SOME RESULTS IN THE EXTENSION WITH A COHERENT SUSLIN TREE

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ABSTRACT. We show that under PFA$(S)$, the coherent Suslin tree $S$ (which is a witness of the axiom PFA$(S)$) forces that there are no $\omega_2$-Aronszajn trees. We also determine the values of cardinal invariants of the continuum in this extension.

1. INTRODUCTION

In [20], Stevo Todorčević introduced the forcing axiom PFA$(S)$, which says that there exists a coherent Suslin tree $S$ such that the forcing axiom holds for every proper forcing which preserves $S$ to be Suslin, that is, for every proper forcing $\mathbb{P}$ which preserves $S$ to be Suslin and $\aleph_1$-many dense subsets $D_\alpha$, $\alpha \in \omega_1$, of $\mathbb{P}$, there exists a filter on $\mathbb{P}$ which intersects all the $D_\alpha$. PFA$(S)[S]$ denotes the forcing extension with the coherent Suslin tree $S$ which is a witness of PFA$(S)$. Since the preservation of a Suslin tree by the proper forcing is closed under countable support iteration (due to Tadatoshi Miyamoto [15]), it is consistent relative to some large cardinal assumption that PFA$(S)$ holds.

The first appearance of such a forcing axiom is in the paper [13] due to Paul B. Larson and Todorčević. In this paper, they introduced the weak version of PFA$(S)$, called Souslin’s Axiom (in which the properness is replaced by the cccness), and under this axiom, the coherent Suslin tree $S$, which is a witness of the axiom, forces a weak fragment of Martin’s Axiom. In [20], it is also proved that under PFA$(S)$, $S$ forces the open graph dichotomy (1) and the $P$-ideal dichotomy. Namely, many consequences of PFA are satisfied in the extension with $S$ under

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PFA($S$). On the other hand, many people proved that some consequences from $\diamond$ are satisfied in the extension with a Suslin tree (e.g. [16, Theorem 6.15.]). In particular, the pseudo-intersection number $p$ is $\aleph_1$ in the extension with a Suslin tree. In fact, the extension with $S$ under PFA($S$) is designed as a universe which satisfied some consequences of $\diamond$ and PFA simultaneously. By the use of this model, Larson and Todorčević proved that the affirmative answer to Katětov’s problem is consistent [13].

In this note, we point out the values of cardinal invariants of the continuum (e.g. in [2, 6]) in the extension with $S$ under PFA($S$). And we show that under PFA($S$), $S$ forces that there are no $\omega_2$-Aronszajn trees. In [19], Todorčević demonstrated that many consequences of PFA are deduced from PID plus $p > \aleph_1$. In [17], the first author proved that PID plus $p > \aleph_1$ implies the failure of $\square_{\kappa,\omega_1}$ whenever $\text{cf}(\kappa) > \omega_1$. It is not yet known whether PID plus $p > \aleph_1$ implies the failure of $\square_{\omega_1,\omega_1}$. Since $\square_{\omega_1,\omega_1}$ is equivalent to the existence of a special $\omega_2$-Aronszajn tree, our result concludes that it is consistent that PID holds, $p = \aleph_1$ and $\square_{\omega_1,\omega_1}$ fails.

At last in the introduction, we introduce a coherent Suslin tree. A coherent Suslin tree $S$ consists of functions in $\omega^{<\omega_1}$ and is closed under finite modifications. That is,

- for any $s$ and $t$ in $S$, $s \leq_S t$ iff $s \subseteq t$,
- $S$ is closed under taking initial segments,
- for any $s$ and $t$ in $S$, the set

$$\{ \alpha \in \min\{ lv(s), lv(t) \}; s(\alpha) \neq t(\alpha) \}$$

is finite (here, $lv(s)$ is the length of $s$, that is, the size of $s$), and
- for any $s \in S$ and $t \in \omega^{lv(s)}$, if the set $\{ \alpha \in lv(s); s(\alpha) \neq t(\alpha) \}$ is finite, then $t \in S$ also.

For a countable ordinal $\alpha$, let $S_\alpha$ be the set of the $\alpha$-th level nodes, that is, the set of all members of $S$ of domain $\alpha$, and let $S_{\leq \alpha} := \bigcup_{\beta \leq \alpha} S_\beta$. For $s \in S$, we let

$$S|s := \{ u \in S; s \leq_S u \}.$$

We note that $\diamond$, or adding a Cohen real, builds a coherent Suslin tree. A coherent Suslin tree has canonical commutative isomorphisms. Let $s$ and $t$ be nodes in $S$ with the same level. Then we define a function $\psi_{s,t}$ from $S|s$ into $S|t$ such that for each $v \in S|s$,

$$\psi_{s,t}(v) := t \cup (v \upharpoonright [lv(s), lv(v)])$$

(here, $v \upharpoonright [lv(s), lv(v)]$ is the function $v$ restricted to the domain $[lv(s), lv(v)]$).

We note that $\psi_{s,t}$ is an isomorphism, and if $s$, $t$, $u$ are nodes in $S$ of
the same level, then $\psi_{s,t}$, $\psi_{t,u}$ and $\psi_{s,u}$ commute. (On a coherent Suslin tree, see e.g. [10, 12].)

2. Cardinal invariants

**Proposition 2.1** ([20, 4.3 Theorem]). $PFA(S)$ implies that $p = \text{add}(\mathcal{N}) = c = \aleph_2$ holds.

*Proof.* A forcing with property $K$ doesn't destroy a Suslin tree ([14, Theorem 11.]). So, since a $\sigma$-centered forcing satisfies property $K$ and $p = \text{m}(\sigma$-centered) (due to Bell, see e.g. in [6, 7.12 Theorem]), $PFA(S)$ implies $p > \aleph_1$.

To see that $PFA(S)$ implies $\text{add}(\mathcal{N}) > \aleph_1$, here we consider the characterization of the additivity of the null ideal by the amoeba forcing $A$ as follows (see [2, 6.1 Theorem] or [3, Theorem 3.4.17]).

$$\text{add}(\mathcal{N}) = \min \left\{|\mathcal{D} : \mathcal{D} \text{ is a set of dense subsets of } A \text{ such that there are no filters of } A \text{ which meet every member of } \mathcal{D}\right\}.$$  

Since the amoeba forcing is $\sigma$-linked (so satisfies property $K$), $PFA(S)$ implies $\text{add}(\mathcal{N}) > \aleph_1$.

A proof that $PFA(S)$ implies $c = \aleph_2$ is same to one for $PFA$ due to Todorcević [5, 3.16 Theorem] (see also [9, Theorem 31.25]). We note that $PFA(S)$ implies OCA ([8, Lemma 4]), so $b = \aleph_2$ holds ([18, 8.6 Theorem], also [9, Theorem 29.8]). In a proof that $b = c$ holds under PFA, an iteration of a $\sigma$-closed forcing and a ccc forcing which is defined by an unbounded family in $\omega^\omega$ is used. A $\sigma$-closed forcing doesn't destroy a Suslin tree (see e.g. [15]). Since the cccness of the second iterand comes from the unboundedness of a family in $\omega^\omega$, this preserves a Suslin tree because a Suslin tree doesn't add new reals. So this iteration doesn't destroy a Suslin tree. Therefore $b = c$ holds under $PFA(S)$. \hfill $\square$

**Proposition 2.2** ([8, Lemma 2.]). $t = \aleph_1$ holds in the extension with a Suslin tree.

*Proof.* Suppose that $T$ is a Suslin tree, and let $\pi$ be an order preserving function from $T$ into the order structure $([\omega]^{\aleph_0}, \supseteq^*)$ such that if members $s$ and $t$ of $T$ are incomparable in $T$, then $\pi(s) \cap \pi(t)$ is finite. Then for a generic branch $G$ through $T$, the set $\{\pi(s) : s \in G\}$ is a $\subseteq^*$-decreasing sequence which doesn't have its lower bound in $[\omega]^{\aleph_0}$ (because $T$ doesn't add new reals). \hfill $\square$

**Proposition 2.3.** Under $PFA(S)$, $S$ forces that $\text{add}(\mathcal{N}) = c = \aleph_2$. 

Proof. Since $S$ doesn't add new reals and preserves all cardinals, by Proposition 2.1, $S$ forces that $c = \aleph_2$ ([20, 4.4 Corollary]).

To see that $S$ forces $\text{add}(\mathcal{N}) > \aleph_1$, here we consider another characterization of the additivity of the null ideal (see [1], also [2, 3]). A function in the set $\prod_{n\in\omega} ([\omega]^{\leq n+1} \setminus \{\emptyset\})$ is called a slalom, and for a function $f$ in $\omega^\omega$ and a slalom $\varphi$, we say that $\varphi$ captures $f$ (denoted by $f \subseteq \varphi$) if for all but finitely many $n \in \omega$, $f(n) \in \varphi(n)$. Then

$$\text{add}(\mathcal{N}) = \min \left\{ |F| : F \subseteq \omega^\omega \right\} \quad \text{and} \quad \forall \varphi \in \prod_{n\in\omega} ([\omega]^{\leq n+1} \setminus \{\emptyset\}) \exists F \in F (f \not\subseteq \varphi) \}.$$

Let $\dot{X}$ be an $S$-name for a set of $\aleph_1$-many functions in $\omega^\omega$. For each $s \in S$, let

$$Y_s := \left\{ f \in \omega^\omega : s \vdash_{S} f \in \dot{X} \right\}.$$

Since $\dot{X}$ is an $S$-name for a set of size $\aleph_1$, $Y_s$ is of size at most $\aleph_1$ for each $s \in S$, so is the set $\bigcup_{s \in S} Y_s$. And we note that

$$\vdash_{S} \" \dot{X} \subseteq \bigcup_{s \in S} Y_s \".$$

By $\text{add}(\mathcal{N}) > \aleph_1$ (Proposition 2.1), there exists a slalom $\varphi$ which captures all functions in the set $\bigcup_{s \in S} Y_s$. Then

$$\vdash_{S} \" \varphi \text{ captures all functions in } \dot{X} \" ,$$

which finishes the proof. \(\square\)

**Proposition 2.4.** Under PFA($S$), $S$ forces that $\mathfrak{h} = \aleph_2$.

**Proof.** By Proposition 2.1, $\mathfrak{h} = \aleph_2$ holds in the ground model because of the inequality $\mathfrak{p} \leq t \leq \mathfrak{h} \leq c$ (see e.g. [6, §6]).

Let $\dot{X}_\alpha$, for each $\alpha \in \omega_1$, be an $S$-name for a dense open subset of $[\omega]^{\aleph_0}$. For $\alpha \in \omega$ and $s \in S$, let

$$Y_{\alpha,s} := \left\{ x \in [\omega]^{\aleph_0} : \exists t \in S (s \leq S t & t \vdash_{S} \" x \in \dot{X}_\alpha \") \right\}.$$

Then we note that each $Y_{\alpha,s}$ is a dense open subset of $[\omega]^{\aleph_0}$, and

$$\vdash_{S} \" \bigcap_{s \in S} Y_{\alpha,s} \subseteq \dot{X}_\alpha \".$$
SOME RESULTS IN THE EXTENSION WITH A COHERENT SUSLIN TREE

Since $\mathfrak{b} > \aleph_1$, for each $\alpha \in \omega_1$, the set $\bigcap_{\alpha \in \omega_1} \bigcap_{s \in S} Y_{\alpha,s}$ is a dense open subset of $[\omega]^{\aleph_0}$, in particular, it is nonempty. Therefore

$$\models_S \left( \bigcap_{\alpha \in \omega_1} \dot{X}_{\alpha} \neq \emptyset \right),$$

which finishes the proof. \qed

We note that $\mathfrak{b}$ is less than or equal to many standard cardinal invariants, like $\mathfrak{a}$, $\mathfrak{s}$, etc. See e.g. [3, 6, 7].

3. $\omega_2$-ARONSZAJN TREES

**Theorem 3.1.** Under PFA($S$), $S$ forces that there are no $\omega_2$-Aronszajn trees.

**Proof.** An outline of the proof is same to the proof due to Baumgartner in [4] (see also [9, Theorem 31.32.]). So this theorem follows from the following two claims.

**Claim 3.2.** Let $\mathbb{P}$ be a $\sigma$-closed forcing notion, and let $\dot{T}$ be an $S$-name for an $\omega_2$-Aronszajn tree. Then $\mathbb{P}$ adds no $S$-names for cofinal chains through $\dot{T}$ whenever $c > \aleph_1$ holds.

**Proof of Claim 3.2.** At first, we see an easy proof by the result of product forcing ([9, Lemma 15.9] or [11, Ch.VIII, 1.4.Theorem]). We note that the two step iteration $\mathbb{P} \ast S$ is equal to the two step iteration $S \ast \mathbb{P}^V$ (2). In the extension with $S$, since $c > \aleph_1$, a $\sigma$-closed forcing $\mathbb{P}^V$ doesn't add a cofinal branch through the value of $\dot{T}$ by the generic of $S$, which is an $\omega_2$-Aronszajn tree (this can be proved as in [9, Lemma 27.10]). Therefore $\mathbb{P}$ doesn't add an $S$-name for a cofinal chain through $\dot{T}$.

At last, we see a direct proof. In fact, we show that if $\mathbb{P}$ is $\sigma$-closed and $\dot{T}$ is an $S$-name for an $\omega_2$-tree, then $\mathbb{P}$ adds no new $S$-names for cofinal chains through $\dot{T}$ whenever $c > \aleph_1$ holds.

Suppose that $\mathbb{P}$ adds a new $S$-name for a cofinal chain through $\dot{T}$, that is, there exists a sequence $\langle \dot{z}_\alpha ; \alpha \in \omega_2 \rangle$ of $\mathbb{P}$-names for $S$-names for members of $\dot{T}$ such that

$$\models \mathbb{P} \left( \models_S \left( \forall \alpha \beta \in \omega_2, \dot{z}_\alpha <^*_T \dot{z}_\beta \right) \right),$$

and for every $S$-name $\dot{B}$ for a subset of $\dot{T}$ (in the ground model),

$$\models \mathbb{P} \left( \models_S \left( \dot{B} \neq \{ \dot{z}_\alpha ; \alpha \in \omega_2 \} \right) \right).$$

2In fact, in the first argument, we use a $\sigma$-forcing $\text{Fn}(\omega_1, \omega_2, \aleph_1)$, which collapses $\omega_2$ to $\omega_1$ by countable approximations. $S$ doesn't add new countable sets, so $\text{Fn}(\omega_1, \omega_2, \aleph_1)$ doesn't change in the extension with $S$. 

76
We note that we look at $\dot{T}$ as an object in the ground model even in the extension with $\mathbb{P}$. So for any $\mathbb{P}$-name $\dot{t}$ for an $S$-name for a member of $\dot{T}$ and $p \in \mathbb{P}$, densely many extensions of $p$ in $\mathbb{P}$ decides the value of $\dot{t}$ as an $S$-name for a member of $\dot{T}$. By induction on $\sigma \in 2^{<\omega}$, we choose a condition $p_\sigma$ in $\mathbb{P}$, an $S$-name $\dot{x}_\sigma$ for a member of $\dot{T}$ and countable $\alpha_{|\sigma|}$ and $\beta_{|\sigma|}$ such that

- for $\sigma$ and $\tau$ in $2^{<\omega}$ with $\sigma \subseteq \tau$, $p_\tau \leq_{\mathbb{P}} p_\sigma$,
- $\models_{\mathbb{P}} \models_S " \dot{x}_\sigma \in \{ \dot{z}_\alpha; \alpha \in \omega_2 \} "$ for each $\sigma \in 2^{<\omega}$,
- $\models_S " \text{both } \dot{x}_\sigma \in \dot{T} \text{ and } \dot{x}_\sigma \in \dot{T} \text{ are above } \dot{x}_\sigma \text{ in } \dot{T} \text{ for each } \sigma \in 2^{<\omega},$
- for each $n \in \omega$ and $\sigma \in 2^{<\omega}$, every $\alpha_n$-th level node of $S$ decides the value of $\dot{x}_\sigma$ which is of level less than $\beta_n$ in $\dot{T}$.

This can be done because of the property of the sequence $\langle \dot{z}_\alpha; \alpha \in \omega_2 \rangle$ and the cccness of $S$ as a forcing notion.

Since $\mathbb{P}$ is $\sigma$-closed, for any $f \in 2^{\omega}$, there is $p_f \in \mathbb{P}$ such that $p_f \leq_{\mathbb{P}} p_{f|n}$ holds for every $n \in \omega$. Since it is forced with $\mathbb{P}$ that $\langle \dot{z}_\alpha; \alpha \in \omega_2 \rangle$ is a cofinal chain through $\dot{T}$, there exists an $S$-name $\dot{x}_f$ for a member of $\dot{T}$ which is of level $\sup_{n \in \omega} \beta_n$ such that

$$p_f \models_{\mathbb{P}} \models_S " \dot{x}_f \in \{ \dot{z}_\alpha; \alpha \in \omega_2 \} " .$$

Then it holds that

$$p_f \models_{\mathbb{P}} \models_S " \dot{x}_f \text{ is above } \dot{x}_f|n \text{ in } \dot{T} \text{ for every } n \in \omega " .$$

We note that the phrase $\models_{\mathbb{P}} \models_S " \dot{x}_f \text{ is above } \dot{x}_f|n \text{ in } \dot{T} \text{ for every } n \in \omega "$ is also true in the ground model, so we conclude that

$$\models_S \{ \dot{x}_f : f \in 2^{\omega} \} \text{ is a subset of the set of the members of } \dot{T} \text{ whose levels are } \sup_{n \in \omega} \beta_n, \text{ and is of size } \kappa \geq \aleph_1 ,$$

which contradicts to the assumption that $\dot{T}$ is an $S$-name for an $\omega_2$-tree.

\begin{claim}
Claim 3.2
\end{claim}

\begin{claim}
Claim 3.3. Let $\dot{T}$ be an $S$-name for a tree of size $\aleph_1$ and of height $\omega_1$ which doesn't have uncountable (i.e. cofinal) chains through $\dot{T}$. Then there exists a ccc forcing notion which preserves $S$ to be Suslin and forces $\dot{T}$ to be special (i.e. to be a union of countably many antichains through $\dot{T}$).

We note that this claim has been known if $\dot{T}$ is an $S$-name for an $\omega_1$-Aronszajn tree.

\end{claim}
**Proof of Claim 3.3.** For simplicity, we assume that $\dot{T}$ is an $S$-name for an order structure on $\omega_1$, that is, $\prec_{\dot{T}}$ is an $S$-name such that

$$\models_S " \dot{T} = \langle \omega_1, \prec_{\dot{T}} \rangle " ,$$

and that for any $s \in S$ and $\alpha, \beta$ in $\omega_1$, if $s \models_S " \alpha \not\prec \beta "$ and $\alpha < \beta$, then $s \models_S " \alpha \prec_{\dot{T}} \beta "$. Since $S$ is a ccc forcing notion, there exists a club $C$ on $\omega_1$ such that for every $\delta \in C$, every node of $S$ of level $\delta$ decides $\prec_{\dot{T}} \cap (\delta \times \delta)$.

We define the forcing notion $\mathbb{Q}(\dot{T}, C) = \mathbb{Q}$ which consists of finite partial functions $p$ from $S$ into the set $\bigcup_{\sigma \in [\omega]^{<\aleph_0}} (\omega_1)^{\sigma}$ such that

1. for every $s \in \text{dom}(p)$ and $n \in \text{dom}(p(s))$, $p(s)(n) \subseteq \sup(C \cap \downarrow s)$
2. for every $s$ and $t$ in $\text{dom}(p)$, if $s \prec_S t$, then for every $n \in \text{dom}(p(s)) \cap \text{dom}(p(t))$, $t \models_S " p(s)(n) \cup p(t)(n) \text{ is an antichain in } \dot{T} "$

ordered by extensions, that is, for each $p$ and $q$ in $\mathbb{Q}$,

$$p \leq \mathbb{Q} q : \iff p \supseteq q.$$ 

We note that $\mathbb{Q}$ adds an $S$-name which witnesses that $\dot{T}$ is special in the extension with $S$. We will show that if $\mathbb{Q} \times S$ has an uncountable antichain, then some node of $S$ forces that $\dot{T}$ has an uncountable chain, which finishes the proof of the claim.

Suppose that a family $\{ \langle p_\xi, s_\xi \rangle : \xi \in \omega_1 \}$ is an uncountable antichain in $\mathbb{Q} \times S$. By shrinking it and extending each member of the family if necessary, we may assume that

1. for each $\xi \in \omega_1$, $\text{dom}(p_\xi) \subseteq S_{\leq \delta_\xi}$ for some $\delta_\xi \in \omega_1$,
2. the sequence $\langle \delta_\xi ; \xi \in \omega_1 \rangle$ is strictly increasing,
3. for each $\xi \in \omega_1$ and $s \in \text{dom}(p_\xi)$, there exists $t \in \text{dom}(p_\xi) \cap S_{\delta_\xi}$ such that $s \leq_S t$,
4. for each $\xi \in \omega_1$ and $s \in \text{dom}(p_\xi)$ and $t \in \text{dom}(p_\xi) \cap S_{\delta_\xi}$, if $s \leq_S t$, then $p_\xi(s) \subseteq p_\xi(t)$,
5. all sets $\text{dom}(p_\xi) \cap S_{\delta_\xi}$ are of size $n$, and say $\text{dom}(p_\xi) \cap S_{\delta_\xi} = \{ t_1^\xi \: i \in n \}.$
for each $i \in n$, all $\text{dom}(p_{\xi}(t_{i}^{\xi}))$ are same, call it $\sigma_{i}$, and for each $k \in \sigma_{i}$, the size of each $p_{\xi}(t_{i}^{\xi})(k)$ is constant, call it $m_{i,k}$ and say $p_{\xi}(t_{i}^{\xi})(k) = \{\alpha_{i,k}^{\xi}(j) : j \in m_{i,k}\}$,

for each $\xi \in \omega_{1}$, $l_{v}(s_{\xi}) > \delta_{\xi}$,

there exists $\gamma \in \omega_{1}$ such that

- for each $\xi$ and $\eta$ in $\omega_{1}$, $s_{\xi}[\gamma] = s_{\eta}[\gamma] =: u_{-1}$,
- for each $\xi \in \omega_{1}$ and $t \in \text{dom}(p_{\xi})$, $t[\gamma, lv(t)] = s_{\xi}[\gamma, lv(t)]$,
- for each $\xi$ and $\eta$ in $\omega_{1}$ and $i \in n$, $t_{i}^{\xi}[\gamma] = t_{i}^{\eta}[\gamma] =: u_{i}$

(this can be done because of the coherency of $S$),

for each $i \in n$ and $k \in \sigma_{i}$, the set $\{p_{\xi}(t_{i}^{\xi})(k) : \xi \in \omega_{1}\}$ is pair-wise disjoint (by ignoring the root of the $\Delta$-system), and

the set $\{s_{\xi} : \xi \in \omega_{1}\}$ is dense above $u_{-1}$ in $S$.

We note that for each distinct $\xi$ and $\eta$ in $\omega_{1}$, since $\langle p_{\xi}, s_{\xi} \rangle \perp_{Q \times S} \langle p_{\eta}, s_{\eta} \rangle$, $s_{\xi} \perp_{S} s_{\eta}$ or there are $i \in n$, $k \in \sigma_{i}$ and $j_{0}$ and $j_{1}$ in $m_{i,k}$ such that $t_{i}^{\xi} \not\perp_{S} t_{i}^{\eta}$ and

$$t_{i}^{\xi} \cup t_{i}^{\eta} \Vdash_{S} "\alpha_{i,k}^{\xi}(j_{0}) \perp_{T} \alpha_{i,k}^{\eta}(j_{1})"$$

(where $t_{i}^{\xi} \cup t_{i}^{\eta}$ is the longer one of $t_{i}^{\xi}$ and $t_{i}^{\eta}$).

Let

$$u_{-1} \Vdash_{S} "\dot{I}_{-1} := \{\xi \in \omega_{1} : s_{\xi} \in \dot{G}\}, \text{ which is uncountable}",$$

and $\dot{U}$ an $S$-name for a uniform ultrafilter on $\dot{I}_{-1}$. We note that $u_{0}$ forces that the $S$-name

$$\psi_{u_{-1},u_{0}}(\dot{I}_{-1}) := \{\xi \in \omega_{1} : u_{0} \cup (s_{\xi}[\gamma, lv(s_{\xi})) \in \dot{G}\}$$

is an uncountable subset of $\omega_{1}$. For each $\xi \in \omega_{1}$, $k \in \sigma_{0}$, $l$ and $j$ in $m_{0,k}$, we define

$$u_{0} \Vdash_{S} "\text{whenever } \xi \in \psi_{u_{-1},u_{0}}(\dot{I}_{-1}), \dot{Y}_{0,k,j}^{\xi,l} := \{\eta \in \psi_{u_{-1},u_{0}}(\dot{I}_{-1}) : t_{0}^{\xi} \cup t_{0}^{\eta} \Vdash_{S} "\alpha_{0,k}^{\xi}(l) \perp_{T} \alpha_{0,k}^{\eta}(j)" \}"$$
(3) and define

\[ u_0 \vdash_{S} \{ \xi \in \psi_{u_{-1}, u_{0}}(\dot{I}_{-1}) : \bigcup_{k \in \sigma_0, l_0, j_0} Y_{0,k,j}^{\xi,l_0,j_0} \not\in \psi_{u_{-1}, u_{0}}(\dot{U}) \} \]

and define $\dot{I}_0 := \{ \xi \in \psi_{u_{-1}, u_{0}}(\dot{I}_{-1}) : \bigcup_{k \in \sigma_0, l_0, j_0} Y_{0,k,j}^{\xi,l_0,j_0} \not\in \psi_{u_{-1}, u_{0}}(\dot{U}) \} \)

\[ u_0 \vdash_{S} \{ \xi \in \psi_{u_{-1}, u_{0}}(\dot{I}_{-1}) : \bigcup_{k \in \sigma_0, l_0, j_0} Y_{0,k,j}^{\xi,l_0,j_0} \not\in \psi_{u_{-1}, u_{0}}(\dot{U}) \} \]

which is in $\psi_{u_{-1}, u_{0}}(\dot{U})$ for some $l_0, k_0$ and $j_0$ otherwise ... case 2.

If the case 2 happens, then we can make an $S$-name for a cofinal chain through $\dot{T}$ (which is forced by some node above $u_0$ in $S$), so we are done. Whenever the case 1 happens, we repeat this procedure, that is, given $\dot{I}_i$ for some $i \in n - 1$, we define, for each $\xi \in \omega_1$, $k \in \sigma_{i+1}$, $l$ and $j$ in $m_{i+1,k}$,

\[ u_{i+1} \vdash_{S} \{ \eta \in \psi_{u_{i}, u_{i+1}}(\dot{I}_{i}) : t_{i+1}^{\xi} \cup t_{i+1}^{\eta} \not\vdash_{S} \alpha_{i+1,k}^{\eta}(j) \angle_{T} \alpha_{i+1,k}^{\eta}(j) \} \]

and define

\[ u_{i+1} \vdash_{S} \{ \xi \in \psi_{u_{i}, u_{i+1}}(\dot{I}_{i}) : \bigcup_{k \in \sigma_{i+1}, l_0, j_0} Y_{i+1,k,j}^{\xi,l_0,j_0} \not\in \psi_{u_{i}, u_{i+1}}(\dot{U}) \} \]

which is in $\psi_{u_{i}, u_{i+1}}(\dot{U})$ for some $l_0, k_0$ and $j_0$ otherwise ... case 2.

We show that for some $i \in n - 1$, the case 2 happens in the construction of $\dot{I}_{i+1}$, which finishes the proof. Suppose that the case 1 happens in the construction of all the $\dot{I}_{i+1}$. We take $v \in S$ and $\xi \in \omega_1$ such that $u_{n-1} \leq_S v$ and

\[ v \vdash_{S} \{ \xi \in \dot{I}_{n-1} (which is in the set \psi_{u_{1}, u_{n-1}}(\dot{U})) \} \]

\[ \text{We note that by the property of the club } C, \text{ for each } \xi \text{ and } \eta \text{ in } \omega_1, \text{ if } t_{0}^{\xi} \cup t_{0}^{\eta} \in S, \text{ then this decides whether } \alpha_{0,k}^{\xi}(l) \perp_{T} \alpha_{0,k}^{\eta}(j) \text{ or not.} \]
Then it follows that
\[ v \geq_S u_{n-1} \cup (s_\xi | \gamma, \mathcal{L}(s_\xi)) \geq_S t_\xi_{n-1}. \]
We take \( v' \in S \) and \( \eta \in \omega_1 \) such that \( v' \geq_S v \) and
\[
v' \forces_S \text{"} \eta \in \psi_{u-1,u_{n-1}}(i_{-1}) \setminus \left( \bigcup_{i \in n} \psi_{u_i,u_{n-1}}(\bigcup l_{i,k,j}) \right) \text{ (which is in the set } \psi_{u-1,u_{n-1}}(\dot{u})) \text{"}.
\]
Then for every \( i \in n, u_i \cup (v' | \gamma, \mathcal{L}(v')) \) is above both \( t_\xi^\xi, t_\eta^\eta, u_i \cup (s_\xi | \gamma, \mathcal{L}(s_\xi)) \) and \( u_i \cup (s_\eta | \gamma, \mathcal{L}(s_\eta)) \). Then it follows that \( s_\xi \not\subseteq s_\eta \), and by the property of the club set \( C \), for every \( i \in n \) and \( k \in \sigma_i \),
\[
t_\xi^\xi \cup t_\eta^\eta \forces_S \text{"} p_\xi(t_\xi^\xi)(k) \cup p_\eta(t_\eta^\eta)(k) \text{ is an antichain in } \dot{T} \text{"}.
\]
Therefore \( \langle p_\xi, s_\xi \rangle \) and \( \langle p_\eta, s_\eta \rangle \) are compatible in \( \mathbb{Q} \times S \), which is a contradiction.

\[ \neg \text{ Claim 3.3 } \]

REFERENCES

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