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On atomic AEC and quasi-minimality

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Abstract

Atomic abstract elementary class have been researched in connection with the model theory of infinitary logic. In recent years, the results were summarized by J.T.Baldin [1]. In that book, categoricity problem of atomic AEC is discussed mainly. I tried some local argument around the problem.

Apology In this note, I do not have exact references to the papers in which the results are originally proved.

1. Atomic AEC and splitting

We recall some definitions.

Definition 1 A class of structures $(K, \prec_K)$ (of a language $L$) is an abstract elementary class (AEC) if the class $K$ and class of pairs satisfying the binary relation $\prec_K$ are each closed under isomorphism and satisfy the following conditions:
A1. If $M \prec_K N$, then $M \subseteq N$.
A2. $\prec_K$ is a partial order on $K$.
A3. If $\{A_i : i < \delta\}$ is a $\prec_K$-increasing chain:
   (1) $\bigcup_{i<\delta} A_i \in K$
   (2) for each $j < \delta$, $A_j \prec_K \bigcup_{i<\delta} A_i$
   (3) if each $A_i \prec_K M \in K$, then $\bigcup_{i<\delta} A_i \prec_K M$.
A4. If $A, B, C \in K$, $A \prec_K C$, $B \prec_K C$ and $A \subseteq B$, then $A \prec_K B$.
A5. There is a Löwenheim-Skolem number $LS(K)$ such that if $A \subseteq B \in K$, there is an $A' \in K$ with $A \subseteq A' \prec_K B$ and $|A'| \leq |A| + LS(K)$.

Definition 2 We say an AEC $(K, \prec_K)$ is atomic if $K$ is the class of atomic models of a countable complete first order theory and $\prec_K$ is first order elementary submodel.

In the following, $K$ denotes an atomic AEC.
Definition 3 Let $T$ be a countable first order theory.
A set $A$ contained in a model $M$ of $T$ is atomic if every finite sequence in $A$ realizes a principal type over the empty set.
Let $A$ be an atomic set.
$S_{at}(A)$ is the collection of $p \in S(A)$ such that if $a \in M$ realizes $p$, $Aa$ is atomic (where $M$ is the big model).
We refer to a $p \in S_{at}(A)$ as an atomic type.

We consider the notion of stability for atomic types.

Definition 4 The atomic class $K$ is $\lambda$–stable if for every $M \in K$ of cardinality $\lambda$, $|S_{at}(M)| = \lambda$.

Example 5 ([1]) 1. Let $K_1$ be the class of atomic models of the theory of dense linear order without endpoints. Then $K_1$ is not $\omega$–stable.
2. Let $K_2$ be the class of atomic models of the theory of the ordered Abelian group of rationals. Then $K_2$ is $\omega$–stable.

The notion of independence by splitting is available in this context.

Definition 6 A complete type $p$ over $B$ splits over $A \subset B$ if there are $b, c \in B$ which realize the same type over $A$ and a formula $\phi(x, y)$ such that $\phi(x, b) \in p$ and $\neg\phi(x, c) \in p$.
Let $A, B, C$ be atomic.
We write $A \perp_C B$ and say $A$ is independent from $B$ over $C$ if for any finite sequence $a \in A$, $tp(a/B)$ does not split over some finite set of $C$.

Fact 7 ([1]) Under the atomic $\omega$–stable assumption of $(K, \prec_K)$ (and some assumption of parameters), the independence relation by splitting (over models) satisfies almost all forking axioms.

Theorem 8 ([1]) If $K$ is $\omega$–stable and has a model of power $\aleph_1$, then it has a model of power $\aleph_2$.

I considered the same problems under some weaker condition.

Definition 9 Let $K$ be an atomic AEC and $M \in K$.
$M$ has no infinite splitting chain if for any $p \in S_{at}(M)$ which is realized outside $M$, there is no increasing sequence $\{A_i\}_{i<\omega}(\subset M)$ such that $p[A_{i+1}$ splits over $A_i$ for all $i < \omega$.
We can prove the next facts.

Fact 10 If $K$ is $\omega$–stable, then no model of $K$ has infinite splitting chain.
Fact 11  Let $K$ have no infinite splitting chain (i.e. every $M \in K$ has no infinite splitting chain).
If $K$ has a model of power $\aleph_1$, then it has a model of power $\aleph_2$.

Fact 12  Under the assumption that $(K, \prec_K)$ has no infinite splitting chain, the independence relation by splitting (over models) satisfies almost all forking axioms except symmetry.

At present, I do not have the definitive result about symmetry of splitting. But we can prove the next fact.

Definition 13  Let $K$ be an atomic AEC and $M \in K$.
$M$ has infinite splitting left-chain if there is a sequence $\{B_i\}_{i<\omega} \subset M$ and $b$ (outside $M$), and $A \subset M$ such that $tp_{at}(B_i/Ab\{B_j : j<i\})$ splits over $A \cup \{B_j : j<i\}$ for all $i < \omega$.

Fact 14  Let $K$ have no infinite splitting chain. Suppose that any countable atomic set is extended to a countable model in $K$.
If the independence by splitting over models is not symmetry, then there is an infinite splitting left-chain.

2. *—excellent AEC and categoricity

We recall some definitions.

Definition 15  The atomic AEC $K$ is *—excellent if
A1. $K$ has arbitrarily large models,
A2. $K$ is $\omega$—stable,
A3. $K$ satisfies the amalgamation property,
A4. Let $p$ be a complete type over a model $M \in K$ such that $p[C]$ is realized in $M$ for each finite $C \subset M$, then there is a model $N \in K$ with $N$ primary over $Ma$ such that $p$ is realized by $a$ in $N$.

Definition 16  Let $M \in K$ and $A \subset M$.
The type $p \in S_{at}(A)$ is big if for any $M' \supset A$ with $M' \in K$, there exists an $N'$ such that $M' \prec_K N'$ and $p$ has a realization in $N' - M'$.
A triple $(M, N, \phi)$ is called a Vaughtian triple if $\phi(M) = \phi(N)$ where $M \prec_K N$ with $M \neq N$ and $L(M)$—formula $\phi$ is big.

The next theorem is the analogous result of Morley’s categoricity theorem for atomic AEC.

Theorem 17 ([1])  Suppose $K$ is an *—excellent atomic AEC.
Then the following are equivalent.
(1) $K$ is categorical in some uncountable cardinality.
(2) $K$ has no Vaughtian triple.
(3) $K$ is categorical in every uncountable cardinality.

**Theorem 18** ([1]) For each $2 \leq k < \omega$, there is an $L_{\omega_1, \omega}$-sentence $\phi_k$ such that:

$\phi_k$ is categorical in $\mu$ if $\mu \leq \aleph_{k-2}$, and

$\phi_k$ is not categorical in any $\mu$ with $\mu > \aleph_{k-2}$.

In the proof of Theorem 17, the geometry of quasi-minimal formula plays the important role. I considered that the argument in [5] makes the proof concise.

**Definition 19** The type $p \in S_{at}(A)$ is quasi-minimal if $p$ is big and for any $M$ containing $A$, $p$ has a unique extension to a type over $M$ which is not realized in $M$.

**Lemma 20** ([1]) Let $K$ be $\omega$-stable.

Then for any $M \in K$, there is a $c \in M$ and a formula $\phi(x, c)$ which is quasi-minimal.

**Definition 21** Let $X$ be an infinite set and $cl$ a function from $\mathcal{P}(X)$ to $\mathcal{P}(X)$ where $\mathcal{P}(X)$ denotes the set of all subsets of $X$. If the function $cl$ satisfies the following properties, we say $(X, cl)$ is a pregeometry.

(I) $A \subset B \Rightarrow A \subset cl(A) \subset cl(B)$,

(II) $cl(cl(A)) = cl(A)$,

(III) (Finite character) $b \in cl(A) \Rightarrow b \in cl(A_0)$ for some finite $A_0 \subset A$,

(IV) (Exchange axiom)

$b \in cl(A \cup \{c\}) - cl(A) \Rightarrow c \in cl(A \cup \{b\})$.

In the proof of Theorem 17, the closure operator $cl$ is defined as follows.

**Definition 22** Let $c \in M \in K$. And suppose $\phi(x, c)$ determines a quasi-minimal type over $M$.

For any elementary extension $N(\in K)$ of $M$, $cl$ is defined on the set of realizations of $\phi(x, c)$ in $N$ by $a \in cl(A)$ if $tp(a/\emptyset)$ is not big.

We recall some result from [5].

**Definition 23** Let $M$ be an uncountable structure and $p(x) \in S_1(M)$.

For all subsets $A \subset M$, the operator $cl_p'$ is defined by $cl_p'(A) = \{a \in M : \text{tp}(a/\emptyset) \text{ is not a realization of } p[A] \}.

The $n$-th closure $cl^n_p(A)$ of $A$ is inductively defined as follows:

$cl_p^0(A) = A$ and $cl_p^{n+1}(A) = cl_p'(cl_p^n(A))$

We put $cl_p(A) = \bigcup_{n \in \omega} cl^n_p(A)$. 

Theorem 24 ([5]) Let $N$ be a quasi-minimal model and $p(x) \in S_1(N)$. Suppose that $p(x)$ does not split over $A$ for some countable $A \subset N$ (and $N \neq cl_p(a)$ for some finite $a \in N$).

Then $cl_p$ is a closure operator and exactly one of the following two holds;
1. Every $cl_p$–free sequence over $A$ is totally indiscernible. In this case, $(N, cl_p)$ is a pregeometry, and $p$ is definable over $A$.
2. Otherwise. In this case, there is a finite extension $A_0$ of $A$ and an $A_0$–definable partial order $\leq$ such that every $cl_p$–free sequence over $A_0$ is strictly increasing.

We can deduce the next lemma.

Lemma 25 Let $K$ be $\omega$–stable and have a sufficiently large model. And let $M \in K$.

Then there is a finite $c \in M$ and a formula $\phi(x, c)$ such that $\phi(x, c)$ determines a quasi-minimal $p(x) \in S_{at}(M)$ and $p(x)$ does not split over $c$, and $cl$ defines a pregeometry in $(p[c](M))$.

3. P–closure in atomic AEC

I considered $P$-closure in the quasi-minimal set $\phi(M, c)$ above. The $P$-closure is the collection of realizations of types that is $P$-analysable and co-foreign to $P$. In this note, I omits the $P$-analysable assumption, resulting in a larger $P$-closure.

In this section, $A \not\triangleright_C B$ means $tp(A/BC)$ does not split over $C$.

Assumptions

Let $K$ be an $\omega$–stable atomic AEC and $M \in K$.

$\phi(x, c)$ determines a quasi-minimal $p(x) \in S_{at}(M)$ and $p(x)$ does not split over $c$. And we may assume that $c = \emptyset$.

The set $P$ of types is defined by $P = \{q \in S(A') : q$ is a conjugate of $p[A$ for some finite $A \subset M]\}$.

Definition 26 In this definition, parameters are finite subset of $\phi(M, c)$ above and types are atomic types.

Now $P$ is an $\emptyset$-invarint family of types.

A complete type $q \in S_{at}(A)$ is foreign to $P$ if for all $a \models q$, $A \subset B$ with $a \not\triangleright_A B$, and realizations $\bar{c}$ of extensions of types in $P$ over $B$, we always have $a \not\triangleright_A \bar{c} B$.

A partial type $q$ is co–foreign to $P$ if every type in $P$ is foreign to $q$.

The $P$ – closure $cl_P(A)$ of a set $A$ is the collection of all element $a$ such that $tp(a/A)$ is co-foreign to $P$.

We can prove the next fact.
Fact 27  Let $K$ be $\omega$–stable and have a sufficiently large model. And let $M \in K$ and $\phi(x, c)$ be a quasi-minimal formula that determines $p(x) \in S_{at}(c)$ for some $c \in M$ as above.
For any $A \subset p(M)$, $cl(A)=cl_{P}(A)$.
And $(p(M), cl_{P})$ is a pregeometry.

References