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Kyoto University
\textbf{\textbeta\text{-expansion}'s Attractors Observed in A/D converters}

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A new class of analog-to-digital (A/D) and digital-to-analog (D/A) converters using a flaky quantiser, called the $\beta$-encoder, as shown in Fig. 1 [1, 2, 3] has been shown to have exponential bit rate accuracy while possessing a self-correction property for fluctuations of the amplifier factor $\beta$ and the quantiser threshold $\nu$. Motivated by the close relationships [4, 5, 6] between $\beta$-transformations and $\beta$-expansion, we have recently observed [9, 10] that (1) such a flaky quantiser is exactly realized by the "multi-valued Rényi-Parry map", defined here so that probabilistic behavior in the "flaky region" is completely explained using dynamical systems theory; (2) a sample $x$ is always confined to a subinterval of the contracted interval while the successive approximation of $x$ is stably performed using $\beta$-expansion even if $\nu$ may vary at each iteration (i.e. a small real-valued quantity, approximately proportional to the quantisation error, does not necessarily converge to any fixed value, e.g., 0 but may oscillate without diverging. Such a phenomenon is precisely the kind of "chaos"; (3) such a subinterval enables us to obtain the decoded sample easily, as it is equal to the midpoint of the subinterval and to prove two classic $\beta$-expansions, known as the greedy and lazy expansions [7, 8] are perfectly symmetrical in terms of their quantisation errors. The subinterval further suggests that $\nu$ should be set to around the midpoint of its associated greedy and lazy values. A switched-capacitor (SC) circuit technique [11, 12] has been proposed for implementing A/D converter circuit based on several types of $\beta$-encoders and SPICE simulations have been given to verify the validity of these circuits against deviations and mismatches of circuit parameters. Our review

Figure 1: A typical $\beta$-encoder with its input $z_0 = y \in [0,1)$ and output $(b_i,\beta)$.

Figure 2: The scale-adjusted $\beta$-map: $S_{\beta,\nu,s}(x)$ and its eventually onto map in which an attractor can be observed.

Figure 3: The scale-adjusted negative $\beta$-map: $R_{\beta,\nu,s}(x)$ and its eventually onto map in which an attractor can be observed.

is twofold. First, the $\beta$-encoder leads us to naturally define the "multi-valued Rényi-Parry map" [4, 5] with its eventually onto map, as it is identical to the Parry's $(\beta,\alpha)$-map [6]. Second, chaos, called "$\beta$-expansion's attractors" can be observed on the onto-map. Two types of $\beta$-expansion's attractors are as follows:

1. \textbf{Scale-Adjusted $\beta$-Map}[9, 11]: Daubechies et al. [1, 2] introduced a "flaky" version of an imperfect quantiser, defined as

\begin{equation}
Q^{\nu}_{\Delta_\beta}(z) = \begin{cases} 0, & \text{if } z \leq \nu_0, \\ 1, & \text{if } z \geq \nu_1, \\ 0 \text{ or } 1, & \text{if } z \in \Delta_\beta = [\nu_0, \nu_1], \nu_0 < \nu_1, \end{cases}
\end{equation}

which is a $\nu$-varying model of a quantiser $Q_{\nu}(z) = \begin{cases} 0, & \text{if } z \leq \nu, \\ 1, & \text{if } z \geq \nu, \end{cases}$ $\nu \in [\nu_0, \nu_1], \nu_0 < \nu_1$. We obtain:

\textbf{Lemma 1}[9]: Let $S_{\beta,\nu,s}(x)$ be the scale-adjusted map with a scale $s$, defined by

\begin{equation}
S_{\beta,\nu,s}(x) = \beta x - s(\beta - 1)Q^{\nu}_{\Delta_\beta}(x) = \begin{cases} \beta x, & x \in [0, \gamma \nu), \\ \beta x - s(\beta - 1), & x \in [\gamma \nu, s), \nu \in (s(\beta - 1), s), s > 0 \end{cases}
\end{equation}
which is referred to as the “multi-valued Rényi-Parry map” on the flaky region \( \Delta_\beta = [s(\beta - 1), s] \) and has its eventually onto Parry’s \((\beta, \alpha)\)-map \([6]\) with the subinterval \([\nu - s(\beta - 1), \nu] \) as shown in Fig.2. This map realises the flaky quantiser \( Q_f^{\ell}[s(1 - \gamma), \gamma]() \). Let \( b_i, s, \nu, \gamma, \) be its associated bit sequence for the threshold sequence \( \nu^i = \nu_1\nu_2 \ldots \nu_L \), defined by

\[
b_i, s, \nu, \gamma = Q_{\gamma \nu}(s^{\nu} - 1, s(x)) = \begin{cases} 0, & s^{\nu} - 1, s(x) \in [0, \gamma \nu_i), \\ 1, & s^{\nu} - 1, s(x) \in [\gamma \nu_i, s). \end{cases}
\]

Then we get \( x = s(\beta - 1) \sum_{i=1}^{L} b_i, s, \nu, \gamma \), \( \gamma + \gamma^{L} s^{(\nu \gamma)^{L}}(x) \) and its decoded value \( \hat{x}_L, s, \nu, \gamma \) defined by

\[
\hat{x}_L, s, \nu, \gamma = s(\beta - 1) \sum_{i=1}^{L} b_i, s, \nu, \gamma \), \( \gamma + \gamma^{L} s^{(\nu \gamma)^{L}}(x) \) and its decoded value \( \hat{x}_L, s, \nu, \gamma \) defined by

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2. Negative \( \beta \)-Map \([10, 12] \) : We get

\[ R_{\beta, \nu, \gamma}(x) = -\beta x + s[1 + (\beta - 1) Q_{\nu}(x)] = \begin{cases} s - \beta x, & x \in [0, \gamma \nu), \\ \beta s - \beta x, & x \in [\gamma \nu, s). \end{cases} \]

which is another “multi-valued Rényi-Parry map” on the flaky region \( \Delta_\beta = [s(\beta - 1), s] \) realising \( Q_f^{\ell}[s(1 - \gamma), \gamma]() \) and has its eventually onto Parry’s \((\beta, \alpha)\)-map \([6]\) with the subinterval \([\nu - s, \beta - \nu] \) as shown in Fig. 3. Let \( b_i, R_{\beta, \nu, \gamma}, \) be the associated bit sequence for the threshold sequence \( \nu^i \), defined by

\[
b_i, R_{\beta, \nu, \gamma} = Q_{\gamma \nu}(R_{\beta, \nu}^{\nu} - 1, s(x)) = \begin{cases} 0, & R_{\beta, \nu}^{\nu} - 1, s(x) \in [0, \gamma \nu_i), \\ 1, & R_{\beta, \nu}^{\nu} - 1, s(x) \in [\gamma \nu_i, s). \end{cases}
\]

Then we get \( x = -(\gamma)^{L} s^{(\nu \gamma)^{L}}(x) - s L \sum_{i=1}^{L} f_i, R_{\beta, \nu}, (\gamma)^{L}, \) where \( f_i, R_{\beta, \nu}^{\nu} = 1 + b_i, R_{\beta, \nu}^{\nu}, (\beta - 1) \). Such a negative \( \beta \)-expansion defines a new A/D converter called a negative \( \beta \)-encoder which facilitates the implementation of stable analog circuits. Figures 2 \([11, 12] \) and 3 \([11, 12] \) show a typical \( \beta \)-expansion’s attractor of Eqs. (2) and (4), respectively.

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References


