DESINGULARIZATION AND SINGULARITIES OF SOME MODULI SCHEME OF SHEAVES ON A SURFACE

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Abstract. Let $X$ be a nonsingular projective surface over $\mathbb{C}$, and $H_-$ and $H_+$ be ample line bundles on $X$ in adjacent chamber of type $(c_1, c_2)$. Let $0 < a_- < a_+ < 1$ be adjacent minichambers, which are defined from $H_-$ and $H_+$, such that the moduli scheme $M(H_-)$ of rank-two $a_-$-stable sheaves with Chern classes $(c_1, c_2)$ is non-singular. We shall construct a desingularization of $M(a_+)$ by using $M(a_-)$. As an application, we study whether singularities of $M(a_+)$ are terminal or not in some cases where $X$ is ruled or elliptic.

1. Introduction

Let $X$ be a projective non-singular surface over $\mathbb{C}$, $H$ an ample line bundle on $X$. Denote by $M(H)$ the coarse moduli scheme of rank-two $H$-stable sheaves with fixed Chern class $(c_1, c_2) \in \text{NS}(X) \times \mathbb{Z}$. In this paper we think about singularities and desingularization of $M(H)$ from the view of wall-crossing problem of $H$ and $M(H)$.

Let $H_-$ and $H_+$ be ample line bundles on $X$ separated by only one wall of type $(c_1, c_2)$. For a parameter $a \in (0, 1)$, one can define the $a$-stability of sheaves in such a way that $a$-stability of sheaves with fixed Chern class equals $H_-$-stability (resp. $H_+$-stability) if $a$ is sufficiently close to 0 (resp. 1), and there is a coarse moduli scheme $M(a)$ of rank-two $a$-stable sheaves with Chern classes $(c_1, c_2)$. Let $a_-$ and $a_+ \in (0, 1)$ be parameters which are separated by only one miniwall. Assume $M_- = M(a_-)$ is non-singular. One can find such $a_-$ when $X$ is ruled or elliptic. We construct a desingularization $\tilde{\pi}_+ : \tilde{M} \to M_+$ of $M_+ = M(a_+)$ by using $M_-$ and wall-crossing methods, and apply it to consider whether singularities of $M_+$ are terminal or not when $X$ is ruled or elliptic.

Let $\overline{M}(H)$ denote the Gieseker-Maruyama compactification of $M(H)$. By [10], when $X$ is minimal and its Kodaira dimension is positive, $\overline{M}(H)$ has the nef canonical divisor if $\dim \overline{M}(H)$ equals its expected dimension and if $H$ is sufficiently close to $K_X$. Thus, to understand minimal models of a moduli scheme of stable sheaves, it can be meaningful to study singularities on $\overline{M}(H)$. As a problem to be solved, it is desirable to extend results in this article to the case where $M_-$ is not necessarily non-singular but its singularities are terminal (Remark 2.5).

Notation. For a $k$-scheme $S$, $X_S$ is $X \times S$ and $\text{Coh}(X_S)$ is the set of coherent sheaves on $X_S$. For $s \in S$ and $E_s \in \text{Coh}(X_S)$, $E_s$ means $E \otimes k(s)$. For $E$ and $F \in \text{Coh}(X)$, $\text{ext}^i(E, F) := \dim \text{Ext}^i_X(E, F)$ and $\text{hom}(E, F) = \dim \text{Hom}_X(E, F)$. $\text{Ext}^i_X(E, E)^0$ indicates $\text{Ker}(\text{tr} : \text{Ext}^i(E, E) \to \text{H}^0(\mathcal{O}_X))$. For $\eta \in \text{NS}(X)$, we define $W^\eta \subset \text{Amp}(X)$ by $\{ H \in \text{Amp}(X) \mid H \cdot \eta = 0 \}$. 

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We begin with background materials. Let $H_-$ and $H_+$ be ample divisors lying in neighboring chambers of type $(c_1, c_2) \in \NS(X) \times \mathbb{Z}$, and $H_0$ an ample divisor in the wall $W$ of type $(c_1, c_2)$ which lies in the closure of chambers containing $H_-$ and $H_+$ respectively. (Refer to [8] about the definition of wall and chamber.) Assume that $M = H_+ - H_-$ is effective. For a number $a \in [0, 1]$ one can define the $a$-stability of a torsion-free sheaf $E$ using

$$P_a(E(n)) = \{ (1 - a)\chi(E(H_-)(nH_0)) + a\chi(E(H_+(nH_0))) \}/ \text{rk}(E).$$

There is the coarse moduli scheme $\overline{M}(a)$ of rank-two $a$-semistable sheaves on $X$ with Chern classes $(c_1, c_2)$. Denote by $M(a)$ its open subscheme of $a$-stable sheaves. When one replace $H_\pm$ by $NH_\pm$ if necessary, $M(0)$ (resp. $M(1)$) equals the moduli scheme of $H_-$-semistable (resp. $H_+$-semistable) sheaves. There exist finite numbers $a_1 \ldots a_t \in (0, 1)$ called minichambers such that $M(a)$ and $M(a)$ changes only when $a$ passes a minwall. Refer to [2, Section 3] for details. Fix numbers $a_-$ and $a_+$ separated by the only one miniwall, and indicate $M_\pm = M(a_\pm)$ and $M_\pm = M(a_\pm)$ for short. From [9, Section 2], the subset

$$M_- \supset P_- = \{ [E] \mid E \text{ is not } a_+ \text{-semistable} \}
\supset \text{resp. } M_+ \supset P_+ = \{ [E] \mid E \text{ is not } a_- \text{-semistable} \}$$

is contained in $M_-$ (resp. $M_+$) and endowed with a natural closed subscheme structure of $M_-$ (resp. $M_+$). Let $\eta$ be a element of

$$A^+(W) = \{ \eta \in \NS(X) \mid \eta \text{ defines } W, 4c_2 - c_1^2 + \eta_2 \geq 0 \text{ and } \eta \cdot H_+ > 0 \}.$$

After [2, Definition 4.2] we define

$$T_\eta = M(1, (c_1 + \eta)/2, n) \times M(1, (c_1 - \eta)/2, m),$$

where $n$ and $m$ are numbers defined by

$$n + m = c_2 - (c_1 - \eta)^2/4 \text{ and } n - m = \eta \cdot (c_1 - K_X)/2 + (2a_0 - 1)\eta \cdot (H_+ - H_-),$$

and $M(1, (c_1 + \eta)/2)$ is the moduli scheme of rank-one torsion-free sheaves on $X$ with Chern classes $((c_1 + \eta)/2, n)$. If $F_{T_\eta}$ (resp. $G_{T_\eta}$) is the pull-back of a universal sheaf of $M(1, (c_1 + \eta)/2, n)$ (resp. $M(1, (c_1 - \eta)/2, m)$) to $X_{T_\eta}$, then we have an isomorphism

$$P_- \simeq \prod_{\eta \in A^+(W)} \mathbb{P}_{T_\eta} \left( \text{Ext}^1_X(F_{T_\eta}, G_{T_\eta}(K_X)) \right)$$

(1)

from [9, Section 5].

**Proposition 2.1** ([9] Proposition 4.9). The blowing-up of $M_-$ along $P_-$ agrees with the blowing-up of $M_+$ along $P_+$. So we have blowing-ups

$$M_- \xrightarrow{\pi_-} B_{P_-}(M_-) = B_{P_+}(M_+) \xrightarrow{\pi_+} M_+.$$

By taking $4c_2 - c_1^2$ to be sufficiently large with respect to $H_-$ and $H_+$, we can assume from [6] and [7] that $M_\pm \supset \text{Sing}(M_\pm) := \{ E \mid \text{ext}^2(E, E)^0 \neq 0 \}$ satisfies $\text{codim}(M_\pm, \text{Sing}(M_\pm)) \geq 2$ and that $P_\pm \subset M_\pm$ is nowhere dense, and hence both $M_-$ and $M_+$ are normal l.c.i. schemes and birationally equivalent. Suppose that
\( A^+(W) = \{ \eta \} \) for simplicity and denote \( T_\eta = T \). From Hironaka’s desingularization theorem, there is a sequence of blowing-ups
\[
M_N \rightarrow M_{N-1} \cdots \rightarrow M_-
\]
along non-singular centers \( Z_i \subset M^i \) such that the ideal sheaf of \( \mathcal{O}_{M_\eta} \) generated by pull-back of the ideal sheaf of \( P_- \subset M_- \) is invertible.

**Claim 2.2.** If we set
\[
l_i = \max \{ \text{ext}^i(F_t, G_t(K_X)) \mid t \in T \},
\]
then we can take the center \( Z_i \) in (2) so that the dimension of \( Z_i \) is not greater than \( l_i - 1 + \dim T \).

**Proof.** Since one can readily show \( \text{ext}^2(F_t, G_t(K_X)) = \text{hom}(G_t, F_t) = 0 \) for all \( t \in T \), (1) implies that \( P_- \) is embedded in a \( \mathbb{P}^1 \)-bundle over \( T \). Thus for \( s \in P_- \), the rank of \( \Omega_{P_-} \otimes k(s) \) is not greater than \( \dim T + l_i - 1 \). From the exact sequence
\[
CN_{P_/|M_-} \rightarrow \Omega_{M_-}|_{P_-} \rightarrow \Omega_{P_-} \rightarrow 0,
\]
we can choose local coordinates \( g_i \in \mathcal{O}_{M_-} \) so that \( g_i \) lies in \( I_{P_-} \) for \( i \leq \dim M_- - (\dim T + l_i - 1) \). From [1, Thm. 1.10], one can choose the center \( Z_i \) in such a way that the ideal sheaf of \( Z_i \) contains the weak transform of \( I_{P_-} \) by \( M_i \rightarrow M_- \), say \( I_t \). If \( y \) is a local generator of the exceptional divisor of \( M_1 \rightarrow M_- \), then \( g_i/y \) \( (i \leq \dim M_- - (\dim T + l_i - 1)) \) are partial coordinating parameters of \( M_i \) and belong to \( I_1 \). Since \( I_{Z_t} \) contains \( I_1 \), the claim holds for \( i = 1 \). For general \( i \), one can verify the claim in the same way. \( \square \)

From Proposition 2.1, we obtain a morphism
\[
M_N \rightarrow B(M) := B_{P_-}(M_-) = B_{P_+}(M_+) \rightarrow M_+
\]
and a diagram
\[
\begin{array}{ccc}
\hat{M} := M_N & \xrightarrow{\pi} & \hat{P}_+ \\
\downarrow \pi_- & & \downarrow \pi_+ \\
M_- & \xrightarrow{\pi_-} & B(M) \xrightarrow{\pi_+} M_+
\end{array}
\]
Therefore we can regard \( \hat{M} \) as a desingularization of \( M_+ \).

Next let us calculate \( \pi_-^* K_{M} - \pi_+^* K_{M_+} \). If we denote by \( D_i \subset \hat{M} \) the pull-back of the exceptional divisor of \( M^i \rightarrow M^{i-1} \), then
\[
K_{\hat{M}} - \pi_+^* K_{M_+} = \sum_i [\dim M_- - \dim Z_i - 1] D_i.
\]

Next consider \( \pi_-^* (K_{M_-}) - \pi_+^* (K_{M_+}) \). By the proof of Proposition 2.1, which uses elementary transform, we have the following.

**Proposition 2.3.** Denote the exceptional divisor \( \pi_-^{-1}(P_-) = \pi_+^{-1}(P_+) \subset B(M) \) by \( D \). Suppose we have a universal family \( E_{M_-} \in \text{Coh}(X_{M_-}) \) of \( M_- \) and a universal family \( E_{M_+} \in \text{Coh}(X_{M_+}) \) of \( M_+ \). If \( p : D \rightarrow P_+ \rightarrow T \) is a natural map, then there
are line bundles \( L_\pm \) on \( P_\pm \) and a line bundle \( L_0 \) on \( B(M) \) such that we have exact sequences
\[
0 \to \pi_* E_{M_+}^+ \otimes L_0 \to \pi_* E_{M_-}^- \to p^* G_T \otimes \pi_* L_+ \to 0 \tag{5}
\]
in \( \text{Coh}(X_{B(M)}) \) and
\[
0 \to \pi_* F_T \otimes \pi_* L_- \to \pi_* (E_{M_-}^-)|_{X_D} \to p^* G_T \otimes \pi_* L_+ \to 0 \tag{6}
\]
in \( \text{Coh}(X_D) \).

The exact sequence (6) is the relative a_±-Harder Narashimhan filtration of \( E_{M_-}^- \).

Here we remark that generally a universal family of \( M_- \) exists only \( \text{etale-locally} \),
but one can generalize this proposition to general case with straightforward labor.
Suppose \( L_\pm \) and \( L_0 \) in this proposition are trivial for simplicity. From (5)
\[
\pi_* K_{M_-} - \pi_* K_{M_+}
= \pi_* \det \mathbf{RHom}_{X_{B(M)/B(M)}}(E_{M_-}^-, E_{M_-}^-) - \pi_* \det \mathbf{RHom}_{X_{B(M)/B(M)}}(E_{M_+}^+, E_{M_+}^+)
= \det \mathbf{RHom}_{X_{B(M)/B(M)}}(\pi_* E_{M_-}^-, \pi_* E_{M_-}^-)
- \det \mathbf{RHom}_{X_{B(M)/B(M)}}(\pi_* E_{M_+}^+, \pi_* E_{M_+}^+)
= \det \mathbf{RHom}_{X_{B(M)/B(M)}}(E_{B(M)}^-, G_D) + \det \mathbf{RHom}_{X_{B(M)/B(M)}}(E_{B(M)}^+, \pi_* G_T)
+ \det \mathbf{RHom}_{X_{B(M)/B(M)}}(\pi_* G_T, E_{B(M)}^+)
= \det \mathbf{RHom}_{X_{B(M)/B(M)}}(E_{B(M)}^-, G_D) + \det \mathbf{RHom}_{X_{B(M)/B(M)}}(G_D, E_{B(M)}^+).
\]
If \( i : D \hookrightarrow B(M) \) is inclusion, then by (6)
\[
\det \mathbf{RHom}_{X_{B(M)/B(M)}}(E_{B(M)}^-, G_D) = \det i_* \mathbf{RHom}_{X_D/D}(E_{B(M)}^-, G_D) =
\det i_* \mathbf{RHom}_{X_D/D}(F_D, G_D) + \det i_* \mathbf{RHom}_{X_D/D}(G_D, G_D). \tag{7}
\]
Since \( \det \mathcal{O}_D = D \), (7) equals \( [\chi(F_t, G_t) + \chi(G_t, G_t)] D \) for any \( t \in D \). By the Serre duality
\[
\det \mathbf{RHom}_{X_{B(M)/B(M)}}(G_D, E_{B(M)}^+)
= \det \mathbf{RHom}_{B(M)}(\mathbf{RHom}_{X_{B(M)/B(M)}}(E_{B(M)}^+, G_D(K_X)), \mathcal{O}_{B(M)})
= - \det \mathbf{RHom}_{X_{B(M)/B(M)}}(E_{B(M)}^+, G_D(K_X))
= - \det i_* \mathbf{RHom}_{X_D/D}(E_{B(M)}^+, G_D(K_X))
= - [\chi(F_t, G_t(K_X)) + \chi(G_t, G_t(K_X))] D = -[\chi(G_t, F_t) + \chi(G_t, G_t)] D.
\]
Therefore
\[
\pi_* K_{M_-} - \pi_* K_{M_+} = [\chi(F_t, G_t) - \chi(G_t, F_t)] D = 2(c_1(F_t) - c_1(G_t)) \cdot K_X.
\tag{8}
\]
Moreover, we put
\[
\tilde{\pi}^* D = \sum_{i=0}^{N-1} \lambda_i D_i. \tag{9}
\]
When \( \dim M_- - (l_1 - 1 + \dim T) > 0 \), all \( \lambda_i \) are 1. Indeed, the proof of Claim 2.2 says that some element \( g \in I_{P_-} \) satisfies that if \( y \) is a local generator of the exceptional divisor of \( M_1 \to M_- \), then \( g/y \) is a partial coordinating parameter of \( M_1 \). Thus the pull-back of \( I_{P_-} \) by \( M_1 \to M_- \) is divided by \( y \), but cannot be divided by \( y^2 \), which implies \( \lambda_1 = 1 \). One can show \( \lambda_i = 1 \) similarly. Consequently, from (4), (8) and (9), we have shown the following.

**Proposition 2.4.** In the diagram (3) it holds that

\[
K_M - \hat{\pi}^* K_{M_+} = \sum_{i=0}^{N-1} \left[ \dim M_- - \dim Z_i - 1 + \lambda_i 2(c_1(F_i) - c_1(G_i)) \cdot K_X \right] D_i. 
\]

with \( \lambda_i \geq 1 \). If \( \dim M_- > l_1 - 1 + \dim T \), then \( \lambda_i = 1 \) and

\[
\dim M_- - \dim Z_i - 1 + 2\lambda_i (c_1(F_i) - c_1(G_i)) \cdot K_X \geq \dim M_- - (l_1 - 1 + \dim T) - 1 + 2(c_1(F_i) - c_1(G_i)) K_X.
\]

One can use this proposition to verify whether singularities in \( M_+ \) is terminal or not.

**Remark 2.5.** It is desirable to extend this article to the case where \( M_- \) is not necessarily non-singular but its singularities are terminal. It is a problem that we can not use (4) since \( M_- \) is not non-singular.

### 3. Examples: ruled or elliptic surface

We shall give examples of \( M_\pm \) with \( M_- \) non-singular. If a surjective morphism \( X \to C \) to a nonsingular curve \( C \) exists, then by [3, p.142] we have a \((c_1, c_2)\)-suitable polarization, that is, an ample line bundle \( H \) such that \( H \) does not lie on any wall of type \((c_1, c_2)\), and for any wall \( W = W^\eta \) of type \((c_1, c_2)\), we have \( \eta \cdot f = 0 \) or \( \text{Sign}(f \cdot \eta) = \text{Sign}(H \cdot \eta) \). From [3, p.159, p.201], if \( X \) is a ruled surface or an elliptic surface, then any rank-two sheaf \( E \) of type \((c_1, c_2)\) which is stable with respect to \((c_1, c_2)\)-suitable polarization is good, i.e. \( \text{Ext}^2(E, E)^0 = 0 \).

(A) First we suppose that \( X \) is a (minimal) ruled surface. When \( c_1 \cdot f \) is odd \( M(H) \) is empty for \((c_1, c_2)\)-suitable polarization. Thus we assume \( c_1 = 0 \). If a rank-two sheaf \( E \) of type \((c_1, c_2)\) is stable with respect to a polarization \( H \) such that \( H \cdot K_X < 0 \), then \( E \) is good and so \( M(H) \) is nonsingular. Hence we assume that \( W^{K_X} \cap \text{Amp}(X) \neq \emptyset \), so \( 2 \leq g = g(C) \) and \( e(X) \leq 2g - 2 \) from the description of \( \text{Amp}(X) \) [4, Prop. V.2.21]. Since \( \dim \text{NS}(X) = 2 \), if we move polarization \( H \) from a \((c_1, c_2)\)-suitable one, then \( M(H) \) may begin to admit singularities when \( H \) passes the wall \( W^{K_X} \). Let \( H_- \) and \( H_+ \) be ample line bundles separated by only one wall \( W^{K_X} \). \( M(H_-) \) is non-singular, and \( E^+ \in P_+ \) has a non-trivial exact sequence

\[
0 \to G = L \otimes I_{Z_1} \to E^+ \to F = L^{-1} \otimes I_{Z_2} \to 0
\]

with \( -2L \sim mK_X \). About this filtration we have \( \text{Ext}^2(E^+, E^+) = 0 \) since \( p_g(X) = 0 \) (See [5, p. 49] for \( \text{Ext}_+ \)), and

\[
\text{ext}^2(E^+, E^+) = \text{ext}^2(E^+, E^+) = \text{ext}^2(L \otimes I_{Z_1}, L^{-1} \otimes I_{Z_2}) = \text{hom}(I_{Z_1}, O(K_X + 2L) \otimes I_{Z_2}).
\]
Since $W^K_X$ defines a wall, $H^0(\mathcal{O}(K_X + 2L)) = 0$ unless $2L + K_X = 0$. Hence $\text{ext}^2(E^+, E^+) \neq 0$ if and only if $-2L = K_X$ and $Z_l \subset Z_r$. As a result when one defines $a$-stability using $H^i_+$,

$$
\chi^a(E^+) - \chi^a(L \otimes I_{Z_l}) = Aa + B + l(Z_l)
$$

for some constant $A$ and $B$, and so the moduli scheme $M(a)$ of $a$-stable sheaves begins to admit singularities just when $a$ passes a miniwall $a_0$ defined by

$$
l(Z_l) = \begin{cases} 
c_2/2 - (g - 1) & \text{if } c_2 \text{ is even} \\
(c_2 - 1)/2 - (g - 1) & \text{if } c_2 \text{ is odd} 
\end{cases}
$$

Let $a_-$ and $a_+$ be minichambers separated by only one miniwall $a_0$. $M(a_+) = M_+$ has singularities along $P_+ \times_T T'$, where

$$
T' = \{(L \otimes I_{Z_l}, L^{-1} \otimes I_{Z_r}) \mid -2L = K_X\}_\text{red} \subset M(1, K_X/2, l(Z_l)) \times M(1, -K_X/2, l(Z_r))
$$

(B) Suppose that $X$ is an elliptic surface with a section $\sigma$ and $c_1 = \sigma$. In contrast to ruled surfaces, $K_X^2 = 0$ and so $W^K_X \cap \text{Amp}(X)$ is always empty, though one can study some singularities appearing in $M(H)$ by Proposition 2.4. Let $\pi : X \to C$ be an elliptic fibration, $f \in \text{NS}(X)$ its fiber class, $d = -\text{deg} R^1 \pi_*(\mathcal{O}_X) - \sigma^2 \geq 0$. We have a natural map to a ruled surface $\kappa : X \to \mathbb{P}(\pi_*(\mathcal{O}(2\sigma))) = \mathbb{P}(\mathcal{E}_2)$. Since $\kappa_* (\sigma)$ is a section of $\mathbb{P}(\mathcal{E}_2)$, and since the pull-back of an ample line bundle by a finite map is ample, $L = af$ satisfies $W^{2L-c_1} \cap \text{Amp}(X) \neq \emptyset$ if $a > 0$ from the description of the ample cone of a ruled surface. Let $c_1$ be $\sigma$ and $c_2 = (c_1 - L) \cdot L = a$. Then any sheaf $E$ with non-trivial exact sequence

$$
0 \longrightarrow F = L \longrightarrow E \longrightarrow G = L^{-1} \otimes c_1 \longrightarrow 0,
$$

(12)
whose Chern class equals $(c_1, c_2)$, is stable with respect to a $(c_1, c_2)$-suitable ample line bundle. Indeed, $(2L-c_1) \cdot f < 0$ and so $\pi_*(\mathcal{O}(2L-c_1)) = 0$ and $R^1 \pi_*(\mathcal{O}(2L-c_1))$ commutes with base change. Thus the exact sequence

$$
0 \longrightarrow H^1(C, \pi_*(\mathcal{O}(2L-c_1))) \longrightarrow H^1(X, \mathcal{O}(2L-c_1)) \longrightarrow H^0(E, R^1 \pi_*(\mathcal{O}(2L-c_1)))
$$

shows that the restriction of the exact sequence (12) to a general fiber is non-trivial, and so a corollary of Artin's theorem for vector bundles on an elliptic curve [3, p. 89] and a basic property of a suitable polarization [3, p. 144] deduce that $E$ is stable with respect to a suitable polarization. Thereby such $E$ is good. Let $H_- and H_+$ be ample line bundles which lie in no wall of type $(c_1, c_2)$ with $(2L-c_1) \cdot H_- < 0 < (2L-c_1) \cdot H_+$. One can define $a$-stability by them. Let $a_0$ be a miniwall such that $\chi^{a_0}(\mathcal{O}(L)) = \chi^{a_0}(\mathcal{O}(2L-c_1))$, $a_- < a_0 < a_+ minichambers$, and $M_\pm = M(a_\pm)$. Then some connected components of $P_+ \subset M_+$ contains any sheaf $E$ with non-trivial exact sequence (12), and some neighborhood of them in $M_-$ is non-singular. It induces a desingularization of some open neighborhood of connected components $K_+$ of $P_+$ consisting of sheaves $E^+$ with a non-trivial exact sequence

$$
0 \longrightarrow L^{-1} \otimes c_1 \longrightarrow E^+ \longrightarrow L \longrightarrow 0
$$
as in Section 2.

We have in case of (A) $\text{ext}^1(G, F) \leq 1$, and in case of (B) $\text{ext}^1(G, F) = h^0(c_1 - 2L + K_X) - \chi(c_1 - 2L) \leq 2c_2 + C(X)$ with some constant $C(X)$ independent of $c_2$ because $h^0(c_1 - 2L + K_X) = 0$ if $a = c_2$ is sufficiently large. Thus in both
cases one can show that, if $c_2$ is sufficiently large, then all singularities of $M_+$ along above-mentioned sheaves are terminal.

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