On categoricity of atomic AEC

前園 久智 (Hisatomo MAESONO)
早稲田大学メディアネットワークセンター
(Media Network Center, Waseda University)

Abstract

In recent years, the results about atomic abstract elementary class were summarized by J.T.Baldin [1]. In that book, categoricity problem of atomic AEC is discussed mainly under the assumption of atomic $\omega$-stability (or $*-$excellence). I tried the argument around the problem under some weaker conditions.

1. Atomic AEC and splitting

We recall some definitions.

Definition 1 A class of structures $(K, \prec_K)$ (of a language $L$) is an abstract elementary class (AEC) if the class $K$ and class of pairs satisfying the binary relation $\prec_K$ are each closed under isomorphism and satisfy the following conditions ;
A1. If $M \prec_K N$, then $M \subseteq N$.
A2. $\prec_K$ is a partial order on $K$.
A3. If $\{ A_i : i < \delta \}$ is a $\prec_K$-increasing chain :
(1) $\bigcup_{i<\delta} A_i \in K$
(2) for each $j < \delta$, $A_j \prec_K \bigcup_{i<\delta} A_i$
(3) if each $A_i \prec_K M \in K$, then $\bigcup_{i<\delta} A_i \prec_K M$.
A4. If $A, B, C \in K$, $A \prec_K C$, $B \prec_K C$ and $A \subseteq B$, then $A \prec_K B$.
A5. There is a Löwenheim-Skolem number $LS(K)$ such that if $A \subseteq B \in K$, there is an $A' \in K$ with $A \subseteq A' \prec_K B$ and $|A'| \leq |A| + LS(K)$.

Definition 2 We say an AEC $(K, \prec_K)$ is atomic if $K$ is the class of atomic models of a countable complete first order theory and $\prec_K$ is first order elementary submodel.

In the following, $K$ denotes an atomic AEC.

Definition 3 Let $T$ be a countable first order theory.
A set $A$ contained in a model $M$ of $T$ is atomic if every finite sequence in
A realizes a principal type over the empty set.
Let \( A \) be an atomic set.
\( S_{\text{at}}(A) \) is the collection of \( p \in S(A) \) such that if \( a \in \mathcal{M} \) realizes \( p \), \( Aa \) is atomic (where \( \mathcal{M} \) is the big model).
We refer to a \( p \in S_{\text{at}}(A) \) as an atomic type.

We consider the notion of stability for atomic types.

**Definition 4** The atomic class \( K \) is \( \lambda \)-stable if for every \( M \in K \) of cardinality \( \lambda \), \( |S_{\text{at}}(M)| = \lambda \).

**Example 5** ([1])
1. Let \( K_1 \) be the class of atomic models of the theory of dense linear order without endpoints. Then \( K_1 \) is not \( \omega \)-stable.
2. Let \( K_2 \) be the class of atomic models of the theory of the ordered Abelian group of rationals. Then \( K_2 \) is \( \omega \)-stable.

The notion of independence by splitting is available in this context.

**Definition 6** A complete type \( p \) over \( B \) splits over \( A \subset B \) if there are \( b, c \in B \) which realize the same type over \( A \) and a formula \( \phi(x, y) \) such that \( \phi(x, b) \in p \) and \( \neg\phi(x, c) \in p \).

Let \( A, B, C \) be atomic.
We write \( A \downarrow_C B \) and say \( A \) is independent from \( B \) over \( C \) if for any finite sequence \( a \in A \), \( \text{tp}_{\text{at}}(a/B) \) does not split over some finite subset of \( C \).

**Fact 7** ([1]) Under the atomic \( \omega \)-stable assumption of \( (K, \prec_K) \) (and some assumption of parameters), the independence relation by splitting (over models) satisfies almost all forking axioms.

**Theorem 8** ([1]) If \( K \) is \( \omega \)-stable and has a model of power \( \aleph_1 \), then it has a model of power \( \aleph_2 \).

**2. Atomic AEC without infinite splitting chain**

In Baldwin's book [1] they argue the categoricity of atomic AEC under \( \omega \)-stability assumption of atomic types. I considered the same problem under some weaker conditions.

**Definition 9** Let \( K \) be an atomic AEC and \( M \in K \).
\( M \) has no infinite splitting chain if for any nonalgebraic \( p \in S_{\text{at}}(M) \), there is no increasing sequence \( \{A_i\}_{i<\omega} \subset M \) such that \( p \upharpoonright A_{i+1} \) splits over \( A_i \) for all \( i < \omega \).

We can prove the next facts.
Fact 10  If $K$ is $\omega$-stable, then no model of $K$ has infinite splitting chain.

Fact 11  Under the assumption that $(K, \prec_K)$ has no infinite splitting chain, the independence relation by splitting (over models) satisfies almost all forking axioms.

3. Existence of pregeometry

In [1], categoricity of atomic AEC are proved by means of the fact that every model is prime and minimal over a basis of some pregeometry given by a quasi-minimal set. So I tried to define pregeometry in the present context.

At first we prove the next proposition which is some modification of Theorem 8 above.

Proposition 12  If there are $N \in K$ with $|N| > \aleph_0$ and a nonalgebraic type $p(x) \in S^{1}_{at}(N)$ such that $N$ has no infinite splitting chain.

Then there are $M \in K$ with $|M| = \aleph_2$ and a nonalgebraic type $q(x) \in S^{1}_{at}(M)$ such that $M$ has no infinite splitting chain and $q$ does not split over some $b \in M$, and $q \upharpoonright b$ has a Morley sequence $I$ in $M$ with $|I| = \aleph_2$.

Moreover if $|N| = \aleph_1$, then we can take $M$ such that $N \prec M$.

In this note, Morley sequence means the sequence constructed by non-splitting extensions. Thus Morley sequences are indiscernible.

Lemma 13  Let $M \in K$ and $p(x) \in S_{at}(M)$.

Suppose that $M$ has no infinite splitting chain and $p$ does not split over some $b \in M$.

And let $I = \{a_i : i < \alpha\}$ be a Morley sequence of $p \upharpoonright b$ in $M$.

Then $I$ is totally indiscernible.

In [8], they characterized generically stable types. We try to modify the notion in this context.

Definition 14  Let $M \in K$.

A nonalgebraic type $p(x) \in S_{at}(M)$ is generically stable in $M$ if for some $A \subset M$, $p$ does not split over $A$ and if $I = \{a_i : i < \alpha\}$ is a Morley sequence of $p \upharpoonright A$ in $M$, then for any $\phi(x) \in L(M)$-formula, $\{i : M \models \phi(a_i)\}$ is either finite or co-finite.

We can prove the next lemma.

Lemma 15  Let $M \in K$ and $q(x) \in S^{1}_{at}(M)$ be in Proposition 12.

Then $q$ is generically stable in $M$.

Moreover if $q$ does not split over $b$, then $q$ is definable over $b$ and $q \upharpoonright b$ is stationary w.r.t. nonsplitting extension.
We recall the definition of pregeometry.

**Definition 16** Let $X$ be an infinite set and $\text{cl}$ a function from $\mathcal{P}(X)$ to $\mathcal{P}(X)$ where $\mathcal{P}(X)$ denotes the set of all subsets of $X$. If the function $\text{cl}$ satisfies the following properties, we say $(X, \text{cl})$ is pregeometry.

(I) $A \subset B \Rightarrow A \subset \text{cl}(A) \subset \text{cl}(B)$,
(II) $\text{cl}(\text{cl}(A)) = \text{cl}(A)$,
(III) (Finite character) $b \in \text{cl}(A) \Rightarrow b \in \text{cl}(A_0)$ for some finite $A_0 \subset A$,
(IV) (Exchange axiom) $b \in \text{cl}(A \cup \{c\}) - \text{cl}(A) \Rightarrow c \in \text{cl}(A \cup \{b\})$.

We define big type which is a modified notion in [1].

**Definition 17** Let $a \in M$ and $A \subset M \in K$.

A nonalgebraic atomic type $\text{tp}_{\text{at}}(a/A)$ is big if there is an atomic model $N \in K$ such that $A \subset N$ and $\text{tp}_{\text{at}}(a/A)$ has a nonalgebraic atomic extension over $N$.

In the following we argue under the existence of uncountable model $M \in K$ and a nonalgebraic type $p(x) \in S^1_{\text{at}}(M)$. We may assume that $p$ has what is called a minimal U-rank, or $\text{U-rank} = 1$.

**Lemma 18** Let $K$ has no infinite splitting chain and $M \in K$. And let $p(x) \in S^1_{\text{at}}(M)$ be nonalgebraic and $p$ does not split over $b$ for some $b \in M$.

Then $p \upharpoonright b$ has an extension $q(x) \in S^1_{\text{at}}(c)$ such that $b \in c \in M$ and $q$ is big, but any splitting extension of $q$ is not big.

We may assume that the type $q$ in Proposition 12 above has such property.

We define some closure operator.

**Definition 19** Let $M \in K$ and $p(x) \in S^1_{\text{at}}(M)$. And let $p$ does not split over $\emptyset$ (or some finite parameter) and $p \upharpoonright \emptyset$ is stationary.

The operator $\text{cl}_p$ is defined by ;

$\text{cl}_p^0(X) = X$ and $\text{cl}_p^{n+1}(X) = \{ a \in (p \upharpoonright \emptyset)(M) | a \notin (p \upharpoonright \text{cl}_p^n(X))(M) \}$,

and $\text{cl}_p(X) = \bigcup_{n<\omega} \text{cl}_p^n(X)$ for any $X \subset (p \upharpoonright \emptyset)(M)$.

We can prove the next fact.

**Theorem 20** Let $K$ has no infinite splitting chain and $M \in K$ ( with $|M| > \aleph_0$ ).

And let $p(x) \in S^1_{\text{at}}(M)$ be a nonalgebraic type such that $p$ does not split over $\emptyset$ and $p \upharpoonright \emptyset$ has no big splitting extension ( or $p$ has a minimal U-rank among such types ).

Then $((p \upharpoonright \emptyset)(M), \text{cl}_p)$ is pregeometry.
4. Constructible sequence of atomic types

In the argument of categoricity for *-excellent AEC, prime models play a crucial role. Now we do not assume the existence of prime models. We try the analogous argument of $F_{\kappa(T)}^a$-prime models in some large atomic model.

First we check the next lemma.

**Lemma 21** (K has no infinite splitting chain.)

Let $M \in K$. And let $A \subset B \subset M$ and $a$ be such that $tp_{at}(a/A)$ has a nonsplitting extension over $B$ (or $A \leq TV B$) and $tp_{at}(a/A)$ is stationary.

Then the following are equivalent:

(i) $tp_{at}(a/A) \vdash tp_{at}(a/B)$

(ii) For any $a'$ such that $tp_{at}(a'/A) = tp_{at}(a/A)$, $tp_{at}(a'/B)$ does not split over $A$.

I define some isolation of atomic types.

**Definition 22** Let $a \in M \in K$ and $A \subset M$.

A type $tp_{at}(a/A)$ is quasi–isolated if there is $b \in M$ such that $tp_{at}(a/b) \vdash tp_{at}(a/A)$.

A sequence $\{c_i : i < \alpha\} \subset M$ is quasi–constructible over $A$ if, for any $\beta < \alpha$, $tp_{at}(c_\beta/A \cup \{c_i : i < \beta\})$ is quasi-isolated.

$M$ is quasi–constructible over $A$ if $M \setminus A$ can be written as a quasi-constructible sequence.

We can prove the next proposition by using Lemma 21 above.

**Proposition 23** Let $K$ has no infinite splitting chain and $N \in K$ (with $|N| > \aleph_0$).

And let a nonalgebraic $p(x) \in S_{at}^1(N)$ be such that $p$ does not split over $\emptyset$ and $p$ has no big splitting extension (or $p$ has a minimal U-rank among such types).

(Suppose that $p \upharpoonright \emptyset$ has a Morley sequence $I$ with $|I| > \aleph_0$ in $N$.)

Then for any basis $J$ of $((p \upharpoonright \emptyset)(N), cl_p)$, there is a quasi-constructible model over $J$ in $N$.

5. Categoricity in some large atomic model

At first we recall the definition of Vaughtian triple from [1]. Note that the notion big is modified here.

**Definition 24** A triple $(M, N, \phi)$ is called a Vaughtian triple if $\phi(M) = \phi(N)$ where $M \prec N \in K$ with $M \neq N$ and $L(M)$–formula $\phi$ is big.
In this chapter, we assume that $K$ has no infinite splitting chain where $K$ is an atomic AEC. Under this condition we can prove some results about the two cardinal problem.

I tried the argument of categoricity in this context by means of quasi-constructible model. But I do not have the settled result yet. At present I can prove the next theorem by the properties of generically stable types.

If we try to extend the categoricity result to the whole $K$, we need some additional conditions, such as amalgamation property of models, and any atomic set is included in an atomic model, and so on.

In the next Theorem 25, $p \mid \emptyset$ has a Morley sequence $I$ in $N$ with $|I| = |N|$.

**Theorem 25** Let $K$ has no infinite splitting chain and $N \in K$ such that ($|N| > \aleph_0$ and ) there is no Vaughtian triple in $N$.

And let $p(x) \in S_{\text{at}}^1(N)$ be nonalgebraic such that $p$ does not split over $\emptyset$ and $p \mid \emptyset$ has no big splitting extension ( or $p$ has a minimal $U$-rank among such types ).

Then for $M_i < N (i < 2)$ with $|M_0| = |M_1|$, $M_0 \cong M_1$.

6. Example of Shelah et al.

Shelah's original work ([4],[5]) showed that categoricity up to $\aleph_\omega$ of a sentence in $L_{\omega_1,\omega}$ implies categoricity in all uncountable cardinalities. Shelah and Hart showed the necessity of the assumption by constructing some example ([6]). This example is adapted by Baldwin and Kolesnikov ([1],[2]).

We can not recall the definition of it and details here.

**Theorem 26** ([1],[2]) For each $k < \omega$, there is a $L_{\omega_1,\omega}$-sentence $\phi_{k+2}$ such that:

- $\phi_{k+2}$ is categorical in $\mu$ if $\mu \leq \aleph_k$, and
- $\phi_{k+2}$ is not categorical in any $\mu$ with $\mu > \aleph_k$.

And they proved the next proposition in [2].

**Proposition 27** ([2]) Let $M$ be the standard model of $\phi_{k+2}$ of size $\aleph_k$.

Then there are $2^{\aleph_k}$ Galois types over $M$.

This structure is expanded to be an atomic model. And we can check the next fact.

**Fact 28** Let $M$ and $\phi_{k+2}$ be the $L_{\omega_1,\omega}$-sentence in the Proposition 27 above. Then $M$ has an infinite splitting chain ( in the expanded language ).
References

[6] B. Hart and S. Shelah, *Categoricity over P for first order T or categoricity for $\phi \in L_{\omega_1,\omega}$ can stop at $\aleph_k$ while holding for $\aleph_0, \cdots, \aleph_{k-1}$*, Israel J. of math, vol. 70, pp. 219-235, 1990