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<th>Title</th>
<th>AN UNSATURATED GENERIC STRUCTURE (Model theoretic aspects of the notion of independence and dimension)</th>
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<tbody>
<tr>
<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2011), 1741: 9-12</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2011-05</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/170914">http://hdl.handle.net/2433/170914</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Kyoto University
AN UNSATURATED GENERIC STRUCTURE

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ABSTRACT. We construct an ab initio generic structure for a predimension function with a positive rational coefficient strictly less than 1 which is unsaturated and has a non-\(\omega\)-stable theory. Superstability of the theory will be discussed in a sequel paper.

1. INTRODUCTION

We consider graph structures. A graph structure has one binary relation as a first order structure. \(X \subseteq_{\text{fin}} Y\) means that \(X\) is a finite subset of \(Y\).

For a graph structure \(A\), let

\[ \delta_{\alpha}(A) = |A| - \alpha e(A). \]

Here, \(\alpha\) is a rational number such that \(0 < \alpha < 1\), \(|A|\) the number of points in \(A\), and \(e(A)\) the number of edges in \(A\). \(\delta_{\alpha}(A)\) is called a predimension function.

Suppose \(A \subseteq_{\text{fin}} B\) (substructure = induced subgraph).

\(A \leq B\) (\(A\) is a strong substructure of \(B\) or \(A\) is closed in \(B\)) if

\[ A \subseteq X \subseteq_{\text{fin}} B \Rightarrow \delta_{\alpha}(A) \leq \delta_{\alpha}(X). \]

In this case, if \(A = \{a\}\) (a singleton) then \(a\) is called a closed point in \(B\).

We say that \(A \leq B\) is minimal if \(A \leq B\), \(A \neq B\), and \(A \leq X \leq B\) implies \(X = A\) or \(X = B\).

With this notation, let

\[ K_{\alpha} = \{A : \text{finite : } A \geq \emptyset\}. \]

Definition 1.1. Suppose \(K \subseteq K_{\alpha}\). A countable graph \(M\) is a generic structure of \(K\) if

- \(A \subseteq_{\text{fin}} M\) \(\Rightarrow\) there exists \(B\) such that \(A \subseteq B \subseteq_{\text{fin}} M\) and \(B \leq M\);
- \(A \subseteq_{\text{fin}} M\) \(\Rightarrow\) \(A \in K\);
- for any \(A, B \in K\),

\[
\begin{array}{c}
\forall \mathcal{L} \\
\downarrow \\
\downarrow \\
A \\
\leq \\
\end{array}
\]

\[
\begin{array}{c}
\leq \\
M \\
\end{array}
\]

\( \leq \)
Definition 1.2. A class $K$ has the amalgamation property (AP, in short) if for any $A, B, C \in K$,

![Diagram of amalgamation property](image)

Fact 1.3. Suppose $K \subseteq K_\alpha$,

1. $\emptyset \in K$,
2. $K$ has the $AP_{f}$,
3. $A \subset B \in K$ implies $A \in K$.

Then $K$ has a generic structure.

Definition 1.4. Suppose $K \subseteq K_\alpha$. $K$ has thrifty amalgamation if whenever $A \leq B$ is minimal, $A \leq C$ with $A, B, C \in K$ then either $B \oplus_A C \in K$ or there is a strong embedding of $B$ into $C$ over $A$.

2. An Amalgamation Class

Definition 2.1. A graph $A$ is a minimal 1-component (in $K_\alpha$) if $|A| \geq 2$, $\delta_\alpha(A) = 1$, and $\delta_\alpha(X) > 1$ for any $X \subset A$ such that $1 < |X| < |A|$.

The following are examples of a minimal 1-component in the case $\alpha = 2/3$.

In the rest of the paper, we fix $\alpha = 2/3$ and $\delta_\alpha$ will be written $\delta$.

![Diagram of minimal 1-component](image)

Let $S_A$ be the set of connected substructures of $(A, a, b)$, i.e., the connected substructures of $A$ containing $a$ and $b$. Let $S_B$ be the set of connected substructures of $(B, a, b)$. Let $S_0 = S_A \cup S_B$.

Let $S_1$ be the smallest class with thrifty amalgamation containing $S_0$.

Lemma 2.2. (1) If $(X, a, b) \in S_0$, then $(X, a, b)$ is $(A, a, b)$, $(B, a, b)$, or $(Y, a, b) \leq (X, a, b)$ for some proper substructure $(Y, a, b)$ of $(B, a, b)$.

(2) If $(X, a, b) \in S_0$ with $1 < \delta(X) < 2$ then $\delta(X) = 4/3$ or $5/3$ and there is $(Y, a, b) \in S_0$ such that $X \leq Y$ and $\delta(Y) \geq 2$.

Definition 2.3. Let $S$ be a class of structures $(X, a, b)$ where $X$ is a graph and $a, b$ are two distinguished points in $X$.

Suppose that there are graphs $A_1, A_2, \ldots, A_n$ and points $a_{i-1}, a_i \in A_i$ such that $(A_i, a_{i-1}, a_i)$ is isomorphic to some element of $S$ for each $i$, and

$$Y = A_1 \oplus_{a_1} A_2 \oplus_{a_2} \cdots \oplus_{a_{n-1}} A_n.$$
We call \( Y \) a \( S \)-chain. \( n \) is called the length of the \( S \)-chain \( Y \). Each \( A_i \) is called an amalgamand of \( Y \). With such \( Y \), if we can write

\[
X = Y/(a_0 = a_n)
\]

then we call \( X \) a \( S \)-cycle. \( n \) is called the length of the \( S \)-cycle \( X \). Each amalgamand of \( Y \) is also called an amalgamand of \( X \).

If \( S \) consists of one graph with two points and one edge, we simply call an \( S \)-chain a chain, and an \( S \)-cycle a cycle.

Let \( K_0 \) be the set of \( S_1 \)-cycles of length greater than \( |B| \).

**Proposition 2.4.** Suppose \( X \in K_0 \).

1. \( \delta(X) = 0 \) if and only if every amalgamand of \( X \) is isomorphic to \((A, a, b)\) or \((B, a, b)\).
2. \( \delta(X) = 1/3 \) if and only if exactly one amalgamand of \( X \) is isomorphic to a proper substructure of \((A, a, b)\) or \((B, a, b)\) with \( \delta = 4/3 \) and each of the remaining amalgamands is isomorphic to \((A, a, b)\) or \((B, a, b)\).
3. \( \delta(X) = 2/3 \) if and only if either exactly one amalgamand of \( X \) is isomorphic to a proper substructure of \((A, a, b)\) or \((B, a, b)\) with \( \delta = 5/3 \) or exactly two amalgamands of \( X \) are isomorphic to a proper substructure of \((A, a, b)\) or \((B, a, b)\) with \( \delta = 4/3 \), and each of the remaining amalgamands is isomorphic to \((A, a, b)\) or \((B, a, b)\).
4. \( 0 < \delta(X) < 1 \) if and only if \( \delta(X) = 1/3 \) or \( \delta(X) = 2/3 \).

**Proposition 2.5.** Suppose \( X \in K_0 \).

1. If \( \delta(X) = 0 \) then there is no proper substructure of \( X \) closed in \( X \).
2. If \( \delta(C) \geq 2 \) for exactly one amalgamand \( C \) of \( X \), and each of the remaining amalgamands of \( X \) is isomorphic to \((A, a, b)\) or \((B, a, b)\), then there is a closed point of \( X \) in \( C \), and all the closed points of \( X \) are in \( C \).
3. If \( \delta(C), \delta(D) \geq 2 \) for exactly two amalgamands \( C \), \( D \) of \( X \), and each of the remaining amalgamands of \( X \) is isomorphic to \((A, a, b)\) or \((B, a, b)\), then there is a closed point of \( X \) in \( C \), and also in \( D \), and all the closed points of \( X \) are in \( C \) or \( D \).

Let \( K_1 \) be the set of \( S_1 \)-chains and its substructures.

Let \( K_2 \) be the smallest set with thrifty amalgamation containing \( K_0 \) and \( K_1 \).

**Proposition 2.6.** Suppose \( X \in K_2 \) and \( X \) is connected. If \( \delta(X) < 1 \) then \( X \in K_0 \).

**Proposition 2.7.** Suppose \( c_1 \) and \( c_2 \) are two closed points in \( X \in K_2 \). Then there is \( Y \in K_2 \) such that \( X \leq Y \) and \( c_1 \) and \( c_2 \) are connected in \( Y \).

**Proof.** If \( c_1 \) and \( c_2 \) are connected then there is nothing to prove. Suppose \( c_1 \) and \( c_2 \) are not connected in \( X \in K_2 \). Let \( X_1 \) be the connected component of \( X \) containing \( c_1 \) and \( X_2 \) the connected component of \( X \) containing \( c_2 \). If \( c_1, c_2 \in U \subset X \), then

\[
\delta(U) \geq \delta(U \cap X_1) + \delta(U \cap X_2) \geq 1 + 1 = 2
\]

since \( c_i \leq U \cap X_i \) for \( i = 1, 2 \). Hence, \( \{c_1, c_2\} \leq X \). Consider a chain \( C_3 \) of length 3 with end points \( c_1 \) and \( c_2 \). then \( \{c_1, c_2\} \leq C_3 \in K_2 \). Hence there is \( Y \in K_2 \) such that \( X \) and \( C_3 \) are strongly embedded in \( Y \) over \( \{c_1, c_2\} \).

\( \square \)
K. IKEDA AND H. KIKYO

3. An Unsaturated Generic Structure

Let $M$ be the generic structure of $K_2$.

Proposition 3.1. $M$ has only one connected component with closed points. The other connected components are exactly $\{A, B\}$-cycles.

Proposition 3.2. $Th(M)$ is not $\omega$-stable.

Proof. In a saturated model of $Th(M)$, we have all $\{A, B\}$-chains of countable length by compactness. Therefore, there are continuumly many types over $\emptyset$. \qed

We will discuss superstability of $Th(M)$ in a sequel paper.

REFERENCES