Unramified extensions and geometric $\mathbb{Z}_p$-extensions of global function fields

By

Tsuyoshi ITOH

Abstract

We study on finite unramified extensions of global function fields (that is, function fields of one variable over a finite field). We show two results. One is an extension of Perret's result about the ideal class group problem. Another is a construction of a geometric $\mathbb{Z}_p$-extension which has a certain property.

§ 1. Main theorems

Throughout the present paper, we fix a prime number $p$ and a finite field $\mathbb{F}$ of characteristic $p$. Let $q$ be the number of elements of $\mathbb{F}$. Recall that a global function field is a function field of one variable over a finite field. Let $k$ be a global function field with full constant field $\mathbb{F}$. We also recall that a finite algebraic extension $K/k$ is geometric if and only if the constant field of $K$ is also $\mathbb{F}$.

It is known that there is a finite abelian group $G$ which is not isomorphic to the divisor class group of degree 0 of any global function field (Stichtenoth [20]). On the other hand, Perret [16] showed the following:

Theorem 1.1 ([16]). For any given finite abelian group $G$, there is a finite separable geometric extension $k/\mathbb{F}(T)$ such that $\text{Cl}(\mathcal{O}) \cong G$, where $\mathcal{O}$ is the integral closure of $\mathbb{F}[T]$ in $k$ and $\text{Cl}(\mathcal{O})$ is the ideal class group of $\mathcal{O}$.

This theorem is shown by using the following:
**Theorem 1.2** ([16]). For any given finite abelian group $G$, there is a global function field $k$ with full constant field $\mathbb{F}$ and a non-empty finite set $S$ of places of $k$ such that $\text{Cl}_S(k) \cong G$, where $\text{Cl}_S(k)$ is the $S$-class group of $k$.

Let $S$ be a non-empty finite set of places of $k$, and $H_S(k)$ the $S$-Hilbert class field of $k$, that is, the maximal unramified abelian extension field of $k$ in which all places of $S$ split completely (see [17]). We note that $\text{Cl}_S(k) \cong \text{Gal}(H_S(k)/k)$ by class field theory. Hence Theorem 1.2 also implies the existence of $k$ and $S$ which satisfy $\text{Gal}(H_S(k)/k) \cong G$. (More precisely, we can take $k$ and $S$ such that $H_S(k)/k$ is a geometric extension. See [16].)

In the present paper, we extend the above result to non-abelian finite groups. We will show the following:

**Theorem 1.3.** For any given finite group $G$, there is a global function field $k$ with full constant field $\mathbb{F}$ and a non-empty finite set $S$ of places of $k$ such that $\text{Gal}(\tilde{H}_S(k)/k) \cong G$, where $\tilde{H}_S(k)$ denotes the maximal unramified Galois extension field of $k$ in which all places of $S$ split completely. Moreover, we can take $k$ and $S$ such that $\tilde{H}_S(k)/k$ is a geometric extension.

See Ozaki [15] for the number field case.

We will prove Theorem 1.3 in section 2. Our proof is due to Perret’s idea (see [16]). That is, we will construct an unramified $G$-extension, and take a sufficiently large set $S$ of places such that $\text{Gal}(\tilde{H}_S(k)/k) \cong G$. (We use the term “$G$-extension” as a Galois extension whose Galois group is isomorphic to $G$.) To construct an unramified $G$-extension, we shall show an analog (Theorem 2.2) of Fröhlich’s classical result [4] for number fields.

In section 3, we shall apply Perret’s idea to Iwasawa theory. Let $k$ be a global function field with full constant field $\mathbb{F}$, $S$ a non-empty finite set of places of $k$. We recall that a $\mathbb{Z}_p$-extension is an infinite Galois extension whose Galois group is topologically isomorphic to the additive group of the ring $\mathbb{Z}_p$ of $p$-adic integers. Let $k_\infty/k$ be a geometric $\mathbb{Z}_p$-extension, that is, $k_\infty/k$ is a $\mathbb{Z}_p$-extension which satisfies that every finite subextension over $k$ is a geometric extension (see, e.g., [7]). (Recall that $p$ is the characteristic of $\mathbb{F}$.) We assume that

(A) only finitely many places of $k$ ramify in $k_\infty/k$, and

(B) all places of $S$ split completely in $k_\infty/k$.

Under these assumptions, we can treat Iwasawa theory for the $S$-class group (see [17]). For a non-negative integer $n$, let $k_n$ be the $n$th layer of $k_\infty/k$. That is, $k_n$ is the unique subfield of $k_\infty$ which is a cyclic extension over $k$ of degree $p^n$. Moreover, let $A_n$ be the
Sylow $p$-subgroup of the $S$-class group of $k_n$. (Here we use the same symbol $S$ as the set of places of $k_n$ lying above $S$.) We put $X_S = \limleftarrow A_n$, where the projective limit is taken with respect to the norm maps. We call $X_S$ the Iwasawa module of $k_\infty/k$.\footnote{Note that $X_S$ is a finitely generated torsion $\Lambda$-module, and the “Iwasawa type formula” holds for $A_n$ (see [17]). That is, there are non-negative integers $\lambda, \mu, \nu$ such that $|A_n| = p^{\lambda n + \mu p^n + \nu}$ for all sufficiently large $n$. Aiba [1] studied these invariants $\lambda, \mu, \nu$ for certain geometric $\Z_p$-extensions.}

There is a natural problem: characterize the $\Lambda$-modules which appear as $X_S$. (For the number field case, the same problem is dealt in, e.g., [14], [5].) Concerning this problem, we shall give the following result including “non-abelian” cases.

\textbf{Theorem 1.4.} For any given finite $p$-group $G$, there exist a global function field $k$ with full constant field $F$, a non-empty finite set $S$ of places of $k$, and a geometric $\Z_p$-extension $k_\infty/k$ satisfying the above assumptions (A) and (B) such that $\Gal(\tilde{L}_S(k_n)/k_n) \cong G$ (as groups) for all $n \geq 0$, where $\tilde{L}_S(k_n)$ is the maximal unramified Galois pro-$p$-extension field of $k_n$ in which all places lying above $S$ split completely.

For the number field case, Ozaki [14] showed that every “finite $\Lambda$-module” appears as $X_S$. (For the number field case, the same problem is dealt in, e.g., [14], [5].) Concerning this problem, we shall give the following result including “non-abelian” cases.

\section{Proof of Theorem 1.3}

\subsection{Function field analog of Fröhlich’s result}

At first, we shall show that for any finite group $G$, there is an unramified geometric extension $K/k$ of global function fields such that $\Gal(K/k) \cong G$. Recall that any finite group can be embedded into a finite symmetric group. Hence it is sufficient to consider the case that $G$ is a finite symmetric group. For the number field case, Fröhlich already showed the following result.

\textbf{Theorem 2.1 ([4])}. For every positive integer $n$, there is an unramified Galois extension $K/k$ of algebraic number fields such that $\Gal(K/k) \cong \mathfrak{S}_n$, where $\mathfrak{S}_n$ denotes the symmetric group of degree $n$.

We will show the following:
Theorem 2.2. For every positive integer \( n \), there is a global function field \( k \) with full constant field \( \mathbb{F} \) and an unramified geometric Galois extension \( K/k \) such that \( \text{Gal}(K/k) \cong \mathfrak{S}_n \). More precisely, there exist a geometric Galois extension \( K/\mathbb{F}(T) \) and a subextension \( k/\mathbb{F}(T) \) of \( K/\mathbb{F}(T) \) such that \( K/k \) is unramified and that \( \text{Gal}(K/k) \cong \mathfrak{S}_n \).

To prove this, we follow Fröhlich’s original argument (see also Malinin [10]). That is, we construct a certain (ramified) \( \mathfrak{S}_n \)-extension over \( \mathbb{F}(T) \) and then we take a certain base change of this extension. Let \( \infty \) be the infinite place of \( \mathbb{F}(T) \).

Lemma 2.3. There is a Galois extension \( k' \) over \( \mathbb{F}(T) \) which satisfies all of the following properties.

- \( k'/\mathbb{F}(T) \) is a geometric extension,
- \( \text{Gal}(k'/\mathbb{F}(T)) \cong \mathfrak{S}_n \), and
- \( \infty \) is unramified in \( k'/\mathbb{F}(T) \).

Proof. At first, we must see that there is an \( \mathfrak{S}_n \)-extension over \( \mathbb{F}(T) \). This follows from the fact that \( \mathbb{F}(T) \) is a Hilbertian field (see, e.g., [3, Corollary 16.2.7]). We put \( A = \mathbb{F}[T] \). For an element \( r \) of \( A \), let \( \text{deg}(r) \) be the degree of \( r \) as a polynomial of \( T \).

Fix a monic separable polynomial \( F(X) \in A[X] \) of degree \( n \) such that the splitting field of \( F(X) \) over \( \mathbb{F}(T) \) is an \( \mathfrak{S}_n \)-extension.

We claim that there is an element \( N_F \in A \) which satisfies the following property: if a monic polynomial \( G(X) \in A[X] \) of degree \( n \) satisfies \( G(X) \equiv F(X) \pmod{N_F} \), then the splitting field of \( G(X) \) over \( \mathbb{F}(T) \) is also an \( \mathfrak{S}_n \)-extension. We shall show this claim. By using the Chebotarev density theorem, we can take an irreducible monic polynomial \( p_1 \) such that if \( G(X) \equiv F(X) \pmod{p_1} \) then \( G(X) \) is irreducible and separable. Similarly, we can take distinct irreducible monic polynomials \( p_2, p_3 \) of \( A = \mathbb{F}(T) \) which are distinct from \( p_1 \) and satisfy the following properties: (i) if \( G(X) \equiv F(X) \pmod{p_2} \) then the Galois group of \( G(X) \) contains a cycle of length \( n - 1 \) (as a subgroup of \( \mathfrak{S}_n \)), and (ii) if \( G(X) \equiv F(X) \pmod{p_3} \) then the Galois group of \( G(X) \) contains a transposition. We put \( N_F = p_1p_2p_3 \). This \( N_F \) satisfies the above claim. Moreover, we can take \( N_F \) which is prime to \( T \) by the Chebotarev density theorem. We also fix such \( N_F \).

To construct a geometric \( \mathfrak{S}_n \)-extension which is unramified at the infinite place, we take \( G(X) \) as follows:

\[
\begin{align*}
G(X) & \equiv F(X) \pmod{N_F}, \\
G(X) & \equiv \text{a product of distinct monic polynomials of degree 1} \pmod{r}, \text{ and} \\
G(X) & \equiv \text{a separable polynomial} \pmod{T},
\end{align*}
\]

where \( r \) is a monic irreducible polynomial of \( A = \mathbb{F}[T] \) such that \( n < q^{\text{deg}(r)} \), \( \text{deg}(r) \) is odd, and \( r \) is prime to \( TN_F \). By the first congruence, we see that the splitting field \( k' \)
of $G(X)$ is an $\mathcal{S}_n$-extension. We shall show that the constant field of $k'$ is $F$. Let $\overline{F}$ be the algebraic closure of $F$. We note that $M := k' \cap \overline{F}(T)$ is a finite cyclic extension over $\overline{F}(T)$. Since $\text{Gal}(k'/\overline{F}(T)) \cong \mathcal{S}_n$, $M$ must be $\overline{F}(T)$ or the unique quadratic subfield in $k'/\overline{F}(T)$. If $M \neq \overline{F}(T)$, then no odd degree place of $\overline{F}(T)$ splits in $M$. However, we see that the place of $\overline{F}(T)$ corresponding to $r$ splits completely in $k'$ by the second congruence. It is a contradiction.

By the third congruence, we see that the place of $\overline{F}(T)$ corresponding to $T$ is unramified in $k'$. We replace the indeterminate $T$ by $U = 1/T$, then the infinite place of $\overline{F}(U)$ is unramified in $k'$ (and the former two conditions are also satisfied).

We shall prove Theorem 2.2. We may assume that $n \geq 2$. Fix a geometric $\mathcal{S}_n$-extension $k'/\overline{F}(T)$ satisfying the properties of Lemma 2.3. We put $m = n!$. We can take a separable monic polynomial $F(X) \in A[X]$ of degree $m$ (as a polynomial of $X$) whose splitting field over $\overline{F}(T)$ is $k'$. Let $M'$ be the unique quadratic subextension field of $\overline{F}(T)$ contained in $k'$.

We define the following notation.

- $\{p_1, \ldots, p_t\}$: the set of distinct places of $\overline{F}(T)$ which ramify in $k'$ (hence are distinct from $\infty$).
- $p_{t+1}$: a place $\neq \infty, p_1, \ldots, p_t$ of $\overline{F}(T)$ which is inert in $M'$ and has degree $> \frac{\log(m)}{\log(q)}$.
- $p_{t+2}$: a place $\neq \infty$ of $\overline{F}(T)$ which splits completely in $k'$ and has odd degree $> \frac{\log(m)}{\log(q)}$ (hence is distinct from $p_1, \ldots, p_{t+1}$).
- $p_1, \ldots, p_{t+2}$: irreducible monic polynomials of $A = \overline{F}[T]$ corresponding to $p_1, \ldots, p_{t+2}$, respectively.

Note that we can take $p_{t+1}$ (resp. $p_{t+2}$) by using Theorem 9.13B of [18], which is an effective version of the Chebotarev density theorem for global function fields. (See also [12], etc.) Indeed, by this theorem, there is a place of $\overline{F}(T)$ of arbitrary sufficiently large degree which is inert in $M'$ (resp. splits completely in $k'$), as $M'/\overline{F}(T)$ is a geometric cyclic extension (resp. $k'/\overline{F}(T)$ is a geometric Galois extension).

By using Lemma 2.3, we can also construct an $\mathcal{S}_m$-extension over $\overline{F}(T)$. Let $H(X)$ be a monic polynomial in $A[X]$ of degree $m$ which gives an $\mathcal{S}_m$-extension. Then there is an element $N_H$ of $A$ having the following property: if a monic polynomial $G(X) \in A[X]$ of degree $m$ satisfies $G(X) \equiv H(X) \pmod{N_H}$, then the splitting field of $G(X)$ over $\overline{F}(T)$ is also an $\mathcal{S}_m$-extension (see the proof of Lemma 2.3). We can also take $N_H$ such that it is prime to $p_1, \ldots, p_{t+2}$.

We take a monic polynomial $G(X)$ of $A[X]$ (having degree $m$) which satisfies the following conditions (2.1)--(2.4).

\begin{equation}
(2.1) \quad G(X) \equiv H(X) \pmod{N_H}.
\end{equation}
If $G(X)$ satisfies (2.1), then $G(X)$ gives an $\mathfrak{S}_m$-extension. Let $L$ be the splitting field of $G(X)$ over $\mathbb{F}(T)$.

(2.2) $G(X) \equiv (a \text{ product of distinct monic polynomials of degree } 1) \pmod{p_{t+1}}$.

If $G(X)$ satisfies (2.1) and (2.2), then we see that $p_{t+1}$ splits in the unique quadratic subextension, say $M_L$, over $\mathbb{F}(T)$ contained in $L$. On the other hand, $p_{t+1}$ is inert in the unique quadratic subextension $M'$ over $\mathbb{F}(T)$ contained in $k'$. We claim that $k' \cap L = \mathbb{F}(T)$. Indeed, suppose that $k' \cap L \neq \mathbb{F}(T)$. Then $k' \cap L$ is a quadratic extension over $\mathbb{F}(T)$. If $n = 2$, this is clear. For $n \geq 3$, we have $\text{Gal}(L/\mathbb{F}(T)) \cong \mathfrak{S}_m$, where $m = n! \geq 5$. Observe also that $k' \cap L \neq L$, as $m > n$. Now, since the alternating group $\mathfrak{A}_m$ is the unique nontrivial proper normal subgroup of $\mathfrak{S}_m$ when $m \geq 5$ (see, e.g., [19]), $k' \cap L$ is a quadratic extension over $\mathbb{F}(T)$. Since this quadratic extension is contained in both $k'$ and $L$, it must coincide with both $M'$ and $M_L$ at a time. This contradicts the above observation on the behavior of $p_{t+1}$ in $M'$ and $M_L$. Thus, we have proved the claim. Then we see $\text{Gal}(Lk'/L) \cong \mathfrak{S}_n$.

(2.3) $G(X) \equiv (a \text{ product of distinct monic polynomials of degree } 1) \pmod{p_{t+2}}$.

If $G(X)$ satisfies (2.1)–(2.3), then the odd degree place $p_{t+2}$ splits completely in $Lk'/\mathbb{F}(T)$. We claim that $Lk'/\mathbb{F}(T)$ is a geometric extension. Note that the degree of a place of $k'$ lying above $p_{t+2}$ is also odd because $p_{t+2}$ splits completely in $k'$. Since $\text{Gal}(Lk'/k') \cong \mathfrak{S}_m$ and an odd degree place splits completely in $Lk'/k'$, we see that $Lk'/k'$ is also a geometric extension. Hence the claim follows. By using Krasner’s lemma, we can see that there is a positive integer $s_i$ for each $i = 1, \ldots, t$ depending only on $F(X)$ such that if $G(X) \equiv F(X) \pmod{p_i^{s_i}}$ then $L \mathbb{F}(T)_{p_i} = k' \mathbb{F}(T)_{p_i}$, where $\mathbb{F}(T)_{p_i}$ is the completion of $\mathbb{F}(T)$ at $p_i$ (see, e.g., [13]). Hence if we take $G(X)$ satisfying (2.1)–(2.3) and

(2.4) $G(X) \equiv F(X) \pmod{p_i^{s_i}}$ for $i = 1, \ldots, t$,

then we can see that $Lk'/L$ is unramified at all places.

We can take $G(X)$ satisfying (2.1)–(2.4). By the above arguments, the extension $Lk'/L$ satisfies the assertion of Theorem 2.2. □

Remark. When $G$ is abelian, an unramified geometric $G$-extension was constructed by Angles [2]. Moret-Bailly [11] also gives a result which is very close to ours. Probably, it seems that one can prove our main theorems by using the result given in [11] instead of Theorem 2.2.

§ 2.2. Proof of Theorem 1.3

Since $G$ is embedded into $\mathfrak{S}_n$ for some $n > 0$, Theorem 2.2 implies that there exists a global function field $k$ with full constant field $\mathbb{F}$ and an unramified geometric Galois extension $K/k$ such that $\text{Gal}(K/k) \cong G$. 
Proposition 2.4. There is a non-empty finite set $S$ of places of $k$ such that (i) all places of $S$ split completely in $K$, and (ii) $\tilde{H}_S(k)/k$ is a finite extension.

Proof. The crucial point of this proposition is choosing a set $S$ to satisfy (ii). For a positive integer $N$, we put

$$B_N = \{p \mid p \text{ is a place of } k \text{ having degree } N, \text{ p splits completely in } K/k\}.$$  

Since $K/k$ is a geometric extension, Theorem 9.13B of [18] implies that

$$|B_N| = q^N/|G| + O\left(q^{N/2}/N\right)$$  

(recall that $q$ is the number of elements of $\mathbb{F}$). In particular, if $N$ is sufficiently large, then we obtain the inequality

$$|B_N| > \frac{q^{N/2} - 1}{N}\text{Max}(g-1,0),$$  

where $g$ is the genus of $k$. We fix an integer $N$ which satisfies the above inequality. According to Ihara's theorem [8, Theorem 1(FF)], if $S \supset B_N$, then $\tilde{H}_S(k)/k$ is a finite extension. Hence we can take $S$ to satisfy the conditions (i) and (ii).

The rest of the proof of Theorem 1.3 is quite similar to Perret’s argument given in [16]. We remark that $K$ is contained in $\tilde{H}_S(k)$. For a nontrivial element $\sigma$ of $\text{Gal}(\tilde{H}_S(k)/K)$, we can take a place $\mathfrak{P}$ of $\tilde{H}_S(k)$ corresponding to $\sigma$ by the Chebotarev density theorem. We can take $\mathfrak{P}$ which is unramified in $\tilde{H}_S(k)/K$. Let $p$ be the place of $k$ which is lying below $\mathfrak{P}$. Since the decomposition field of $\mathfrak{P}$ in $\tilde{H}_S(k)/k$ contains $K$ and $K/k$ is a Galois extension, we see that $p$ splits completely in $K/k$. Then we see $\tilde{H}_S(k) \supset \tilde{H}_{S\cup\{p\}}(k) \supset K$. Replacing $S \cup \{p\}$ by $S$ and repeating the above operation, we can see that $\tilde{H}_S(k) = K$ for some finite set $S$. This implies $\text{Gal}(\tilde{H}_S(k)/K) \cong G$.

We recall that $K/k$ is a geometric extension. Hence the final part of the theorem follows.

§ 3. Proof of Theorem 1.4

Firstly, we shall show the following:

Theorem 3.1. Let $k$ be a finite Galois extension over $\mathbb{F}(T)$. Then, there exist a non-empty finite set $S$ of places of $\mathbb{F}(T)$ and a geometric $\mathbb{Z}_p$-extension $F_\infty/\mathbb{F}(T)$ which satisfy the following properties.
• $F_\infty \cap k = F(T)$,

• all places of $S$ split completely in $k$,

• both of $F_\infty /\mathbb{F}(T)$ and $F_\infty k/k$ satisfy the assumptions (A) and (B) in section 1, and

• the Sylow $p$-subgroup of $\text{Cl}_S(F_n k)$ is trivial for all $n \geq 0$,

where $F_n$ is the $n$th layer of $F_\infty /\mathbb{F}(T)$. (We use the same symbol $S$ as the set of places lying above $S$.)

Proof. We take a place $p_0$ of $\mathbb{F}(T)$ which splits completely in $k$. We also take a place $r$ of $\mathbb{F}(T)$ which is distinct from $p_0$ and unramified in $k$. We claim that there is a geometric $\mathbb{Z}_p$-extension $F_\infty /\mathbb{F}(T)$ unramified outside $r$ which satisfies that

• $r$ is totally ramified, and

• $p_0$ splits completely.

We shall show this claim. Let $M$ be the maximal pro-$p$ abelian extension over $\mathbb{F}(T)$ which is unramified outside $r$. We know that $\text{Gal}(M/\mathbb{F}(T))$ is isomorphic to a countable infinite product of the additive group of $\mathbb{Z}_p$ (see [21], [9]). Hence there are infinitely many geometric $\mathbb{Z}_p$-extensions which satisfy the above conditions.

By the above choice of $F_\infty$, we see $F_1 \cap k = \mathbb{F}(T)$. We put $k_1 = F_1 k$. Then $k_1/\mathbb{F}(T)$ is a Galois extension, and $p_0$ splits completely in $k_1$. We set $S_0 = \{p_0\}$, and we use the same symbol to denote the set of places lying above $p_0$. We can see that $H_{S_0}(k_1)$ is a finite Galois extension over $\mathbb{F}(T)$. We take a nontrivial element $\sigma_1$ of $\text{Gal}(H_{S_0}(k_1)/k_1)$.

By using the above argument, we can take a geometric $\mathbb{Z}_p$-extension $F'_\infty /\mathbb{F}(T)$ unramified outside $r$ which satisfies

• $F'_\infty \cap F_\infty = \mathbb{F}(T)$,

• $r$ is totally ramified in $F'_\infty F_\infty$, and

• $p_0$ splits completely in $F'_\infty$.

Let $F'_1$ be the initial layer of $F'_\infty /\mathbb{F}(T)$. Then we see that $F'_1 \cap k_1 = \mathbb{F}(T)$ and $k_1 F'_1 \cap H_{S_0}(k_1) = k_1$. We note that

\[
\text{Gal}(F'_1 H_{S_0}(k_1)/k_1) \cong \text{Gal}(F'_1 k_1/k_1) \times \text{Gal}(H_{S_0}(k_1)/k_1), \quad \text{Gal}(F'_1 k_1/k_1) \cong \text{Gal}(F'_1/\mathbb{F}(T)).
\]

Hence there is an isomorphism

\[
\text{Gal}(F'_1/\mathbb{F}(T)) \times \text{Gal}(H_{S_0}(k_1)/k_1) \xrightarrow{\sim} \text{Gal}(F'_1 H_{S_0}(k_1)/k_1).
\]
Let \( \tau \) be a generator of the cyclic group \( \text{Gal}(F'_1/\mathbb{F}(T)) \), and \( \tau_1 \) an element of \( \text{Gal}(F'_1H_{S_0}(k_1)/k_1) \) which is the image of \((\tau, \sigma_1)\) under the above isomorphism. We can regard \( \tau \) as an element of \( \text{Gal}(F'_1H_{S_0}(k_1)/\mathbb{F}(T)) \). By the Chebotarev density theorem, there is a place \( \mathfrak{p}_1 \) of \( F'_1H_{S_0}(k_1) \) which corresponds to \( \tau_1 \). Let \( p_1 \) be the place of \( \mathbb{F}(T) \) lying below \( \mathfrak{p}_1 \). We can take \( \mathfrak{p}_1 \) such that \( p_1 \) is not ramified in \( F'_1H_{S_0}(k_1) \). Then we see that \( p_1 \) splits completely in \( k_1 \) and is inert in \( F'_1 \). We put \( S_1 = S_0 \cup \{ p_1 \} \).

In general, \( p_1 \) may not split completely in \( F_\infty \). This is a problem because we need the assumption (B). We remark that \( F_\infty F'_\infty/\mathbb{F}(T) \) is a \( \mathbb{Z}_p^2 \)-extension unramified outside \( \tau \). We recall that \( p_1 \) does not split in \( F'_1 \). Hence the decomposition field of \( F_\infty F'_\infty/\mathbb{F}(T) \) for \( p_1 \) is a \( \mathbb{Z}_p \)-extension over \( \mathbb{F}(T) \). We denote it by \( F''_\infty \). We also note that \( F''_\infty/\mathbb{F}(T) \) is the unique \( \mathbb{Z}_p \)-extension contained in \( F_\infty F'_\infty \) such that \( p_1 \) splits completely. Then the initial layer of \( F''_\infty/\mathbb{F}(T) \) must coincide with \( F_1 \). We replace \( F_\infty \) by \( F''_\infty \).

We note that \( H_{S_0}(k_1) \supseteq H_{S_1}(k_1) \) by the definition of \( p_1 \). Similarly, we can choose a place \( p_2 \), put \( S_2 = S_1 \cup \{ p_2 \} \), and modify the \( \mathbb{Z}_p \)-extension such that all places of \( S_2 \) splits completely. Repeating this operation, we see that \( H_{S_0}(k_1) = k_1 \) for some finite set \( S_1 \). From the above construction, we see that \( F_\infty \cap k = \mathbb{F}(T) \) and that \( F_\infty k/k \) satisfies the assumptions (A) and (B).

Finally, we shall give an Iwasawa-theoretic argument. In \( F_\infty k/k \), all ramified places (which are lying above \( \tau \)) are totally ramified. From this, we also see \( H_{S_0}(k) = k \). Let \( A_n \) be the Sylow \( p \)-subgroup of \( \text{Cl}_{S_1}(kF_n) \). By the above results, we see that both of \( A_0 \) and \( A_1 \) are trivial. In this situation, we can use the method given in Fukuda [6]. Namely, if all places which ramify in \( F_\infty k/k \) are totally ramified and both of \( A_0 \) and \( A_1 \) are trivial, then \( A_n \) is trivial for all \( n \geq 0 \). (See [6, Theorem 1]. We note that the same method is also applicable for our situation.) Hence we see that \( A_n \) is trivial for all \( n \geq 0 \).

We shall show Theorem 1.4. We fix a finite \( p \)-group \( G \). By using Theorem 2.2, we can take a geometric Galois extension \( K/\mathbb{F}(T) \) and a subextension \( k/\mathbb{F}(T) \) of \( K/\mathbb{F}(T) \) such that \( K/k \) is unramified and \( \text{Gal}(K/k) \cong G \). By Theorem 3.1, we can take a geometric \( \mathbb{Z}_p \)-extension \( F_\infty/\mathbb{F}(T) \) and a set \( S \) of places of \( \mathbb{F}(T) \) such that \( F_\infty \cap K = \mathbb{F}(T) \), all places of \( S \) split completely in \( K \), both of \( F_\infty/\mathbb{F}(T) \) and \( F_\infty K/K \) satisfy the assumptions (A) and (B), and \( A_n \) is trivial for all \( n \geq 0 \) (where \( A_n \) is the Sylow \( p \)-subgroup of \( \text{Cl}_{S}(F_n K) \), and \( F_n \) is the \( n \)th layer of \( F_\infty/\mathbb{F}(T) \)). We note that \( F_\infty k/k \) also satisfies the assumptions (A) and (B). We claim that \( \tilde{L}_S(F_n K) = F_n K \) for all \( n \geq 0 \). Indeed, if \( \tilde{L}_S(F_n K)/F_n K \) is nontrivial, then there is a nontrivial finite Galois \( p \)-subextension over \( F_n K \). Moreover, there is a nontrivial finite abelian \( p \)-subextension over \( F_n K \) because every \( p \)-group is solvable. Since \( A_n \) is trivial, it is a contradiction. We have shown the above claim. This implies that \( \tilde{L}_S(F_n k) = F_n K \) because \( F_n K/F_n k \) is unramified and all places of \( F_n k \) lying above \( S \) split completely in \( F_n K \). Hence
Gal(\(L_S(F_nk)/F_nk\)) \(\cong G\) for all \(n \geq 0\). Then the theorem follows.

References