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Morley’s theorem on Omitting types

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Abstract

We think about Morley’s omitting types theorem for countable first-order theory. Then I introduce the result of having been related to Morley’s theorem shown by [4].

1 Introduction

Definition.(\omega-number) \mathbb{N}_{0} = \omega, \mathbb{N}_{\alpha+1} = 2^{\mathbb{N}_{\alpha}}, \mathbb{N}_{\delta} = \sup_{\alpha<\kappa} \mathbb{N}_{\alpha}.

Fact.(Erdős-Rado) Let \alpha be infinite cardinal and n < \omega. Then \mathbb{N}_{n} \rightarrow (\omega^{+})_{\omega}^{+1}, \mathbb{N}_{\alpha+n} \rightarrow (\mathbb{N}_{\alpha})^{n+1}.

Note. \alpha \rightarrow (\beta)^{n}_{\gamma} means whenever |X| = \alpha and given any function f from \{X\}^{n} into \gamma, there exists a subset Y of X with |Y| = \beta and an i < \gamma such that for all y \in |Y|^{n}, f(y) = i.

Theorem.(Stretching) Let \mathcal{L} be countable language, M be a model of theory T of \mathcal{L}, \langle A, \prec \rangle be an infinite set of indiscernibles in M, and \langle B, \prec \rangle be an arbitrary infinite linearly ordered set. Then there exist a model \mathcal{N} of T such that \langle B, \prec \rangle is a set of indiscernibles in \mathcal{N}, and for any a_{1} < \cdots < a_{n} \in A and b_{1} < \cdots < b_{n} \in B, \text{tp}(a_{1}, \ldots, a_{n}) = \text{tp}(b_{1}, \ldots, b_{n}).

Proof. Put \Sigma := \{t(x_{1}, \ldots, x_{n}) : t \text{ is term in } \mathcal{L}\}. We define an equivalence relation \sim on \Sigma as follows. If t(x_{1}, \ldots, x_{n}), t'(x_{1}, \ldots, x_{n}) \in \Sigma, define t \sim t' iff for any a_{1} < \cdots < a_{n} \in A, M \models t(a_{1}, \ldots, a_{n}) = t'(a_{1}, \ldots, a_{n}). Put \overline{N} := \{t(b_{1}, \ldots, b_{n}) : t(x_{1}, \ldots, x_{n}) \in \Sigma, b_{1} < \cdots < b_{n} \in B\}. We define an equivalence relation \approx on \overline{N} as follows. If t(b_{1}, \ldots, b_{n}), t'(b_{1}', \ldots, b_{m}') \in \overline{N}, define t \approx t' iff t_{0}(z_{1}, \ldots, z_{s}) \sim t'_{0}(z_{1}, \ldots, z_{s}), where \{z_{1}, \ldots, z_{s}\} := \{x_{1}, \ldots, x_{n}\} \cup \{x'_{1}, \ldots, x'_{m}\} and t_{0}(z_{1}, \ldots, z_{s}) := t(x_{1}, \ldots, x_{n}), t'_{0}(z_{1}, \ldots, z_{s}) := t'(x'_{1}, \ldots, x'_{m}). Put \mathcal{N} := \{t(\overline{b})^{\approx} : t(\overline{b}) \in \overline{N}\}. Note that for any t_{1}(\overline{b}_{1}), \ldots, t_{n}(\overline{b}_{n})^{\approx} \in \mathcal{N} there exists, for some \overline{b} \in B and t' \in \Sigma such that t_{i}(\overline{b}_{i})^{\approx} = t_{i}'(\overline{b})^{\approx}. We treat B as a subset of \mathcal{N} by identifying each b \in B with \overline{b}^{\approx}.

\mathcal{N} can be made into a \mathcal{L}-structure by defining constants, functions and relations as follows:

(Constatns) \mathcal{N} \models c_{\mathcal{N}} = c^{\approx}.
(Functions) \mathcal{N} \models F(t_{1}(\overline{b})^{\approx}, \ldots, t_{n}(\overline{b})^{\approx}) = (F(t_{1}(\overline{b}), \ldots, t_{n}(\overline{b})))^{\approx}.
(Relations) \mathcal{N} \models R(t_{1}(\overline{b})^{\approx}, \ldots, t_{n}(\overline{b})^{\approx})
\overset{\text{def}}{=} \text{ for all } a_{1} < \cdots < a_{m} \in A, M \models R(t_{1}(\overline{a}), \ldots, t_{n}(\overline{a})).

This definition does not depend on the choice of representatives of the equivalence classes under \sim.
By induction on the complexity of formulas and use Skolem function it can be shown that for any \( b_1 < \cdots < b_n \in B \) and \( \phi(x_1, \ldots, x_n) \in \mathcal{L} \),
\[
N \models \phi(b_1, \ldots, b_n) \iff \text{ for all } a_1 < \cdots < a_n \in A, \\
M \models \phi(a_1, \ldots, a_n).
\]

By indiscernibility of \( \langle A, < \rangle \), \( \langle B, < \rangle \) is a set of indiscrernibles in \( N \) and for any \( a_1 < \cdots < a_n \in A \) and \( b_1 < \cdots < b_n \in B \), \( tp(a_1, \ldots, a_n) = tp(b_1, \ldots, b_n) \). In particular, \( N \equiv M \), hence \( N \) is a model of \( T \).

2 Morley's Theorem

**Theorem.** (Morley's omitting types theorem) Let \( T \) be a theory of countable language \( \mathcal{L} \), \( \Gamma \) a set of partial types in finitely many variables over \( \emptyset, \mu = (2^\omega)^+ \). Suppose \( \{ M_\alpha : \alpha < \mu \} \) is a sequence of models of \( T \) such that

1. \( |M_\alpha| > \beth_\omega \),
2. \( M_\alpha \) omits each member of \( \Gamma \).

Then for every \( \lambda \geq \omega \), there is a model \( N \) with \( |N| = \lambda \) of \( T \) such that \( N \) omits each member of \( \Gamma \).

**Proof.** Assume to simplify an argument \( T \) has built-in Skolem functions and the set of formulas \( \Gamma \) in the unary. Let \( C = \langle c_i : i < \omega \rangle \) be a sequence of new constant symbols, \( \mathcal{L}^* = \mathcal{L} \cup C \).

Now we construct the consistent \( \mathcal{L}^* \)-theory \( \Phi \) as following properties:

1. \( T \cup \{ c_i \neq c_j : i < j < \omega \} \subset \Phi; \)
2. for each term \( t(x_1, \ldots, x_n) \) and \( \phi \in \phi \), there is a \( \phi_p \in p \) such that for all \( i_1 < \cdots < i_n < \omega \),
\[
-\phi_p(t(c_{i_1}, \ldots, c_{i_n})) \in \Phi;
\]
3. for any \( \psi(x_1, \ldots, x_n) \in \mathcal{L} \) if \( i_1 < \cdots < i_n < \omega \) and \( j_1 < \cdots < j_n < \omega \),
\[
\psi(c_{i_1}, \ldots, c_{i_n}) \leftrightarrow \psi(c_{j_1}, \ldots, c_{j_n}) \in \Phi.
\]

**Notation.** \( F := \{(M_\alpha, A_\alpha) : \alpha < \mu \} \) is a sequence such that \( M_\alpha \) is satisfied the hypotheses of the theorem and \( A_\alpha \) is subset of \( M_\alpha \) with \( |A_\alpha| > \beth_\alpha \).

We say that \( F' = \{(M'_\alpha, B_\alpha) : \alpha < \mu \} \) is subsequence of \( F \) if for each \( M'_\alpha \) there is \( \beta \geq \alpha \) such that \( M'_\alpha = M_\beta \) and \( B_\alpha \subset A_\beta \) with \( |B_\alpha| > \beth_\alpha \).

Fix a linear ordering of each \( M_\alpha \) in an arbitrary fashion denoting them all by \( < \).

**Claim 1.** Fix a term \( t(x_1, \ldots, x_n) \). There is subsequence \( F' \) of \( F \) as following property: for each \( \phi \in \phi \) there is a \( \phi_p \in p \) such that for any \( (M'_\alpha, B'_\alpha) \in F' \), if \( i_1 < \cdots < i_n < \omega \) and \( b_{i_j} \in B'_\alpha \) then \( M'_\alpha \models -\phi_p(t(b_{i_1}, \ldots, b_{i_n})) \).
Proof of claim 1. Note $|\Gamma| \leq 2^\omega$. Let $N_\alpha = M_{\alpha+n}$. Define, for all $\alpha < \mu$, $f_\alpha : \{A_{\alpha+n}\}^n \to \mathcal{L}^\Gamma$ ($\overline{a} \mapsto f_\alpha(\overline{a})$) where $f_\alpha(\overline{a}) : \Gamma \to \mathcal{L}$ ($p \mapsto (f_\alpha(\overline{a}))(p) = \phi_{\alpha,p} \in p$) such that $N_\alpha \models \neg \phi_{\alpha,p}(t(\overline{a}))$ such a $\phi_{\alpha,p}$ exists since $N_\alpha$ omits $p$.

Now $|A_{\alpha+n}| > \mathfrak{s}_{\alpha+n}$ and for $\alpha \geq 3$, $\mathfrak{s}_{\alpha} \geq |\mathcal{L}^\Gamma|$.

By Erdős-Rado Theorem, $(\mathfrak{s}_{\alpha+n})^+ \to (\mathfrak{s}_{\alpha})^n_{|\mathcal{L}^\Gamma|}$. Thus we obtain $B_\alpha \subset A_{\alpha+n}$ and $\phi_{\alpha,p} \in \mathcal{L}$ such that

1. $|B_\alpha| > \mathfrak{s}_\alpha$,
2. for all $\overline{b} \in [B_\alpha]^n$, $N_\alpha \models \neg \phi_{\alpha,p}(t(\overline{b}))$.

Namely, for all $\overline{b} \in [B_\alpha]^n$, $f_\alpha(\overline{b})$ is constant.

As $\mu = (2^\omega)^+$, by Erdős-Rado, there is subsequence $\{M'_\alpha : \alpha < \mu\}$ of $\{N_\alpha : \alpha < \mu\}$ such that for all $\overline{b} \in [B_\alpha]^n$ and $p \in \Gamma$, $(f_\alpha(\overline{b}))(p) = \text{constant}$. Thus $\{(M'_\alpha, B'_\alpha) : \alpha < \mu\}$ and $\phi_p := (f_\alpha(\overline{b}))(p)$ are required.

Claim 2. Fix a $\mathcal{L}$-formula $\psi(x_1, \ldots, x_n)$. There is subsequence $F'$ of $F$ as following property: for any $(M'_\alpha, B_\alpha) \in F'$ if $i_1 < \cdots < i_n < \mu$, $j_1 < \cdots < j_n < \mu$ and $b_{i_1}, b_{j_1} \in B_\alpha$

$$M'_\alpha \models \psi(b_{i_1}, \ldots, b_{i_n}) \leftrightarrow \psi(b_{j_1}, \ldots, b_{j_n}).$$

Proof of Claim 2. Define, for all $\alpha < \mu$, $h_\alpha : [A_\alpha]^n \to 2$ as follows:

$$h_\alpha(\overline{a}) = \begin{cases} 0 & \text{if } M_\alpha \models \psi(\overline{a}), \\ 1 & \text{otherwise}. \end{cases}$$

By Erdős-Rado theorem, there is $B_\alpha \subset A_\alpha$ such that $|B_\alpha| > \mathfrak{s}_\alpha$ and for all $\overline{b} \in [B_\alpha]^n$, $h_\alpha(\overline{b})$ is constant. Thus, $\{(M_\alpha, B_\alpha) : \alpha < \mu\}$ is required.

Let $\{t_i : i < \omega, t_i \text{ is a term of } \mathcal{L}\}$ and $\{\psi_i : i < \omega, \psi_i \text{ is a } \mathcal{L}\text{-formula}\}$ be enumerations of all the terms of $\mathcal{L}$ and all the $\mathcal{L}$-formula, respectively. Now we construct $\Phi$ by induction on $i < \omega$. Suppose $F_0 := \{(M_\alpha, M_\alpha) : \alpha < \mu\}$ and $\Phi_0 := T \cup \{c_i \neq c_j : i < j < \omega\}$. Clearly, for any $(M_\alpha, M_\alpha) \in F_0$, $M_\alpha \models \Phi_0$ and $|\mathcal{L}| > \mathfrak{s}_{\alpha}$.

Case 1 ($i < \omega$ is even). Assume we have found $F_i$ and $\Phi_i$. We take new term $t(x_1, \ldots, x_n) \in \{t_i : i < \omega, t_i \text{ is a term of } \mathcal{L}\}$, by claim 1, there is subsequence $F_{i+1}$ of $F_i$ as following property: for each $p \in \Gamma$, there is a $\phi_p \in p$ such that for any $(M'_\alpha, B_\alpha) \in F_{i+1}$, if $i_1 < \cdots < i_n < \omega$ and $b_{i_1} \in B_\alpha$,

$$M'_\alpha \models \neg \phi_p(t(b_{i_1}, \ldots, b_{i_n})).$$

We put $\Phi_{i+1} = \Phi_i \cup \{\neg \phi_p(t(c_{i_1}, \ldots, c_{i_n})) : p \in \Gamma, i_1 < \cdots < i_n < \omega\}$.

Case 2 ($i < \kappa$ is odd). Assume we have found $F_i$ and $\Phi_i$. We take new formula $\psi(x_1, \ldots, x_n) \in \{\psi_i : i < \omega, \psi_i \text{ is a } \mathcal{L}\text{-formula}\}$, by claim 2, there is subsequence $F_{i+1}$ of $F_i$ as following property: for any $(M'_\alpha, B_\alpha) \in F_{i+1}$ if $i_1 < \cdots < i_n < \mu$, $j_1 < \cdots < j_n < \mu$ and $b_{i_1}, b_{j_1} \in B_\alpha$

$$M'_\alpha \models \psi(b_{i_1}, \ldots, b_{i_n}) \leftrightarrow \psi(b_{j_1}, \ldots, b_{j_n}).$$
We put $\Phi_{i+1} := \Phi_i \cup \{\psi(c_{i_1}, \ldots, c_{i_n}) \leftrightarrow \psi(c_{j_1}, \ldots, c_{j_n}) : i_1 < \cdots < i_n < \omega, j_1 < \cdots < j_n < \omega\}$.

If put $\Phi := \cup_{i<\omega}\Phi_i$ then it is required $\mathcal{L}^\ast$-theory. We take any $(M_\alpha, A_\alpha) \in F := \cap_{i<\omega}F_i$. By construction $M_\alpha \models \Phi$.

Let $A$ be the set of all interpretation $C = \{c_i : i < \omega\}$ in $M_\alpha$, $N$ be Skolem closure of $A$ in $M_\alpha$. Thus $N$ is model of $T$, omitting $\Gamma$, indiscernibles in $M_\alpha$, and $|N| = \omega$.

Take any $\lambda \geq \omega$. By stretching theorem, there is a model of $T$ which the cardinality of $\lambda$ such that omitting $\Gamma$. Note that if $|\Gamma| \leq \omega$ then it is sufficient $\mu = \omega_1$, see [2].

It is known that Morley's theorem is proved in infinitary logic, and it is effective means to show existence of models in infinitary logic that the compactness theorem is false generally, see [1], [3].

### 3 Related Result

The following result is related to Morley's omitting type theorem. This theorem says the thing that is stronger than Morley's theorem under a certain condition.

**Theorem.** (Tsuboi) Let $T$ be a countable complete $\mathcal{L}$-theory and $\Gamma$ a set of complete types with $|\Gamma| < 2^\omega$. Suppose that for each $\alpha < \omega_1$, there is a model $M_\alpha \models T$ with the following properties:

1. $|M_\alpha| > \beth_\alpha$,
2. $M_\alpha$ omits each member of $\Gamma$.

Then for each $\lambda \geq \omega$ there is a model $N$ omitting $\Gamma$ and with $|N| = \lambda$.

**Proof.** Let $X = \omega_1$ and $\{I_i : i \in X\}$ be a set of infinite indiscernible sequences and $\{t_n : n < \omega\}$ be an enumeration of all the $\mathcal{L}$-terms. We may assume that $t_n$ has $n$-variables. We will say that the set $\{I_i : i \in X\}$ is $t_n$-uniform if the following condition holds: If $i, j \in X$, then $tp(t_n(I_i)) = tp(t_n(I_j))$ where $tp(t_n(I_i)) := tp(t_n(a_0, \ldots, a_{n-1}))$ $(a_0 < \cdots < a_{n-1} \in I_i)$. We will say that $\{I_i : i \in X\}$ is essentially $t_n$-uniform if there is an uncountable subset $Y$ of $X$ such that $\{I_i : i \in Y\}$ is $t_n$-uniform. For a formula $\phi(x)$, define $X^\phi, t_n := \{i \in X : \phi(x) \in tp(t_n(I_i))\}$. Put $X_\emptyset = \omega_1$, and for each $i \in X_\emptyset$ we fix a sequence $I_{\emptyset}(i)$ enumerating the universe $M_i$.

Using the argument in the paper([4]), for $\eta \in 2^{<\omega}$ and $k < \omega$, we can inductively choose $X_\eta \subset \omega_1$, $\{I_\eta(i) : i \in X_\eta\}$ and formulas $\phi_{\eta, k}$ with the following properties:

1. If $\eta < \nu$, then
   \begin{enumerate}
     \item $X_\nu$ is an uncountable subset of $X_\eta$;
     \item $I_\nu(i)$ is a subsequence of $I_\eta(i)$ for each $i \in X_\nu$.
   \end{enumerate}
2. $i < j \Rightarrow |I_\eta(i)| < |I_\eta(j)|$, and $\sup\{|I_\eta(i)| : i \in X_\eta\} \geq \beth_{\omega_1}$.
3. If $\eta \in 2^n$ then
   (a) each $I_\eta(i)$ is an infinite indiscernible sequence;
   (b) $\{I_\eta(i) : i \in X_\eta\}$: essentially $t_n$-uniform $\Rightarrow$ it is $t_n$-uniform.

4. If $\eta \in 2^n$ and $k \leq n$ then
   $\{I_\eta(i) : i \in X_\eta\}$: not $t_n$-uniform
   $\Rightarrow X_{\eta^0} \subset (X_\eta)^{(-\phi_{\nu,k})}.t_k$ and $X_{\eta^1} \subset (X_\eta)^{(\phi_{\nu,k})}.t_k$.

For all $\nu \in 2^\omega$, we define the following:

1. $K_\nu$ is the set of all $n < \omega$ such that $\{I_{\nu|n}(i) : i \in X_{\nu|n}\}$ is not $t_n$-uniform;
2. for $n \in K_\nu$, $\Delta_\nu^n(x) := \bigcup_{n \leq m < \omega} \{\phi_{\nu|m,n}(x) : \nu(m) = 0\} \cup \bigcup_{n \leq m < \omega} \{\neg\phi_{\nu|m,n}(x) : \nu(m) = 1\};$
3. $\Phi_\nu := \{\{x_i\}_{i<\omega} \text{ is indiscernible}\} \cup \bigcup_{n \in K_\nu} \Delta_\nu^n(t_n(\overline{x}_n)) \cup \bigcup_{n \not\in K_\nu} p_{\nu|n}(t_n(\overline{x}_n));$
4. $F_\nu := \{(M_\nu^i, I_\nu(i)) : i \in X_\nu\} \quad (I_\nu(i) \subset M_\nu^i)$.

\[
\begin{array}{c}
2^\omega \\
\{ F_{(0\ldots)} : (M_0^{(0\ldots)}, I_{(0\ldots)}(0)), \ldots, (M_i^{(0\ldots)}, I_{(0\ldots)}(i)), \ldots \models \Phi_{(0\ldots)} \\
\vdots & \vdots & \vdots \vspace{1ex} \\
F_\nu : (M_0^\nu, I_\nu(0)), \ldots, (M_i^\nu, I_\nu(i)), \ldots \models \Phi_\nu \\
\vdots & \vdots & \vdots \vspace{1ex} \\
F_{(1\ldots)} : (M_0^{(1\ldots)}, I_{(1\ldots)}(0)), \ldots, (M_i^{(1\ldots)}, I_{(1\ldots)}(i)), \ldots \models \Phi_{(1\ldots)} \vspace{1ex} \}
\end{array}
\]

We can take $\nu \in 2^\omega$ well, see [4], such that if $\{c_i : i < \omega\}$ realizing $\Phi_\nu$ in $M_0^\nu$, and $N$ is Skolem closure of $\{c_i : i < \omega\}$ in $M_0^\nu$ then $N$ omits $\Gamma$. The rest of the statement is clear from Stretching Theorem.

References


