NOTES ON MIYAMOTO’S FORCING AXIOMS

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ABSTRACT. In this short paper, we prove that Miyamoto’s forcing axiom \( FA'_{\sigma}\) (\( \sigma \)-closed) is equivalent to \( MA^{+}_{\sigma} \). We also study some variants of \( FA'_{\sigma} \).

1. INTRODUCTION

Throughout this note, \( \theta \) will denote a sufficiently large regular cardinal. For submodels \( M, N \prec H_{\theta}, M \prec_{\omega_{1}} N \) means that \( M \subseteq N \) and \( M \cap \omega_{1} = N \cap \omega_{1} \).

Miyamoto ([1]) introduced the following forcing axiom \( FA'_{\sigma} \):

**Definition 1.1.** \( FA'_{\sigma} \) is the assertion that for every \( \sigma \)-closed poset \( \mathbb{P} \), every countable \( M \prec H_{\theta} \) with \( \mathbb{P} \in M \), every \( (M, \mathbb{P}) \)-generic condition \( p \), and every dense subsets \( \langle D_{i} : i < \omega_{1} \rangle \) in \( \mathbb{P} \), there exists a directed set \( F \subseteq \mathbb{P} \) such that:

1. \( |F| \leq \omega_{1} \),
2. \( F \cap D_{i} \neq \emptyset \) for every \( i < \omega_{1} \),
3. \( q \in F \) for some \( q \leq p \),
4. \( M \prec_{\omega_{1}} M(F) \).

Where \( M(F) = \{ h(F) : h : \mathcal{P}(\mathbb{P}) \to H_{\theta}, h \in M \} \prec H_{\theta} \). \( M(F) \) is the minimal elementary submodel of \( H_{\theta} \) containing \( M \cup \{F\} \).

He showed that \( FA'_{\sigma} \) implies \( MA^{+}_{\sigma} \), where:

**Definition 1.2.** \( MA^{+}_{\sigma} \) is the assertion that for every \( \sigma \)-closed poset \( \mathbb{P} \), every \( \mathbb{P} \)-name \( \dot{S} \) of a stationary subset of \( \omega_{1} \), and every dense subsets \( \langle D_{i} : i < \omega_{1} \rangle \) in \( \mathbb{P} \), there exists a filter \( F \subseteq \mathbb{P} \) such that \( F \cap D_{i} \neq \emptyset \) for \( i < \omega_{1} \) and the set \( S = \{ \alpha < \omega_{1} : \exists p \in F (p \Vdash \alpha \in \dot{S}) \} \) is stationary in \( \omega_{1} \).

We show that the converse direction is also true, hence we have:

**Theorem 1.3.** \( FA'_{\sigma} \iff MA^{+}_{\sigma} \).

On the other hand, we shall consider some variants of \( FA'_{\sigma} \), which are also suggested by Miyamoto.

**Definition 1.4.** \( FA'_{\sigma} \) is the assertion that for every \( \sigma \)-closed poset \( \mathbb{P} \), every countable \( M \prec H_{\theta} \) with \( \mathbb{P} \in M \), and every dense subsets \( \langle D_{i} : i < \omega_{1} \rangle \) in \( \mathbb{P} \), there exists a directed set \( F \subseteq \mathbb{P} \) such that
(1) \( |F| \leq \omega_1 \),
(2) \( F \cap D_i \neq \emptyset \) for every \( i < \omega_1 \),
(3) \( F \) contains an \((M, \mathbb{P})\)-generic condition,
(4) \( M \prec_{\omega_1} M(F) \).

**Definition 1.5.** FA\(^0\)(\(\sigma\)-closed) is the assertion that for every \(\sigma\)-closed poset \(\mathbb{P}\), every countable \(M \prec H_\theta\) with \(\mathbb{P} \in M\), and every dense subsets \(\langle D_i : i < \omega_1 \rangle\) in \(\mathbb{P}\), there exists a directed set \(F \subseteq \mathbb{P}\) such that:

(1) \( |F| \leq \omega_1 \),
(2) \( F \cap D_i \neq \emptyset \) for every \( i < \omega_1 \),
(3) \( M \prec_{\omega_1} M(F) \).

Obviously FA\(^*\)(\(\sigma\)-closed) \(\Rightarrow\) FA\('\)(\(\sigma\)-closed) \(\Rightarrow\) FA\(^0\)(\(\sigma\)-closed).

We prove that FA\('\)(\(\sigma\)-closed) and FA\(^0\)(\(\sigma\)-closed) can be characterized by known reflection principles.

**Definition 1.6** (Shelah [2]). For \(\kappa > \omega_1\) and \(S \subseteq [\kappa]^\omega\), let \(\hat{S} = \{x \in [\kappa]^\omega : \exists a \in S (a \subseteq x \text{ and } a \cap \omega_1 = x \cap \omega_1)\}\). SSR (Semi-Stationary Reflection principle) is the assertion that for every cardinal \(\kappa > \omega_1\) and every stationary \(S \subseteq [\kappa]^\omega\), there is \(X \subseteq \kappa\) such that \(|X| = \omega_1 \subseteq X\) and \(\hat{S} \cap [X]^\omega\) is stationary in \([X]^\omega\).

**Definition 1.7.** IRP (Internally approachable Reflection Principle) is the assertion that for every \(\theta\) and every stationary \(S \subseteq [H_\theta]^{\omega}\), there is an internally approachable continuous sequence \(\langle M_i : i < \omega_1 \rangle\) of countable submodels of \(H_\theta\) such that \(\{i < \omega_1 : M_i \in S\}\) is stationary in \(\omega_1\).

We will show the following equivalences:

**Theorem 1.8.**
(1) \( FA'(\sigma\text{-closed}) \iff \text{IRP} \).
(2) \( FA^0(\sigma\text{-closed}) \iff \text{SSR} \).

2. Proofs

**Proof of Theorem 1.3.** The direction FA\(^*\)(\(\sigma\)-closed) \(\Rightarrow\) MA\(^+\)(\(\sigma\)-closed) is known, so we show only the converse direction. Suppose MA\(^+\)(\(\sigma\)-closed). Let \(\mathbb{P}\) be a \(\sigma\)-closed poset. Let \(S\) be the set of all countable \(M \prec H_\theta\) such that \(\mathbb{P} \in M\) but there are an \((M, \mathbb{P})\)-generic condition \(p_M\) and dense subsets \(\langle D_i^M : i < \omega_1 \rangle\) in \(\mathbb{P}\) such that there is no directed set \(F \subseteq \mathbb{P}\) satisfying (1)–(4) in the definition of FA\(^*\)(\(\sigma\)-closed). We see that \(S\) is non-stationary in \([H_\theta]^\omega\), this suffices to prove the theorem.

Suppose otherwise. Let \(\dot{T}\) be a \(\mathbb{P}\)-name such that \(\Vdash \dot{T} = \{M \in S : p_M \in \dot{G}\}\).

**Claim 2.1.** There is \(p \in \mathbb{P}\) such that \(p \Vdash \text{"\(\dot{T}\) is stationary in \([H_\theta]^\omega\)."} \)
Proof. Suppose otherwise. Then there is a $\mathbb{P}$-name $\dot{f}$ such that $\forces " \dot{f} : [H^\omega_\theta]^{<\omega} \rightarrow H^\omega_\theta"$ such that there is no $x \in \check{T}$ closed under $\dot{f}$. Take a sufficiently large another regular cardinal $\chi > \theta$. Take a countable $N < H_\chi$ such that $N \cap H_\theta \in S$ and $\dot{f} \in N$. Then $p_M$ is also an $(N, \mathbb{P})$-generic condition. Thus $p_M \forces " N \cap H_\theta \in \check{T} \text{ but is closed under } \dot{f}"$. This is a contradiction. \hfill $\square$

Pick $p \in \mathbb{P}$ such that $p \forces " \check{T} \text{ is stationary in } [H^\omega_\theta]^\omega."$. By replacing $\mathbb{P}$ by the suborder $\{q \in \mathbb{P} : q \leq p\}$ and $p_M$ by $p_M \wedge p$ for $M \in S$, we may assume that $\forces " \check{T} \text{ is stationary in } [H^\omega_\theta]^\omega."$. Let $\dot{Q}$ be a $\mathbb{P}$-name of a $\sigma$-closed poset which adds a bijection from $\omega_1$ to $H^\omega_\theta$. Take a $(V, \mathbb{P} \ast \dot{Q})$-generic $G \ast H$ and work in $V[G \ast H]$. Let $T$ be the interpretation of $\check{T}$ by $G \ast H$. Since $\mathbb{Q}$ is $\sigma$-closed, $T$ remains stationary in $[H^\omega_\theta]^\omega$. We know $|H^\omega_\theta| = \omega_1$, thus we can find a club $\langle \dot{M}_i : i < \omega_1 \rangle$ in $[H^\omega_\theta]^\omega$ such that $E = \{ i < \omega_1 : i = M_i \cap \omega_1, M_i \in T \}$ is stationary in $\omega_1$. Let $\dot{E}$ be a name of $E$, and $\dot{M}_i$ of $M_i$.

Return to $V$. Notice that for $(p, q) \in \mathbb{P} \ast \dot{Q}$, $i < \omega_1$, and $M \in S$, if $(p, q) \forces " i \in \dot{E} \text{ and } M = \dot{M}_i \"$, then $p$ is compatible with $p_M$. For $i < \omega_1$, let $D_i = \{ (p, q) \in \mathbb{P} \ast \dot{Q} : (p, q) \text{ decides } "i \in \dot{E}" \text{ and } (p, q) \forces " i \in \dot{E} \" \text{ and } \exists M \in S (\langle p, q \rangle \forces " \dot{M}_i = M \" \text{ and } p \leq p_M) \}$. $D_i$ is dense in $\mathbb{P} \ast \dot{Q}$. For $i, j < \omega_1$, let $D_{ij} = \{ (p, q) \in D_i : \exists M \in S (\langle p, q \rangle \forces " i \in \dot{E} \wedge \dot{M}_i = M \")\text{, then } p \in D_j^M \}$. Each $D_{ij}$ is also dense in $\mathbb{P} \ast \dot{Q}$. By MA$^{+}$($\sigma$-closed), we can find a directed set $F' \subseteq \mathbb{P} \ast \dot{Q}$ such that $|F'| = \omega_1$, $D_i \cap F' \neq \emptyset$, $D_{ij} \cap F' \neq \emptyset$ for each $i, j < \omega_1$, and $E^* = \{ i < \omega_1 : \exists (p, q) \in F' \langle (p, q) \forces " i \in \dot{E} \" \} \text{ is stationary in } \omega_1$. Let $F$ be the projection of $F'$ into $\mathbb{P}$. $F$ is a directed set in $\mathbb{P}$ with $|F| = \omega_1$. For $i \in E^*$, there is $M_i \in S$ and $(p, q) \in F^*$ such that $(p, q) \forces " i \in \dot{E} \text{ and } \dot{M}_i = M_i \"$ and $p \leq p_{M_i}$. Thus we may assume that $p_{M_i} \in F$ for $i \in E^*$. Moreover, for $i \in \text{lim}(E^*) \cap E^*$, it is easy to check that $M_i = \bigcup_{j \in E^* \cap i} M_j$, and for each $i \in E^*$ and $j < \omega_1$, we have $D_{ji}^M \cap F \neq \emptyset$. Now take a countable $N < H_\theta$ such that $N \cap \omega_1 \in E^*$ and $\langle M_i : i \in \text{lim}(E^*), F \rangle \in N$. Let $i^* = N \cap \omega_1$. Then $M_{i^*} < \omega_1 N$, thus we have $M_{i^*} < \omega_1 M_{i^*}(F)$. This is a contradiction. \hfill $\square$

Proof of (1) of Theorem 1.8. First we prove $\text{FA}'(\sigma$-closed$) \Rightarrow IRP$. Fix a stationary $S \subseteq [H_\theta]^\omega$. Let $\mathbb{P}$ be a $\sigma$-closed poset adding a bijection from $\omega_1$ to $H_\theta$. Fix $\mathbb{P}$-names $\langle \dot{M}_i : i < \omega_1 \rangle$ such that $\forces " \langle \dot{M}_i : i < \omega_1 \rangle \text{ is an internally approachable club in } [H^\omega_\theta]^\omega."$

Fix another large regular cardinal $\chi > \theta$ and $N < H_\chi$ with $\theta, S, \mathbb{P}, \langle \dot{N}_i : i < \omega_1 \rangle \in N$ and $N \cap H_\theta \in S$. Let $M = N \cap H_\theta$ and $i^* = M \cap \omega_1$. Note that if $p \in \mathbb{P}$ is an $(N, \mathbb{P})$-generic condition, then $p \forces " M = \dot{M}_{i^*} \"$. For $i < \omega_1$, let $D_i = \{ p \in \mathbb{P} : \exists M'(p) \forces " M' = \dot{M}_i \" \}$. $D_i$ is dense in $\mathbb{P}$. Applying $\text{FA}'(\sigma$-closed$)$, we can find a directed set $F \subseteq \mathbb{P}$ such that $F \cap D_i \neq \emptyset$, $F$ contains an $(N, \mathbb{P})$-generic condition, and $N < \omega_1 N(F)$. For each $i < \omega_1$ define $M_i$ as $\exists p \in F (p \forces " M_i = M_i \")$. Then
\begin{align*}
\langle M_i : i < \omega_1 \rangle & \text{ is an internally approachable continuous sequence and } \langle M_i : i < \omega_1 \rangle \in N(F). \text{ Moreover, since } F \text{ contains an } (N, \mathbb{P})\text{-generic condition, we have } M \ast = M. \text{ Finally we see that } \{i < \omega_1 : M_i \in S\} \text{ is stationary, but this follows from that } M \ast \cap \omega_1 = M \cap \omega_1 = N \cap \omega_1 \text{ and } N \sim_{\omega_1} N(F). \\
\text{Next we prove the converse direction IRP } \Rightarrow FA'(\sigma\text{-closed}). \text{ Fix a } \sigma\text{-closed poset } \mathbb{P}. \text{ Fix a well-ordering } \Delta \text{ on } H_\theta. \text{ Let } S \text{ be the set of all countable } M < \langle H_\theta, \in, \Delta \rangle \text{ such that: there are dense open subsets } \langle D^M_i : i < \omega_1 \rangle \text{ of } \mathbb{P} \text{ such that there is no directed set } F \text{ satisfying (1)–(4) in the definition of } FA'(\sigma\text{-closed}). \text{ We claim that } S \text{ is non-stationary, which completes our proof. Suppose to contrary that } S \text{ is stationary. By IRP, we can find an internally approachable continuous sequence } \langle N_i : i < \omega_1 \rangle \text{ with } E = \{i < \omega_1 : N_i \in S\} \text{ stationary. Since it is internally approachable, we may assume that for each } i < j < \omega_1, \langle D^{N_i}_k : k < \omega_1 \rangle \in N_j, \text{ where we are letting } D^{N_i}_k = \mathbb{P} \text{ for } i \notin E. \text{ By induction on } i < \omega_1, \text{ we would define a descending sequence } \langle p_i : i < \omega_1 \rangle \text{ such that } p_i \text{ is } (N_i, \mathbb{P})\text{-generic and } p_{i+1} \in D^{N_i}_k \text{ for } k, j \leq i. \\
\text{First, take the } \Delta\text{-least } (N_0, \mathbb{P})\text{-generic condition } p_0. \text{ Since } N_0 \in N_1, \text{ we have } p_0 \in N_1. \text{ Moreover, since } \langle D^{N_0}_k : k < \omega_1 \rangle \in N_1, \text{ there is the } \Delta\text{-least condition } p_1 \leq p_0 \text{ with } p_1 \in D^{N_0}_0. \text{ Repeating this procedure } \omega_1\text{-times; When } i < \omega_1 \text{ is limit and } \langle p_j : j < i \rangle \text{ was defined, then } p_i \text{ is the } \Delta\text{-least lower bound of the } p_j\text{'s. Then we know that } p_i \text{ is } (N_i, \mathbb{P})\text{-generic and } p_i \in N_{i+1}. \text{ Thus we can take } p_{i+1} \in N_{i+1} \text{ as intended.} \\
\text{Now we have a descending sequence } \langle p_i : i < \omega_1 \rangle \text{ such that } p_i \text{ is } (N_i, \mathbb{P})\text{-generic and for every } i, k < \omega_1, \text{ there is some } j \text{ with } p_j \in D^j_k. \text{ Let } F = \{p_i : i < \omega_1\}. \text{ } F \text{ is a directed set with size } \leq \omega_1. \text{ Take a countable } M < \langle H_\theta, \in, \Delta \rangle \text{ such that } F, \langle N_i : i < \omega_1 \rangle \in M \text{ and } M \cap \omega_1 \in E. \text{ Let } i^* = M \cap \omega_1. \text{ Then } N_i \ast \sim_{\omega_1} M, \text{ in particular we have } N_i \ast \sim_{\omega_1} N_i \ast(F). \text{ However } F \text{ contradicts } N_i \ast \in S. \quad \square
\end{align*}
$M \cap \kappa \subseteq X$. Then $M(F) \cap X \in \hat{S} \cap [X]^{\omega}$. By the elementarity of $M(F)$, $\hat{S} \cap [X]^{\omega}$ is stationary in $[X]^{\omega}$. This completes the proof of one direction.

For the converse, let $\mathbb{P}$ be a $\sigma$-closed poset. Let $S$ be the set of all countable $M \prec H_\theta$ such that $\mathbb{P} \in M$ but there are dense subsets $\langle D_i : i < \omega_1 \rangle$ in $\mathbb{P}$ such that there is no directed set $F$ satisfying (1)–(3) in the definition of $\text{FA}_0^{\omega}(\sigma$-closed). We see that $S$ is non-stationary in $[H_\theta]^{\omega}$, which completes the proof.

Suppose to contrary that $S$ is stationary. By SSR, we can find $X \subseteq H_\theta$ such that $|X| = \omega_1 \subseteq X$ and $\hat{S} \cap [X]^{\omega}$ is stationary in $[X]^{\omega}$. Let $N$ be the Skolem hull of $X$ under $\langle H_\theta, \in, \Delta \rangle$. Then we can take a club $\langle M_i : i < \omega_1 \rangle$ in $[N]^{\omega}$ such that $M_i \prec H_\theta$ for $i < \omega_1$. Since $\hat{S} \cap [X]^{\omega}$ is stationary, we have $E = \{i < \omega_1 : M_i \cap X \in \hat{S} \}$ is stationary in $\omega_1$. Notice that for each $i \in E$, $M_i$ belongs to $S$; since $M_i \cap X \in \hat{S}$, there is $M' \prec H_\theta$ with $M' \prec_{\omega_1} M_i$. If $\langle D_j : j < \omega_1 \rangle$ are dense subsets in $\mathbb{P}$ witnessing $M' \in S$, then it also witnesses that $M_i \in S$. For each $i \in E$, fix dense subsets $\langle D_{j}^i : j < \omega_1 \rangle$ witnessing $M_i \in S$. Now take a countable $N \prec H_\theta$ such that $\langle M_i : i < \omega_1 \rangle, E, \langle D_{j}^i : i \in E, j < \omega_1 \rangle \in N$ and $N \cap \omega_1 \in E$. Let $\tilde{i} = N \cap \omega_1$. Then $M_{\tilde{i}} \prec_{\omega_1} N$. Since $\mathbb{P}$ is $\sigma$-closed, we can find a directed set $F \in N$ such that $|F| = \omega_1$ and $F \cap D_{j}^i \neq \emptyset$ for $i \in E$ and $j < \omega_1$. Then, since $M_{\tilde{i}} \prec_{\omega_1} N$ and $F \in N$, we have that $M_{\tilde{i}} \prec_{\omega_1} M_{\tilde{i}}(F)$. This is a contradiction.

REFERENCES


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