An integral formula for powers of the Bergman kernel on representative bounded homogeneous domains

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Abstract. The representative domain gives a nice realization for a bounded homogeneous domain. For the classical domain, its representative domain is a constant multiple of the standard realization. We show that the integral of the negative power $K^{-s}$ of the normalized Bergman kernel $K$ of the domain equals the reciprocal of a polynomial of $s$, called the Hua polynomial, whose roots are negative rational numbers determined explicitly from structure of the holomorphic automorphism group of the domain.

Introduction.

In [5], Hua proved fascinating formulas about harmonic analysis on classical domains. For instance, if we write $R_I(m,n)$ $(1 \leq n \leq m)$ for the classical domain \{ $Z \in \text{Mat}(m,n; \mathbb{C})$; $I - ZZ^*$ is positive definite \} of type I, we find the following integral evaluation in [5, p. 40]:

$$\int_{R_I(m,n)} \det(I - ZZ^*)^\lambda \, dV(Z) = \pi^{mn} \cdot \frac{\prod_{j=1}^{n} \Gamma(\lambda + j) \prod_{k=1}^{m^{n}} \Gamma(\lambda + k)}{\prod_{l=1}^{m+n} \Gamma(\lambda + l)} (\lambda > -1),$$

(1)

where $dV$ denotes the Lebesgue measure with respect to the natural complex coordinate. In particular, we get the volume $\text{Vol}(R_I(m,n))$ of the domain $R_I(m,n)$ by putting $\lambda = 0$. Furthermore, Hua showed similar integral formulas for the other classical domains, where the results are always expressed as quotients of products of the Gamma functions. Now we observe that the right-hand side of (1) is rewritten as

$$\pi^{mn} \prod_{j=1}^{n} \frac{\Gamma(\lambda + j)}{\Gamma(\lambda + m + n + 1 - j)} = \frac{\pi^{mn}}{\prod_{j=1}^{n} (\lambda + j)_{m+n+1-2j}},$$

where $(a)_p$ denotes the Pochhammer polynomial: $(a)_p = a(a+1)\cdots(a+p-1)$. Note that the denominator is a polynomial of $\lambda$ with the degree being $\sum_{j=1}^{n} (m +
\( n + 1 - 2j = mn = \dim_{\mathbb{C}} R_{I}(m, n) \). This observation is valid for each classical domain. Indeed, using theory of Jordan triple system, Yin, Lu and Roos [13] generalized Hua’s result to bounded symmetric domains as follows. Let \( \mathcal{S} \) be the Harish-Chandra realization of an irreducible bounded symmetric domain of dimension \( N \), and \( \mathcal{N}(Z,W) \) be the associated generic minimal polynomial (if \( \mathcal{S} = R_{I}(m, n) \), then \( \mathcal{N}(Z,W) = \det(I - ZW^{*}) \)). Then it is shown [13, (2.5)] that

\[
\int_{\mathcal{S}} \mathcal{N}(Z,Z)^{\lambda} dV(Z) = \frac{p(0)}{p(\lambda)} \text{Vol}(\mathcal{D}) \quad (\Re \lambda > -1),
\]

where \( p(\lambda) \) is a polynomial of degree \( N \), called the Hua polynomial, whose roots are negative half integers determined explicitly.

In this article, we shall consider further generalization of Hua’s result to a bounded homogeneous domain (BHD) \( \mathcal{U} \). Since there is no Jordan triple system corresponding to a non-symmetric BHD, it is a non-trivial question what the generalization should be. We recall that, for the symmetric case \( \mathcal{U} = \mathcal{S} \), the Bergman kernel \( K_{S}(Z,W) \) equals \( \text{Vol}(\mathcal{S})^{-1} \mathcal{N}(Z,W)^{-\gamma_{S}} \) where \( \gamma_{S} \) is a certain positive integer. Thus, for a general BHD \( \mathcal{U} \), we substitute the reciprocal \( \{\text{Vol}(\mathcal{U})K_{U}(Z,W)\}^{-1} \) of the normalized Bergman kernel for the generic minimal polynomial \( \mathcal{N}(Z,W) \). On the other hand, results in [6] suggest that the representative domain can be regarded as a standard realization of BHD like the Harish-Chandra realization of bounded symmetric domain. Eventually, we obtain the following result: Let \( \mathcal{U} \) be a representative BHD of dimension \( N \). Then we can determine rational numbers \( a_{1}, a_{2}, \ldots, a_{N} \) so that

\[
\int_{\mathcal{U}} \{\text{Vol}(\mathcal{U})K_{U}(\zeta,\zeta)\}^{-s} dV(\zeta) = \frac{\text{Vol}(\mathcal{U})}{F(s)} \quad (\Re s > -\min a_{i}),
\]

(2)

where

\[
F(s) := \prod_{i=1}^{N}(1 + \frac{s}{a_{i}}).
\]

(3)

Let \( \mathcal{D} \) be a (not necessarily bounded) domain biholomorphic to the representative BHD \( \mathcal{U} \). Thanks to a canonical nature of the Bergman kernel \( K_{U} \) (Theorem 1), the formula (2) is equivalent to

\[
\int_{\mathcal{D}} |F(s)K_{D}(z,w)^{s+1}|^{2}K_{D}(z,z)^{-s}dV(z) = F(s)K_{D}(w,w)^{s+1}
\]

(4)

(w \in \mathcal{D}, \Re s > -\min a_{i}),

which implies that the weighted Bergman space \( L_{a}^{2}(\mathcal{D},K_{D}(z,z)^{-s}dV(z)) \) has the reproducing kernel given by \( F(s)K_{D}(z,w)^{s+1} \). We should notice that the statement
in this form is already known essentially in [4] (see also [10]) where $\mathcal{D}$ is a homogeneous Siegel domain, and $F(s)$ is expressed as a quotient of products of the Gamma functions (see Section 3). Nevertheless, we think that the formulation (2) in terms of the representative domain as well as the expression of $F(s)$ as a polynomial is worth claiming to be new.

§1. Preliminaries.

1.1. Let $\mathcal{D} \subset \mathbb{C}^N$ be a bounded complex domain, and $K_\mathcal{D}$ the Bergman kernel of $\mathcal{D}$. If $K_\mathcal{D}(z, w) \neq 0$ for $z, w \in \mathcal{D}$, we set

$$T_\mathcal{D}(z, w) := \left( \frac{\partial^2}{\partial z_i \partial \overline{w}_j} \log K_\mathcal{D}(z, w) \right)_{i,j} \in \text{Mat}(N, \mathbb{C}).$$

Take $p \in \mathcal{D}$ and assume that $K_\mathcal{D}(z, p) \neq 0$ for all $z \in \mathcal{D}$. Then we define the Bergman mapping $\sigma_p : \mathcal{D} \to \mathbb{C}^N$ by

$$\sigma_p(z) := T_\mathcal{D}(p, p)^{-1/2} \text{grad}_\overline{w} \log \frac{K_\mathcal{D}(z, w)}{K_\mathcal{D}(p, w)}|_{w=p} \quad (z \in \mathcal{D}),$$

where $\text{grad}_\overline{w} f(w) := (\frac{\partial f}{\partial \overline{w}_1}, \frac{\partial f}{\partial \overline{w}_2}, \ldots, \frac{\partial f}{\partial \overline{w}_N})$ for an anti-holomorphic function $f$ on $\mathcal{D}$. A domain $\mathcal{U}$ is called a representative domain if it is the image $\sigma_p(\mathcal{D})$ of some Bergman mapping $\sigma_p : \mathcal{D} \to \mathbb{C}^N$.

1.2. In what follows, we assume that a bounded domain $\mathcal{D}$ is homogeneous, that is, the holomorphic automorphism group $\text{Aut}(\mathcal{D})$ acts on $\mathcal{D}$ transitively. The notion of the representative domain works very well for such BHDs. Since $K_\mathcal{D}(z, p) \neq 0$ for any $z, p \in \mathcal{D}$ in this case, the Bergman mapping $\sigma_p : \mathcal{D} \to \mathbb{C}^N$ is always well-defined. It is shown in [12, Theorem 4.7] and [6, Theorem 3.3] that $\sigma_p(\mathcal{D})$ is a bounded domain and $\sigma_p$ gives a biholomorphism from $\mathcal{D}$ onto $\sigma_p(\mathcal{D})$. Thus, any BHD $\mathcal{D}$ is realized as a representative BHD $\mathcal{U}$, which is unique up to unitary linear transform by [6, Proposition 2.1, Lemma 3.2]. A representative BHD $\mathcal{U}$ is characterized by the following properties: (U1) $0 \in \mathcal{U}$, and (U2) $T_\mathcal{U}(\zeta, 0) = I_N \ (\forall \zeta \in \mathcal{U})$. For example, $\sqrt{2} \Delta = \{ z \in \mathbb{C}; |z| < \sqrt{2} \}$ is a representative domain. In general, the Harish-Chandra realization of an irreducible bounded symmetric domain (e.g. a classical domain) coincides with a constant multiple of the representative domain.

1.3. For a representative BHD $\mathcal{U}$, we see from [6, Proposition 3.8] that

$$K(\zeta, 0) = \frac{1}{\text{Vol}(\mathcal{U})} \quad (\forall \zeta \in \mathcal{U}), \quad (5)$$
which is equivalent to the mean value property

\[ f(0) = \frac{1}{{\text{Vol}(\mathcal{U})}} \int_{\mathcal{U}} f(\zeta) dV(\zeta) \quad (f \in L^{2}_{a}(\mathcal{U})). \]

From this observation, we can deduce the following general formula.

**Theorem 1.** For a (not necessarily bounded) domain \( \mathcal{D} \) biholomorphic to a representative BHD \( \mathcal{U} \) and a biholomorphism \( \Phi : \mathcal{D} \to \mathcal{U} \), putting \( a := \Phi^{-1}(0) \in \mathcal{D} \), one has

\[ K_{\mathcal{U}}(\Phi(z), \Phi(w)) = \frac{1}{{\text{Vol}(\mathcal{U})}} \frac{K_{\mathcal{D}}(z, w)K_{\mathcal{D}}(a, a)}{K_{\mathcal{D}}(z, a)K_{\mathcal{D}}(a, w)} \quad (z, w \in \mathcal{D}). \] (6)

**Proof.** By the transformation rule of the Bergman kernel, we have

\[ K_{\mathcal{D}}(z, w) = K_{\mathcal{U}}(\Phi(z), \Phi(w)) \det J(\Phi, z)\overline{\det J(\Phi, w)}. \]

In particular, putting \( w = a \), we have by (5)

\[ K_{\mathcal{D}}(z, a) = \frac{\det J(\Phi, z)\overline{\det J(\Phi, a)}}{{\text{Vol}(\mathcal{U})}}. \]

Similarly, we see that

\[ K_{\mathcal{D}}(a, w) = \frac{\det J(\Phi, a)\overline{\det J(\Phi, w)}}{{\text{Vol}(\mathcal{U})}}. \]

Furthermore, for the case \( z = w = a \), we have

\[ K_{\mathcal{D}}(a, a) = \frac{|\det J(\Phi, a)|^{2}}{{\text{Vol}(\mathcal{U})}}. \]

Substituting these equalities, we obtain (6).

\[ \square \]

**§2. Main result.**

For a representative BHD \( \mathcal{U} \), structure of the holomorphic automorphism group \( \text{Aut}(\mathcal{U}) \) is rather complicated in general, while the Lie algebra \( b \) of the Iwasawa subgroup (maximal connected split solvable Lie subgroup) \( B \subset \text{Hol}(\mathcal{U}) \) has a specific root space decomposition (Theorem 2). The subgroup \( B \) is unique up to inner automorphisms in \( \text{Aut}(\mathcal{U}) \), so that the structure of \( B \) and \( b \) are canonically determined from the BHD \( \mathcal{U} \). Our main result is stated in terms of the dimensions of the root subspaces of \( b \).
2.1. Since the group $B$ acts on the domain $\mathcal{U}$ simply transitively ([11]), we have the linear isomorphism $\iota : b \ni Y \mapsto Y \cdot 0 \in T_0 \mathcal{U} \equiv \mathbb{C}^N$. Let us transfer the complex structure and the Bergman metric $(ds_\mathcal{U}^2)_0$ on $T_0 \mathcal{U}$ to $b$ by means of $\iota$. Let $j : b \rightarrow b$ be a linear map defined in such a way that $\iota(jY) = \sqrt{-1} \iota(Y)$ ($Y \in b$), and $(\cdot | \cdot)_b$ an inner product on $b$ given by $(Y_1 | Y_2)_b := ds_\mathcal{U}^2(\iota(Y_1), \iota(Y_2))_0$ ($Y_1, Y_2 \in b$). Let $a$ be the orthogonal complement of the subspace $[b, b] \subset b$ with respect to $(\cdot | \cdot)_b$. Then $a$ is a commutative Cartan subalgebra of the solvable Lie algebra $b$. For $\alpha \in a^*$, we denote by $b_\alpha$ the root subspace $b_\alpha := \{Y \in b ; [C, Y] = \alpha(C)Y (\forall C \in a)\}$. The number $r := \dim a$ is called the rank of $b$.

**Theorem 2 ([9, Chapter 2, Section 3]).** There exists a basis $\{\alpha_1, \ldots, \alpha_r\}$ of $a^*$ such that $b = b(1) \oplus b(1/2) \oplus b(0)$,

$$b(0) = a \oplus \sum_{1 \leq k \leq m \leq r} \oplus b_{(\alpha_m - \alpha_k)/2}, \quad b(1/2) = \sum_{1 \leq k \leq r} \oplus b_{\alpha_k/2},$$

$$b(1) = \sum_{1 \leq k \leq r} \oplus b_{\alpha_k} \oplus \sum_{1 \leq k \leq m \leq r} \oplus b_{(\alpha_m + \alpha_k)/2}.$$

Let $\{A_1, \ldots, A_r\}$ be the basis of $a$ dual to $\{\alpha_1, \ldots, \alpha_r\}$, and put $E_k := -jA_k$ ($k = 1, \ldots, r$). Then $b_{\alpha_k} = \mathbb{R}E_k$. One has $jb(0) = b(1)$, $jb(1/2) = b(1/2)$ and

$$[b(p), b(q)] \subset b(p + q) \quad (\text{if } p > 1, \text{ then } b(p) := \{0\}) \quad (7)$$

for $p, q = 0, 1/2, 1$.

We note that some root spaces $b_{(\alpha_m \pm \alpha_k)/2}$ or $b_{\alpha_k/2}$ may be zero.

2.2. For $k = 1, \ldots, r$, we set

$$p_k := \sum_{i < k} \dim b_{(\alpha_k - \alpha_i)/2}, \quad q_k := \sum_{m > k} \dim b_{(\alpha_m - \alpha_k)/2}, \quad b_k := (\dim b_{\alpha_k/2})/2.$$

Then we state our main result as follows.

**Theorem 3.** Putting

$$P(s) := \prod_{k=1}^{r} (s(2 + p_k + q_k + b_k) + 1 + q_k/2)_{1+p_k+b_k}, \quad (8)$$

one has

$$\int_{\mathcal{U}} \{Vol(\mathcal{U})K_\mathcal{U}(\zeta, \zeta)\}^s dV(\zeta) = Vol(\mathcal{U}) \frac{P(0)}{P(s)}, \quad (9)$$

where $s$ is a complex number for which the real part of every factor of $P(s)$ is positive.
The polynomial $F(s)$ in (2) is $P(s)/P(0)$. Indeed, the degree of $P(s)$ is $\sum_{k=1}^{r}(1 + p_k + q_k) = \dim b(0) + (\dim b(1/2))/2 = (\dim b)/2$, which is nothing but $N = \dim C U$. For the case $U$ is (a constant multiple of) $R_I(m,n)$, we have $p_k = 2(k - 1)$, $q_k = 2(n - k)$ and $b_k = m - n$, so that Theorem 3 is compatible with (1).

§3. Evaluation of integrals on a homogeneous Siegel domain.

The solvable group $B$ acts on the representative BHD $U$ simply transitively, while we shall see that the same $B$ acts on a certain Siegel domain $D$ as an affine transformation group. The domain $D$ is biholomorphic to $U$. This is a generalization of the relation between the upper half plane and the unit disc in the complex plane $\mathbb{C}$. In this section, making use of Theorem 1, we reduce the integral (9) over $U$ to integrals over the Siegel domain $D$, whose evaluation is essentially due to Gindikin [3] and [4].

3.1. Thanks to (7), we see that $b(0)$ and $b(1)$ are a subalgebra and a commutative ideal of $b$ respectively, and that the group $B(0) := \exp b(0)$ of $B$ acts on $b(1)$ by the adjoint representation. Putting $E := E_1 + \cdots + E_r \in b(1)$, we set $\Omega := B(0) \cdot E \subset b(1)$. Then $\Omega$ is a regular open convex cone in $b(1)$, on which the group $B(0)$ acts simply transitively. The linear map $j|_{b(1/2)}$ gives a complex structure on the space $b(1/2)$. We definite the Hermitian map $Q : b(1/2) \times b(1/2) \rightarrow b(1)_{\mathbb{C}}$ on the complex vector space $(b(1/2), j)$ by $Q(u, u') := ([ju, u'] + i[u, u'])/4$. Let us consider the Siegel domain $D \subset b(1)_{\mathbb{C}} \times (b(1/2), j)$ given by

\[ D := \{ Z = (z, u) \in b(1)_{\mathbb{C}} \times (b(1/2), j) ; \Im z - Q(u, u) \in \Omega \}. \]

An action of the solvable group $B$ on $D$ is defined by

\[ b_0 \cdot (z, u) := (h_0 \cdot z + x_0 + iQ(h_0 \cdot u, u_0) + iQ(u_0, u_0)/2, h_0 \cdot u + u_0) \quad ((z, u) \in D) \]

for $b_0 = \exp(x_0 + u_0)h_0 \in B$ ($x_0 \in b(1), u_0 \in b(1/2), h_0 \in B(0)$). It is easy to check that the point $a_0 := (iE, 0)$ belongs to $D$. Then we can describe the Bergman mapping $C := \sigma_{a_0} : D \sim U$ concretely ([6], [8]).

Noting that $b(0) = a \oplus [b(0), b(0)]$, we define a one-dimensional representation $\chi_{\underline{\sigma}} : B(0) \rightarrow \mathbb{C}^\times$ for $\underline{\sigma} = (\sigma_1, \ldots, \sigma_r) \in \mathbb{C}^r$ by $\chi_{\underline{\sigma}}(\exp C) := e^{\Sigma \sigma_i \alpha_i(C)}$ ($C \in a$). Let $\Delta_{\underline{\sigma}}$ be a smooth function on the cone $\Omega$ given by $\Delta_{\underline{\sigma}}(h \cdot E) := \chi_{\underline{\sigma}}(h)$ ($h \in B(0)$). This $\Delta_{\underline{\sigma}}$ can be expressed as a product of powers of rational functions, and it can be extended as a holomorphic function on the complex domain $\Omega + ib(1)$. Define
\( d = (d_1, \ldots, d_r) \) by \( d_k := 1 + (p_k + q_k)/2 \) \((k = 1, \ldots, r)\). Then \( \Delta_{-d}(x)\, dx \) is an invariant measure on \( \Omega \) with respect to the action of \( B(0) \).

**Proposition 4 ([3, Lemma 5.1])**. The Bergman kernel \( K_{\mathcal{D}} \) of the homogeneous Siegel domain \( \mathcal{D} \) is given by

\[
K_{\mathcal{D}}(Z, Z') = C_{\mathcal{D}} \Delta_{-(2\underline{d} + \underline{b})}(\frac{z - \bar{z}'}{2i} - Q(u, u')) \quad (Z = (z, u), Z' = (z', u') \in \mathcal{D}),
\]

where \( C_{\mathcal{D}} \) is a constant independent of \( Z \) and \( Z' \).

3.2. Let \( E^* \in \mathfrak{b}(1)^* \) be the linear form on \( \mathfrak{b}(1) \) given by \( \langle x, E^* \rangle = \sum_{k=1}^{r} x_{kk} \) for elements \( x = \sum_{k=1}^{r} x_{kk} E_k + \sum_{1 \leq k < m \leq r} X_{mk} \in \mathfrak{b}(1) \) \((x_{kk} \in \mathbb{R}, X_{mk} \in \mathfrak{b}_{(\alpha_m + \alpha_k)/2})\).

Then \( E^* \) belongs to the dual cone \( \Omega^* := \{ \xi \in \mathfrak{b}(1)^*; (x, \xi) > 0 \text{ for all } x \in \overline{\Omega} \setminus \{0\} \} \) of \( \Omega \). Moreover, for any \( \xi \in \Omega^* \), there exists a unique \( h \in B(0) \) for which \( \xi = E^* \circ h \). Therefore, we can define a function \( \delta_{\underline{\sigma}} \) by \( \delta_{\underline{\sigma}}(E^* \circ h) := \chi_{\underline{\sigma}}(h) \) \((h \in B(0))\).

**Proposition 5 ([3, Theorem 2.1, Proposition 2.3])**. (i) For a parameter \( \underline{\sigma} = (\sigma_1, \ldots, \sigma_r) \in \mathbb{C}^r \), the integral \( \Gamma_{\Omega}(\underline{\sigma}) := \int_{\Omega} e^{-\langle x, E^* \rangle} \Delta_{\underline{\sigma} - \underline{d}}(x)\, dx \) converges if and only if \( \Re \sigma_k > p_k/2 \) \((k = 1, \ldots, r)\). In this case, one has \( \Gamma_{\Omega}(\underline{\sigma}) = C_{\Gamma} \prod_{k=1}^{r} \Gamma(\sigma_k - p_k/2) \), where \( C_{\Gamma} \) is a constant independent of \( \underline{\sigma} \).

Moreover, one has

\[
\delta_{\underline{\sigma}}(\xi) = \frac{1}{\Gamma_{\Omega}(\underline{\sigma})} \int_{\Omega} e^{-\langle x, \xi \rangle} \Delta_{\underline{\sigma} - \underline{d}}(x)\, dx \quad (\xi \in \Omega^*). \tag{10}
\]

(ii) The integral \( \gamma_{\Omega^*}(\underline{\sigma}) := \int_{\Omega^*} e^{-\langle E, \xi \rangle} \delta_{\underline{\sigma} - \underline{d}}(\xi)\, dx \) converges if and only if \( \Re \sigma_k > q_k/2 \) \((k = 1, \ldots, r)\), and in this case, \( \gamma_{\Omega^*}(\underline{\sigma}) = \Gamma_{\Omega}(\underline{\sigma} + (p - q)/2) = C_{\Gamma} \prod_{k=1}^{r} \Gamma(\sigma_k - q_k/2) \). Moreover, one has

\[
\Delta_{-\underline{\sigma}}(z) = \frac{1}{\gamma_{\Omega^*}(\underline{\sigma})} \int_{\Omega} e^{-\langle z, \xi \rangle} \delta_{\underline{\sigma} - \underline{d}}(\xi)\, d\xi \quad (z \in \Omega + ib(1)). \tag{11}
\]

(iii) For \( \xi \in \Omega^* \), one has

\[
\int_{b(1/2)} e^{-\langle Q(u, u), \xi \rangle}\, dV(u) = C_Q \delta_{-\underline{b}}(\xi), \tag{12}
\]

where \( C_Q \) is a constant independent of \( \xi \).

3.3. By the transformation rule of the Bergman kernels, we have \( K_{\mathcal{U}}(\zeta, \zeta)\, dV(\zeta) = K_{\mathcal{D}}(Z, Z)\, dV(Z) \) for the change of variable \( \zeta = C(Z) \) \((Z \in \mathcal{D})\). This together with Theorem 1 tells us that the left-hand side of (9) equals

\[
\frac{\text{Vol}(\mathcal{U})}{K_{\mathcal{D}}(a_0, a_0)^{s+1}} \int_{\mathcal{D}} |K_{\mathcal{D}}(Z, a_0)^{s+1}|^2 K_{\mathcal{D}}(Z, Z)^{-s}\, dV(Z),
\]
which is rewritten as
\[ C_D \text{Vol}(U) \int_D |\Delta_{-(s+1)(2d+b)}(\frac{z+iE}{2i})|^2 \Delta_{s(2d+b)}(\frac{z-\bar{z}}{2i} - Q(u, u)) dV(Z) \]
owing to Proposition 4. In order to evaluate this integral, we consider the change of variable
\[ Z = (x + iy + iQ(u, u), u) \in D \quad (x \in b(1), y \in \Omega, u \in b(1/2)). \]
For simplicity, we assume that the real part of $s$ are large enough for the convergence of the integrals in Proposition 5. First of all, by (11) and the Plancherel formula, we have
\[ \int_{b(1)} |\Delta_{-(s+1)(2d+b)}(\frac{z+iE}{2i})|^2 dx = \frac{(4\pi)^{N_1}}{\gamma_{\Omega^*}((s+1)(2d+b))^2} \int_{\Omega^*} e^{-\langle E+y+Q(u,u),\xi\rangle} \delta_{2(s+1)(2d+b)-2d}(\xi) d\xi, \]
where $N_1 := \dim b(1)$. Next, by (12) we have
\[ \int_{b(1/2)} \int_{b(1)} |\Delta_{-(s+1)(2d+b)}(\frac{z+iE}{2i})|^2 dx dV(u) = \frac{(4\pi)^{N_1} C_Q}{\gamma_{\Omega^*}((s+1)(2d+b))^2} \int_{\Omega^*} e^{-\langle E+y,\xi\rangle} \delta_{(2s+1)(2d+b)}(\xi) d\xi. \]
Furthermore, we see from (10) that
\[ \int_{\Omega} \int_{b(1/2)} \int_{b(1)} |\Delta_{-(s+1)(2d+b)}(\frac{z+iE}{2i})|^2 \Delta_{s(2d+b)}(y) dx dV(u) dy = \frac{(4\pi)^{N_1} C_Q \Gamma_{\Omega}(s(2d+b)+d)}{\gamma_{\Omega^*}((s+1)(2d+b))^2} \int_{\Omega^*} e^{-\langle E,\xi\rangle} \delta_{(s+1)(2d+b)-d}(\xi) d\xi \]
where we use Proposition 5 (ii) for the second equality. Therefore, the left-hand side of (9) is equal to
\[ \frac{\Gamma_{\Omega}(s(2d+b)+d)}{\gamma_{\Omega^*}((s+1)(2d+b))} \]
up to a constant multiple, and this is nothing but the reciprocal of $P(s)$ in (8) thanks to Proposition 5 (i) and (ii). Hence we obtain Theorem 3.
References


