of the initial mass of $u$ holds, that is,

$$
(1.3) \quad \|u(\cdot, t)\|_1 = \|u_0(\cdot)\|_1 \quad \text{for } 0 < t < T,
$$

where $\| \cdot \|_p$ denotes the standard $L^p(\Omega)$ norm for $1 \leq p \leq \infty$. A solution $(u, v)$ is said to blow-up at $t = T < \infty$ if $(u, v)$ is classical in $\Omega \times (0, T)$ and satisfies

$$
\limsup_{t \to T} \|u(\cdot, t)\|_\infty = \infty. \quad \text{A simple argument shows that if } u \text{ blows up at a finite time } t = T \text{ then}
$$

$$
\liminf_{t \to T} (T - t) \|u(\cdot, t)\|_\infty > 0.
$$

We say that the blow-up is of type I if $u$ satisfies

$$
\limsup_{t \to T} (T - t) \|u(\cdot, t)\|_\infty < \infty.
$$

The blow-up is called type II if it is not type I. We note that self-similar solutions, given by (2.1) below, blow up in type I rate.

We briefly review some known results concerning blow-up behavior for (1.1) and related systems. In the case $N = 2$, Herrero and Velázquez [15] considered the system

$$
(1.4) \quad \begin{cases}
    u_t = \nabla \cdot (\nabla u - u \nabla v) & \text{in } \Omega \times (0, T) \\
    \tau v_t = \Delta v - v + u & \text{in } \Omega \times (0, T), \\
    \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{in } \partial \Omega \times (0, T),
\end{cases}
$$

together with initial conditions

$$
u(x, 0) = u_0(|x|) \quad \text{and} \quad v(x, 0) = v_0(|x|) \quad \text{for } x \in \Omega,
$$

where $\Omega = \{ x \in \mathbb{R}^2 : |x| < L \}$ and $\tau > 0$. It was shown in [15] that (1.4) has radially symmetric solutions such that $u$ develops a Dirac delta-type singularity at the origin in a finite time, and that $u$ blows up in type II rate. See also [13], [14]. Senba and Suzuki [25] considered the system (1.4) in the case where $\Omega$ is a bounded smooth domain in $\mathbb{R}^2$ and $\tau = 0$ together with the initial condition $u(x, 0) = u_0(x)$ for $x \in \Omega$. Denote by $T$ the maximal existence time of the solution to (1.4). It was shown in [25] that, if $T < \infty$, then the solution $(u, v)$ of (1.4) satisfies

$$
(1.5) \quad u(x, t) \to \sum_{q \in \mathbb{B}} m(q) \delta_q + f
$$
in the sense of measures as $t \to T$, where $\mathcal{B}$ is the set of blow-up points, $\delta_q$ is the delta function whose support is the point $q \in \overline{\Omega}$, $m(q)$ is the constant satisfying $m(q) \geq 8\pi$ if $q \in \Omega$, and $m(q) \geq 4\pi$ if $q \in \partial\Omega$, and $f$ is a nonnegative function in $L^1(\Omega) \cap C(\overline{\Omega} \setminus \mathcal{B})$. Furthermore, Senba [24] showed that, if $\Omega = \{x \in \mathbb{R}^2 : |x| < L\}$, and if a radial solution $(u, v)$ blows up at $t = T < 1$, then (1.5) holds with $\mathcal{B} = \{0\}$ and $m(0) = 8\pi$, and $u$ blows up in type II rate. For nonradial case, see [26].

In the case $N = 3$, Herrero et al [11], [12] have investigated the blow-up behavior of solutions by using matched asymptotic expansions. In [12] they showed that (1.2) has a sequence of self-similar blow-up solutions, and they in [11] showed the existence of Burgers like blow-up solutions which are not self-similar. These solutions consist of an imploding smoothed out shock wave that collapses into a Dirac mass when the singularity is formed, and blow up in type II rate. Later, Brenner et al [4] investigated the problem in the case $3 \leq N \leq 9$ by a numerical approach, and showed the existence and stability of both self-similar blow-up solutions and Burgers like blow-up solutions.

For a solution $(u, v)$ to (1.2), putting $n(x, t) = \Theta u(x, \Theta t)$ and $\phi(x, t) = -\Theta v(x, \Theta t)$ with $\Theta = 1/\|u_0\|_1$, we find that $(n, \phi)$ solves the problem

$$
\begin{cases}
  n_t = \nabla \cdot (\Theta \nabla n + n \nabla \phi) & \text{in } \Omega \times (0, T) \\
  \Delta \phi = n & \text{in } \Omega \times (0, T) \\
  \Theta \frac{\partial n}{\partial \nu} + n \frac{\partial \phi}{\partial \nu} = 0 \text{ and } v = 0 & \text{on } \partial\Omega \times (0, T) \\
  n(x, 0) = n_0(x) & \text{in } \Omega,
\end{cases}
$$

(1.6)

where $n_0(x) = \Theta u_0(|x|)$ for $x \in \Omega$. Note that $n_0$ satisfies $\|n_0\|_1 = 1$. Guerra and Peletier [10] considered the problem (1.6) in the case $3 \leq N \leq 9$. They in [10] characterize the blow-up behavior of solutions in terms of initial data, and showed that the solution behaves like a self-similar solution near the blow-up point.

In this note, we consider the system (1.2) in the case $3 \leq N \leq 9$, and derive criteria of the blow-up rate of solutions. In particular, we will identify an explicit class of initial data for which the blow-up is of type I rate.

To state our results, define $U_0$ and $V_0$, respectively, by

$$
U_0(r) = \frac{1}{r^{N-2}} \int_0^r s^{N-1} u_0(s) ds \quad \text{for } 0 \leq r \leq L
$$

(1.7)

and

$$
V_0(r) = \frac{U_0(r)}{r^2} = \frac{1}{r^N} \int_0^r s^{N-1} u_0(s) ds \quad \text{for } 0 \leq r \leq L.
$$

(1.8)
For the initial condition, we assume that
\[(1.9) \quad V_0'(r) \leq 0 \quad \text{for } 0 \leq r \leq L,\]
where $' = d/dr$. It is easy to see that (1.9) holds if $u_0$ satisfies
\[(1.10) \quad u_0 \in C^1[0, L] \quad \text{and} \quad u_0'(r) \leq 0 \quad \text{for } 0 \leq r \leq L.

Our first result is the following.

**Theorem 1.1.** Let $3 \leq N \leq 9$, and assume that (1.9) holds.

(i) Suppose $U_0$ satisfies $U_0(r) \leq 2$ for $0 \leq r \leq L$. Then a solution $(u, v)$ of (1.2) does not blow up in finite time.

(ii) Suppose that $U_0(r) - 2$ has exactly one zero for $0 \leq r < L$ and $U_0(L) > 2$. If a solution $(u, v)$ of (1.2) blows up in finite time, then the blow-up is of type I.

It should be mentioned that more general criteria will be given in Theorem 3.1 below.

As a consequence of Theorem 1.1, we obtain the following corollary.

**Corollary 1.2.** Let $3 \leq N \leq 9$, and assume that (1.9) holds. Suppose that $U_0$ is increasing for $0 < r < L$. If a solution $(u, v)$ of (1.2) blows up in finite time, then the blow-up is of type I.

Note that $U_0$ satisfies
\[(r^{N-1}U_0'(r))' = r^{N-1}(2u_0(r) + ru_0'(r)) \quad \text{for } 0 < r < L.

Assume that $u_0$ satisfies (1.10) and
\[(1.11) \quad ru_0'(r) + 2u_0(r) \quad \text{has at most one zero for } 0 \leq r \leq L.

Then one easily see that $U_0'(r)$ has at most one zero for $(0, L]$, and that $U_0'(r) > 0$ for $0 < r < r_0$ and $U_0'(r) < 0$ for $r_0 < r \leq L$ if $U_0'(r)$ has a zero at $r_0 \in (0, L)$. Assume, in addition, that
\[(1.12) \quad \int_0^L s^{N-1}u_0(s)ds > 2L^{N-2}.

Then $U_0(L) > 2$ and $U_0(r) - 2$ has exactly one zero for $0 \leq r < L$. By Theorem 1.1 (ii) we obtain the following:

**Corollary 1.3.** Let $3 \leq N \leq 9$, and assume that $u_0$ satisfies (1.10), (1.11) and (1.12). If a solution of (1.2) blows up in finite time, then the blow-up is of type I.
We recall here some sufficient conditions for blow-up in finite time by [2], [21].

**Proposition 1.4.** Let \( N \geq 3 \). Assume that one of the following (i)-(iii) holds:

(i) \( U_0(L) > 2N; \)

(ii) \( U_0(L) \geq 4 \) and \( u_0 \) satisfies, with some \( T_0 > 0, \)

\[
U_0(r) \geq \frac{4r^2}{2(N-2)T_0 + r^2} \quad \text{for } 0 \leq r \leq L;
\]

(iii) \( u_0 \) satisfies

\[
\int_{|x| \leq L} |x|^N u_0(|x|)dx < \left( \frac{\|u_0\|_1^{(2N-2)/N}}{4(N-1)\omega_N} \right)^{N/(N-2)},
\]

where \( \omega_N \) is the surface area of the unit sphere in \( \mathbb{R}^N \).

Then a solution \( (u, v) \) of (1.2) blows up in finite time \( t = T < \infty \). Furthermore, in the case (ii), the solution blows up at time \( T \) with \( T \cdot T_0 \).

The blow-up of solutions was shown in the case (i) by Biler [3, Theorem 3]. We can show the blow-up of solutions in the case (ii) by the comparison argument, and in the case (iii) by following the argument due to Nagai [21, Theorem 3.1]. For the proof of Proposition 1.4, see [20].

As a consequence of Corollaries 1.2 and 1.3 and Proposition 1.4, we can show the existence of solutions which blow up with type I rate. As a simple example, let \( u_0(r) \equiv \ell \) with \( \ell > 2N/L \). Then a solution \( (u, v) \) of (1.2) blows up in finite time with type I rate by Corollary 1.2 and Proposition 1.4 (i). (See [10, Corollary 1.2].) For another example, let \( u_0(r) = \ell G(r, \tau) \) with \( \tau > 0 \) and \( \ell > 0 \), where \( G(r, t) = (4\pi t)^{-N/2}e^{-r^2/4t} \) is the heat kernel. Then (1.10) and (1.11) hold, and it is easy to see that

\[
\|u_0\|_1 = \omega_N \int_0^L s^{N-1}u_0(s)ds \to \ell \quad \text{and} \quad \int_{|x| \leq L} |x|^N u_0(|x|)dx \to 0
\]
as \( \tau \to 0 \). Combining Corollary 1.3 and Proposition 1.4 (iii), we obtain the following:

**Corollary 1.5.** Let \( 3 \leq N \leq 9 \), and let \( u_0(r) = \ell G(r, \tau) \) with \( \tau > 0 \) and \( \ell > 2L^{N-2}\omega_N \). Then there exists \( \tau_0 > 0 \) such that, if \( \tau \in (0, \tau_0] \), then a solution \( (u, v) \) of (1.2) blows up in finite time with type I rate.

We note that, in Corollary 1.5, the initial function \( u_0 \) converges to a Dirac delta function in the sense of measure as \( \tau \to 0 \). Thus this corollary suggests that self-similar blow-up may be seen even if initial function is close to a Dirac delta function.
Next, we consider the local blow-up profile of solutions to (1.2). Assume that $V_0$, defined by (1.8), satisfies
\begin{equation}
(V_0)_{rr} + \frac{N+1}{r}(V_0)_r + N(V_0)^2 + rV_0(V_0)_r \geq 0 \quad \text{for } 0 \leq r \leq L.
\end{equation}
Guerra and Peletier [10] showed that, when $N \geq 3$ and (1.9) and (1.15) hold, any type I blow-up solution behaves like a self-similar solution near the singularity $x = 0$.

Our result is the following.

**Theorem 1.6.** Let $3 \leq N \leq 9$, and assume that (1.9) and (1.15) hold. If a solution $(u, v)$ of (1.2) blows up in finite time, then the blow-up is of type I.

**Remark.** Define the average density function $V$ by
\begin{equation}
V(r, t) = \frac{1}{r^N} \int_0^r s^{N-1}u(s, t)\,ds \quad \text{for } 0 \leq r \leq L, \quad 0 \leq t < T.
\end{equation}
It was shown by Guerra and Peletier [10, Theorem 2.3] that when $N \geq 3$ and (1.9) and (1.15) hold, if a solution $(u, v)$ blows up in finite time $t = T$ with type I rate, then $V$ satisfies
\begin{equation}
\lim_{t \to T} (T - t)V(\rho \sqrt{T - t}, t) = \Phi(\rho)/\rho^2
\end{equation}
uniformly on compact set $|\rho| \leq C$ for every $C > 0$, where $\Phi$ is a certain positive function. Combining with Theorem 1.6, we find that when $3 \leq N \leq 9$ and (1.9) and (1.15) hold, if a solution $(u, v)$ blows up in finite time, then (1.17) holds. Note here that the condition (1.15) ensures that $V_t \geq 0$ for all $0 < t < T$. (See (3.6) below.) It is still an open problem whether (1.17) holds for type I blow-up solutions without the condition (1.15).

We note that similar results hold for the well-studied problem for semilinear heat equation
\begin{equation}
\begin{cases}
  u_t - \Delta u = u^p & \text{in } \Omega \times (0, T), \\
  u = 0 & \text{on } \partial \Omega \times (0, T), \\
  u(x, 0) = u_0(|x|) & \text{in } \Omega,
\end{cases}
\end{equation}
where $p > 1$, $\Omega = \{x \in \mathbf{R}^N : |x| < L\}$, and $u_0(r)$ is nonnegative and nonincreasing for $0 \leq r \leq L$. A simple comparison argument shows that any blow-up solution satisfies
\begin{equation}
\liminf_{t \to T} (T - t)^{(p-1)/p} \|u(\cdot, t)\|_{\infty} > 0.
\end{equation}
Assume that $u_0 = u_0(|x|)$ satisfies
\begin{equation}
\Delta u_0 + u_0^p \geq 0 \quad \text{in } \Omega.
\end{equation}
By the maximum principle, the condition (1.19) implies that \( u_t \geq 0 \) for all \( 0 < t < T \). Friedman and McLeod [6] showed that, if (1.19) holds, then any blow-up solution satisfies

\[
\limsup_{t \to T} (T - t)^{1/(p-1)} \|u(\cdot, t)\|_{\infty} < \infty.
\]

Bebernes and Eberly showed in [1] that, under the condition (1.19), finite time blow-up solutions are asymptotically self-similar. Precisely, any solution \( u \) of (1.18) which blows up in finite time \( t = T \) satisfies

\[
\lim_{t \to T} (T - t)^{1/(p-1)} u((T - t)^{1/2} y, t) = \kappa
\]

uniformly on compact set \( |y| \leq C \) for every \( C > 0 \) with \( \kappa = (p-1)^{-1/(p-1)} \). It should be mentioned that Matos [19] later showed that any blow-up solution which satisfies (1.20) is asymptotically self-similar in the supercritical case without the condition (1.19). For the precise characterization of the behavior of blow-up solutions to (1.18), we refer to Giga and Kohn [7, 8, 9] in the subcritical case and Matano and Merle [17, 18] in the supercritical case.

This note is organized as follows: In Section 2, we will show the existence of a sequence of self-similar solutions to (1.1) with \( \Omega = \mathbb{R}^N \). In Section 3, we derive criteria of the blow-up rate of solutions, and give the proof of Theorems 1.1 and 1.6.

§ 2. Backward self-similar solutions

The proof of Theorems 1.1 and 1.6 are based on the study of the properties of backward self-similar solutions to the system (1.1) with \( \Omega = \mathbb{R}^N \). The system (1.1) with \( \Omega = \mathbb{R}^N \) is invariant under the scaling

\[(u, v) \mapsto (u_\lambda, v_\lambda) = (\lambda^2 u(\lambda x, \lambda^2 t), v(\lambda x, \lambda^2 t))\]

for \( \lambda > 0 \). A solution \((u, v)\) is called self-similar if \((u, v) = (u_\lambda, v_\lambda)\) for each \( \lambda > 0 \), and is called backward if \((u, v)\) is defined for all \( t < 0 \). By the transformation in the time, a backward self-similar solution has the form

\[
u(x, t) = \frac{1}{T-t} \phi(x / \sqrt{T-t}) \quad \text{and} \quad v(x, t) = \psi(x / \sqrt{T-t})
\]

for \( x \in \mathbb{R}^N \) and \( t < T \), where \((\phi, \psi)\) satisfies

\[
\begin{cases}
\Delta \phi - \nabla \cdot \left( \phi \left( \frac{x}{2} + \nabla \psi \right) \right) + \frac{N-2}{2} \phi = 0, & x \in \mathbb{R}^N \\
0 = \Delta \psi + \phi, & x \in \mathbb{R}^N.
\end{cases}
\]

We will obtain the existence of a sequence of self-similar solutions to (2.2) together with the properties of solutions. For the proof, see [20].
**Theorem 2.1.** Let $3 \leq N \leq 9$. Then the system \((2.2)\) has radially symmetric solutions \(\{ (\phi_j, \psi_j) \}_{j=1}^{\infty} \) such that \(\phi_j(r) > 0 \) for \(r \geq 0\) and \(\phi_j(0) \to \infty\) as \(j \to \infty\). For each \(j = 1, 2, \ldots\), define

\[
(2.3) \quad \Phi_j(r) = \frac{1}{r^{N-2}} \int_0^r s^{N-1} \phi_j(s) \, ds, \quad r > 0.
\]

Then \(\Phi_j(r) - 2\) has exactly \(2j\) zeros on \((0, \infty)\) and no zeros on \((R_0, \infty)\) with \(R_0 = 2\sqrt{N-1}\). Furthermore, there exists a sequence \(\{ \alpha_j \}_{j=1}^{\infty} \) satisfying \(0 < \cdots < \alpha_{j+1} < \alpha_j < \cdots < \alpha_1\) and \(\alpha_j \to 0\) as \(j \to \infty\) such that the following (i) and (ii) hold.

(i) For any constant \(c > 0\),

\[
\inf_{0 < r < \alpha_j} \frac{\Phi_j(r)}{r^2} \to \infty \quad \text{as} \quad j \to \infty.
\]

(ii) For any \(\varepsilon > 0\), there exist a constant \(c_0 = c_0(\varepsilon) > 0\) and an integer \(j_0 = j_0(\varepsilon) \in \mathbb{N}\) such that if \(j \geq j_0\) then

\[
\sup_{r \geq c_0 \alpha_j} |\Phi_j(r) - 2| < \varepsilon \quad \text{and} \quad \sup_{r \geq c_0 \alpha_j} \{|r \Phi_j'(r)| \} < \varepsilon.
\]

**Remark.** For each fixed \(r > 0\), we have

\[
(2.4) \quad r^2 \phi_j(r) \to 2(N - 2) \quad \text{as} \quad j \to \infty.
\]

In fact, it follows from \((2.3)\) that

\[
r^2 \phi_j(r) = (N - 2) \Phi_j(r) + r \Phi_j'(r) \quad \text{for} \quad r > 0.
\]

Since (ii) holds and \(\alpha_j \to 0\) as \(j \to \infty\), we obtain \((2.4)\).

The existence of self-similar solutions was already shown by [11, 4, 23]. It seems, however, that the properties on the location of zeros and properties (i) and (ii) are new, and these properties play an important role in the proof of the theorems.

§ 3. Proof of Theorems 1.1 and 1.6 (sketch)

We restrict our attention to radially symmetric solutions to \((1.2)\) of the form \(u = u(r, t)\) and \(v = v(r, t)\), \(r = |x|\), and consider the system

\[
\begin{cases}
  r^{N-1} u_r = (r^{N-1} u_r)_r - (r^{N-1} u v_r)_r, & 0 < r < L, \ 0 < t < T, \\
  0 = (r^{N-1} u_r)_r + r^{N-1} u, & 0 < r < L, \ 0 < t < T, \\
  u_r(0, t) = u_r(L, t) - u(L, t) v_r(L, t) = 0, & 0 < t < T, \\
  v_r(0, t) = v(L, t) = 0, & 0 < t < T, \\
  u(r, 0) = u_0(r), & 0 \leq r \leq L.
\end{cases}
\]

(3.1)
Put
\[ M = \int_0^L r^{N-1} u_0(r) dr. \]
Then, by the third formula in (3.1), it follows that
\[ \int_0^L r^{N-1} u(r, t) dr = M \quad \text{for} \quad 0 \leq t < T. \]

Put \( \hat{u} \) by
\[ \hat{u}(r, t) = \int_0^r s^{N-1} u(s, t) ds = -r^{N-1} v_r(r, t). \]
Here we have used the second formula in (3.1). Then the system (3.1) can be reduced to a single equation
\[ \hat{u}_t = r^{N-1} (r^{1-N} \hat{u}_r)_r + r^{1-N} \hat{u} \hat{u}_r. \]
Define
\[ U(r, t) = r^{2-N} \hat{u}(r, t) = \frac{1}{r^{N-2}} \int_0^r s^{N-1} u(s, t) ds. \]
Then \( U \) satisfies
\[ U_t = U_{rr} + \frac{N - 3}{r} U_r - \frac{2(N - 2)}{r^2} U + \frac{(N - 2) U^2 + r U U_r}{r^2} \]
for \( 0 < r < L, 0 < t < T \) and
\[ U(0, t) = \lim_{r \to 0} U(r, t) = 0 \quad \text{and} \quad U(L, t) = ML^{2-N} \quad \text{for} \quad 0 \leq t < T. \]
Note here that, by using l’Hospital’s rule, we obtain
\[ \lim_{r \to 0} \frac{U(r, t)}{r^2} = \lim_{r \to 0} \frac{\int_0^r s^{N-1} u(s, t) ds}{r^N} = \frac{u(0, t)}{N} \quad \text{for} \quad 0 < t < T. \]
Put \( V(r, t) = U(r, t)/r^2. \) Then \( V \) satisfies
\[ V_t = V_{rr} + \frac{N + 1}{r} V_r + NV^2 + rVV_r \quad \text{for} \quad 0 < r < L, 0 < t < T \]
and \( V(L, t) = ML^{-N} \) for \( 0 < t < T. \) We will show here that
\[ V_r(0, t) = \lim_{r \to 0} V_r(r, t) = 0 \quad \text{for} \quad 0 < t < T. \]
In fact,
\[ V_r(r, t) = \frac{1}{r} \left( u(r, t) - N r^{-N} \int_0^r s^{N-1} u(s, t) ds \right) \]
\[ = \frac{u(r, t) - u(0, t)}{r} - N \int_0^r \frac{s^{N-1} (u(s, t) - u(0, t)) ds}{r^{N+1}}. \]
By using l’Hospital’s rule, we obtain
\[
\lim_{r \to 0} \int_0^r s^{N-1}(u(s, t) - u(0, t))ds = \frac{1}{N+1} \lim_{r \to 0} \frac{u(r, t) - u(0, t)}{r} = \frac{u_0(0, t)}{N+1}.
\]
Since \(u_0(0, t) = 0\) for \(0 \leq t < T\), we obtain (3.7).

Let \((\phi_j, \psi_j)\) be a radially symmetric solutions of (2.2) obtained in Theorem 2.1, and put \(\Phi_j\) by (2.3). Take \(T > 0\), and put
\[
U_j(r, t) = \Phi_j(r/\sqrt{t-t_0}) = \frac{1}{r^{N-2}(T-t)} \int_0^r s^{N-1}\phi_j(s/\sqrt{t-t_0})ds
\]
for \(0 \leq r \leq L\), \(0 \leq t < T\). Then \(U = U_j\) solves (3.4) and
\[
U_j(0, t) = 0 \quad \text{and} \quad U_j(L, t) = \Phi_j(L/\sqrt{T-t}) \quad \text{for} \quad 0 < t < T.
\]
By the similar argument as in (3.5) we obtain
\[
\lim_{r \to 0} \frac{U_j(r, t)}{r^2} = \frac{\phi_j(0)}{N(T-t)} \quad \text{for} \quad 0 < t < T.
\]
Put \(V_j(r, t) = U_j(r, t)/r^2\). Then \(V = V_j\) solves (3.6) and
\[
(V_j)_r(0, t) = 0 \quad \text{and} \quad V_j(L, t) = \Phi_j(L/\sqrt{T-t})L^{-2} \quad \text{for} \quad 0 < t < T.
\]

By using the zero number properties of solutions for linear parabolic equations [5], we will derive criteria of the blow-up rate of solutions to (1.2) in terms of the function \(U\) defined by (3.3). We obtain Theorems 1.1 and 1.6 as a consequence of the following result.

**Theorem 3.1.** Let \(3 \leq N \leq 9\), and assume that (1.9) holds. Let \((u, v)\) be a radially symmetric solution of (1.2) for \(0 \leq t < T\), and define \(U\) by (3.3).

(i) Assume that there exist \(t_0 \in [0, T)\) and \(r_0 \in (0, L]\) such that
\[
U(r, t_0) \leq 2 \quad \text{for} \quad 0 \leq r \leq r_0 \quad \text{and} \quad U(r_0, t) \leq 2 \quad \text{for} \quad t_0 \leq t < T.
\]
Then the solution \((u, v)\) does not blow up at \(t = T\).

(ii) Assume that there exist \(t_0 \in [0, T)\) and \(r_0 \in (0, L]\) such that \(U(r, t_0) - 2\) has exactly one zero for \(0 \leq r \leq r_0\) and \(U(r_0, t) > 2\) for \(t_0 \leq t < T\). If the solution \((u, v)\) blows up at \(t = T < \infty\) then the blow-up is of type I.

It is clear that \(U_0\), defined by (1.7), satisfies \(U_0(r) = U(r, 0)\) for \(0 \leq r \leq L\). By the property (3.2) we see that \(U(L, t) = U_0(L)\) for \(0 \leq t < T\). Then, by applying Theorem 3.1 with \(r_0 = L\) and \(t_0 = 0\), we obtain Theorem 1.1 immediately.
Proof of Theorem 1.6. Assume that \((u, v)\) blows up at \(t = T < \infty\). By Theorem 3.1 (i) we have \(\{ r \in (0, L) : U(r, t) > 2 \} \neq \emptyset \) for any \(0 \leq t < T\). It is easy to see that there exist \(t_0 \in (0, T)\) and \(r_0 \in (0, L)\) such that \(U(r_0, t_0) > 2\) and \(U(r, t_0) - 2\) has exactly one zero for \(0 < r < r_0\). Note that the condition (1.15) ensures that \(V_t \geq 0\) for all \(t \in (0, T)\). Then \(U(r_0, t)\) is nondecreasing in \(t \in (0, T)\), and hence \(U(r_0, t) > 2\) for \(t_0 \leq t < T\). By Theorem 3.1 (ii), the blow-up is of type I. □

We will give a sketch of the proof of Theorem 3.1. For the detail, see [20].

Proof of Theorem 3.1. (Sketch). (i) By using the comparison argument, we may assume that \(U(r, t) < 2\) for \(0 \leq r < r_0\) and \(t_0 < t \leq T\). Take \(\hat{T} > T\), and define \(\hat{U}_j\) by

\[
\hat{U}_j(r, t) = \Phi_j \left( \frac{r}{\sqrt{\hat{T} - t}} \right) \quad \text{for} \quad 0 \leq r \leq L, t_0 \leq t < \hat{T},
\]

where \(\{ \Phi_j \}_{j=1}^\infty\) is a sequence of function obtained in Theorem 2.1. By using the properties in Theorem 2.1, we will find that there exists \(j_0 \in \mathbb{N}\) such that, if \(j = j_0\), then

\[
(3.10) \quad \hat{U}_j(r, t_0) > U(r, t_0) \quad \text{for} \quad 0 < r < r_0,
\]

and

\[
(3.11) \quad \hat{U}_j(r_0, t) > U(r_0, t) \quad \text{for} \quad t_0 \leq t \leq T.
\]

Put \(V\) and \(\hat{V}_j\) by

\[
V(r, t) = \frac{U(r, t)}{r^2} \quad \text{and} \quad \hat{V}_j(r, t) = \frac{\hat{U}_j(r, t)}{r^2},
\]

respectively. Then \(V\) and \(\hat{V}\) solve (3.6) and satisfy \(V_r(0, t) = \hat{V}_r(0, t) = 0\) for \(t_0 \leq t < T\). It follows from (3.10) and (3.11) that

\[
\hat{V}_j(r, t_0) > V(r, t_0) \quad \text{for} \quad 0 \leq r \leq r_0 \quad \text{and} \quad \hat{V}_j(r_0, t) > V(r_0, t) \quad \text{for} \quad t_0 \leq t < T.
\]

Then, by the maximum principle, we obtain \(V(r, t) < \hat{V}_j(r, t)\) for \(0 \leq r \leq r_0, t_0 \leq t < T\). From (3.5) and (3.9) we see that

\[
\lim_{r \to 0} V(r, t) = \lim_{r \to 0} \frac{U(r, t)}{r^2} = \frac{u(0, t)}{N} \quad \text{and} \quad \lim_{r \to 0} \hat{V}_j(r, t) = \lim_{r \to 0} \frac{\hat{U}_j(r, t)}{r^2} = \frac{\phi_j(0)}{N(T - t)}.
\]

This implies that \(u(0, t) < \phi_j(0)/(\hat{T} - t)\) for \(t_0 \leq t < T < \hat{T}\). Note that (1.9) implies \(u(0, t) = \|u(\cdot, t)\|_\infty < \infty\), and hence \((u, v)\) does not blow up at \(t = T\).
(ii) For a continuous function $\psi$ defined on an interval $J$, we define the zero number of the function $\psi$ on $J$ by $Z_J[\psi] = \#\{r \in J : \psi(r) = 0\}$. We will find that $Z_{[0,r_0]}[U(\cdot,t) - 2] = 1$ for $t_0 \leq t < T$, and we may assume that $T - t_0 > 0$ is small enough so that

\begin{equation}
\frac{r_0}{\sqrt{T-t_0}} > R_0,
\end{equation}

where $R_0$ is the constant which appears in Theorem 2.1.

Define $U_j$ by (3.8) for $j = 1, 2, \ldots$. First we show that, for each $j = 1, 2, \ldots$,

\begin{equation}
U_j(r_0, t) < U(r_0, t) \text{ for } t_0 \leq t < T.
\end{equation}

Since $\Phi_j(r) - 2$ has exactly $2j$ zeros on $(0, R_0]$ and no zeros on $(R_0, \infty)$ by Theorem 2.1, we see that that $\Phi_j(r) < 2$ for $r > R_0$. From (3.12) we obtain $U_j(r_0, t) < 2$ for $t_0 \leq t < T$. Since $U(r_0, t) > 2$ for $t_0 \leq t < T$, we obtain (3.13).

By using the properties in Theorem 2.1, we will find that there exists $j_0 \in \mathbb{N}$ such that, if $j \geq j_0$, then $U_j(r, t_0) - U(r, t_0)$ has exactly one zero for $0 \leq r < r_0$. Put $V$ and $V_j$ by $V(r, t) = U(r, t)/r^2$ and $V_j(r, t) = U(r, t)/r^2$, respectively. Then $Z_{[0,r_0]}[V_j(\cdot, t_0) - V(\cdot, t_0)] = 1$. By the zero number property, we obtain $Z_{[0,r_0]}[V_j(\cdot, t) - V(\cdot, t)] = 1$ for $t_0 < t < T$. We denote by $\tilde{r}(t)$ a unique zero of $V_j(r, t) - V(r, t)$. Then $\tilde{r}(t) \in (0, r_0)$ for $t_0 \leq t < T$, and $V(r, t) < V_j(r, t)$ for $0 \leq r < \tilde{r}(t)$. From (3.5) and (3.9) we obtain, for each $t \in [t_0, T)$,

\[
\lim_{r \to 0} V(r, t) = \lim_{r \to 0} \frac{U(r, t)}{r^2} = \frac{u(0, t)}{N} \quad \text{and} \quad \lim_{r \to 0} V_j(r, t) = \lim_{r \to 0} \frac{U_j(r, t)}{r^2} = \frac{\phi_j(0)}{N(T-t)}.\]

Then it follows that $(T-t)u(0, t) \leq \phi_j(0)$ for $t_0 \leq t < T$. This implies that the blow-up is of type I.  

\[\square\]

References


