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LITTLEWOOD-RICHARDSON COEFFICIENTS AND EXTREMAL WEIGHT CRYSTALS

JAE-HOON KWON

ABSTRACT. We describe the tensor product of two extremal weight crystals of type $A_{+\infty}$ by constructing an explicit bijection between the connected components in the tensor product and a set of quadruples of Littlewood-Richardson tableaux.

1. INTRODUCTION

Let $\mathfrak{g}_{>0}$ be the infinite rank affine Lie algebra of type $A_{+\infty}$ and $U_q(\mathfrak{g}_{>0})$ its quantized enveloping algebra. For an integral weight $\Lambda$, there exists an integrable $U_q(\mathfrak{g}_{>0})$-module called the extremal weight module with extremal weight $\Lambda$. The notion of extremal weight modules introduced by Kashiwara [5] is a generalization of integrable highest weight and lowest weight modules. An extremal weight module has a crystal base, which we call an extremal weight crystal for short, and two extremal weight crystals are isomorphic if their extremal weights are in the same Weyl group orbit.

Let $\mathcal{P}$ be the set of partitions. The Weyl group orbit of $\Lambda$ is naturally in one-to-one correspondence with a pair of partitions $(\mu, \nu) \in \mathcal{P}^2$, where $(\mu, \emptyset)$ (resp. $(\emptyset, \nu)$) corresponds to a dominant (resp. anti-dominant) weight. Let us denote by $\mathcal{B}_{\mu, \nu}$ the extremal weight crystal with extremal weight corresponding to $(\mu, \nu) \in \mathcal{P}^2$.

In [9], it is shown that the tensor product of two extremal weight crystals is isomorphic to a finite disjoint union of extremal weight crystals and the Grothendieck ring associated with the category of $\mathfrak{g}_{>0}$-crystals whose object is a finite union of extremal weight crystals, is isomorphic to the Weyl algebra of infinite rank. Using this characterization, it is shown that the multiplicity of $\mathcal{B}_{\zeta, \eta}$ in $\mathcal{B}_{\mu, \nu} \otimes \mathcal{B}_{\sigma, \tau}$ for $(\mu, \nu), (\sigma, \tau), (\zeta, \eta) \in \mathcal{P}^2$ is

\begin{equation}
\sum_{\alpha, \beta, \gamma \in \mathcal{P}} c^\zeta_{\alpha, \beta} c^\mu_{\beta, \gamma} c^\tau_{\gamma, \nu},
\end{equation}

which is a sum of products of four Littlewood-Richardson coefficients.

The main purpose of this note is to construct an explicit crystal isomorphism

\begin{equation}
\mathcal{B}_{\mu, \nu} \otimes \mathcal{B}_{\sigma, \tau} \sim \bigsqcup_{(\zeta, \eta) \in \mathcal{P}^2} \sum_{\alpha, \beta, \gamma \in \mathcal{P}} \mathcal{B}_{\zeta, \eta} \times LR^\zeta_{\alpha, \beta} \times LR^\mu_{\beta, \gamma} \times LR^\tau_{\gamma, \nu},
\end{equation}

which gives a bijective proof of (1.1). Here $LR^\lambda_{\mu, \nu}$ denotes the set of Littlewood-Richardson tableaux of shape $\lambda/\mu$ with content $\nu$ for $\lambda, \mu, \nu \in \mathcal{P}$. We remark that the decomposition of $\mathcal{B}_{\mu, \nu} \otimes \mathcal{B}_{\sigma, \tau}$ is given in [9] by generalizing the insertion algorithm of Stembridge’s rational

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tableaux [13, 14] for \( \mathfrak{g}_n \), but the associated recording tableaux which parameterize the connected components in \( \mathcal{B}_{\mu,\nu} \otimes \mathcal{B}_{\sigma,\tau} \) do not imply (1.1) directly.

The multiplicity (1.1) has another representation theoretical interpretation, that is, it coincides with a generalization of Littlewood-Richardson coefficients introduced in [2], whose positivity is equivalent to the existence of a long exact sequence of 6 finite abelian \( p \)-groups with types \( \sigma, \zeta, \mu, \tau, \eta, \nu \). The author would like to thank Alexander Yong for pointing out this connection.

This note is organized as follows. In Section 2, we recall briefly the notion of crystals and a combinatorial realization of \( \mathcal{B}_{\mu,\nu} \). In Section 3, we review some combinatorics of Littlewood-Richardson tableaux and an insertion algorithm for \( \mathcal{B}_{\mu,\nu} \). Finally, in Section 4, we construct the isomorphism (1.2).

2. Extremal weight crystals

2.1. Let \( \mathfrak{g}_{>0} \) denote the Lie algebra of complex matrices \( (a_{ij})_{i,j \in \mathbb{N}} \) with finitely many non-zero entries. Let \( E_{ij} \) be the elementary matrix with 1 at the \( i \)-th row and the \( j \)-th column and zero elsewhere. Then \( \{ E_{ij} | i, j \geq 1 \} \) is a linear basis of \( \mathfrak{g}_{>0} \).

Let \( h = \bigoplus_{i \geq 1} \mathbb{C}E_{ii} \) be the Cartan subalgebra of \( \mathfrak{g}_{>0} \) and \( \langle \cdot, \cdot \rangle \) the natural pairing on \( h^* \times h \). Let \( \Pi' = \{ h_i = E_{ii} - E_{i+1,i+1} | i \geq 1 \} \) be the set of simple coroots and \( \Pi = \{ \epsilon_i = \epsilon_i - \epsilon_{i+1} | i \geq 1 \} \) the set of simple roots of \( \mathfrak{g}_{>0} \), where \( \epsilon_i \in h^* \) is determined by \( \langle \epsilon_i, E_{jj} \rangle = \delta_{ij} \).

Let \( P = \bigoplus_{i \geq 1} \mathbb{Z} \epsilon_i \) be the weight lattice of \( \mathfrak{g}_{>0} \) and \( P_+ = \{ \Lambda \in P | \langle \Lambda, h_i \rangle \geq 0 \ (i \geq 1) \} \) the set of dominant integral weights. The map \( \lambda = (\lambda_i)_{i \geq 1} \mapsto \omega_\lambda = \sum_{i \geq 1} \lambda_i \epsilon_i \) gives a bijection between \( \mathcal{P} \) and \( P_+ \), where \( \mathcal{P} \) denotes the set of partitions.

For \( i \geq 1 \), let \( r_i \) be the simple reflection given by \( r_i(\Lambda) = \Lambda - \langle \Lambda, h_i \rangle \alpha_i \) for \( \Lambda \in h^* \). Let \( W \) be the Weyl group of \( \mathfrak{g}_{>0} \), that is, the subgroup of \( GL(h^*) \) generated by \( r_i \) for \( i \geq 1 \). Let \( P/W \) be the set of \( W \)-orbits in \( P \). For \( \Lambda = \sum_{i \geq 1} \lambda_i \epsilon_i \in P \), let \( \mu \) and \( \nu \) be the partitions determined by \( \{ \Lambda_i | \Lambda_i > 0 \} \) and \( \{-\Lambda_i | \Lambda_i < 0 \} \), respectively. Then the map \( W\Lambda \mapsto (\mu, \nu) \) is a bijection from \( P/W \) to \( \mathcal{P}^2 \).

2.2. Let us recall briefly the notion of crystals based on [6]. A \( \mathfrak{g}_{>0} \)-crystal is a set \( B \) together with the maps \( \text{wt} : B \to P \), \( \epsilon_i, \varphi_i : B \to \mathbb{Z} \cup \{-\infty\} \) and \( \tilde{e}_i, \tilde{f}_i : B \to B \cup \{0\} \) (\( i \in \mathbb{N} \)) such that for \( b \in B \)

\[
\begin{align*}
(1) \quad & \varphi_i(b) = \langle \text{wt}(b), h_i \rangle + \epsilon_i(b), \\
(2) \quad & \epsilon_i(\tilde{e}_i b) = \epsilon_i(b) - 1, \ \varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1, \ \text{wt}(\tilde{e}_i b) = \text{wt}(b) + \alpha_i \text{ if } \tilde{e}_i b \neq 0, \\
(3) \quad & \epsilon_i(\tilde{f}_i b) = \epsilon_i(b) + 1, \ \varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1, \ \text{wt}(\tilde{f}_i b) = \text{wt}(b) - \alpha_i \text{ if } \tilde{f}_i b \neq 0, \\
(4) \quad & \tilde{f}_i b = b' \text{ if and only if } b = \tilde{e}_i b' \text{ for } b, b' \in B, \\
(5) \quad & \tilde{e}_i b = \tilde{f}_i b = 0 \text{ if } \varphi_i(b) = -\infty, 
\end{align*}
\]

where \( 0 \) is a formal symbol and \( -\infty \) is the smallest element in \( \mathbb{Z} \cup \{-\infty\} \) such that \( -\infty + n = -\infty \) for all \( n \in \mathbb{Z} \).
A crystal $B$ is an $\mathbb{N}$-colored oriented graph where $b \overset{i}{\rightarrow} b'$ if and only if $b' = \tilde{f}_{i}b$ for $b, b' \in B$ and $i \geq 1$. We say that $B$ is connected if it is connected as a graph and regular if $\epsilon_{i}(b) = \max\{k \mid \epsilon_{i}^{k}b \neq 0\}$ and $\varphi_{i}(b) = \max\{k \mid \tilde{f}_{i}^{k}b \neq 0\}$ for $b \in B$ and $i \geq 1$.

The dual crystal $B^{\vee}$ of $B$ is defined to be the set $\{b^{\vee} \mid b \in B\}$ with

$$\begin{align*}
\mathrm{wt}(b^{\vee}) &= -\mathrm{wt}(b), \\
\epsilon_{i}(b^{\vee}) &= \varphi_{i}(b), \\
\tilde{e}_{i}(b^{\vee}) &= \left(\tilde{f}_{i}b\right)^{\vee}, \\
\tilde{f}_{i}(b^{\vee}) &= \left(\tilde{e}_{i}b\right)^{\vee},
\end{align*}$$

for $b \in B$ and $i \geq 1$. Here we assume that $0^{\vee} = 0$.

Let $B_{1}$ and $B_{2}$ be crystals. The tensor product of $B_{1}$ and $B_{2}$ is defined to be the set

$$B_{1} \otimes B_{2} = \{b_{1} \otimes b_{2} \mid b_{i} \in B_{i} \ (i = 1, 2)\}$$

with

$$\begin{align*}
\mathrm{wt}(b_{1} \otimes b_{2}) &= \mathrm{wt}(b_{1}) + \mathrm{wt}(b_{2}), \\
\epsilon_{i}(b_{1} \otimes b_{2}) &= \max\{\epsilon_{i}(b_{1}), \epsilon_{i}(b_{2}) - \langle \mathrm{wt}(b_{1}), h_{i}\rangle\}, \\
\varphi_{i}(b_{1} \otimes b_{2}) &= \max\{\varphi_{i}(b_{1}) + \langle \mathrm{wt}(b_{2}), h_{i}\rangle, \varphi_{i}(b_{2})\}, \\
\tilde{e}_{i}(b_{1} \otimes b_{2}) &= \begin{cases} \\
\tilde{e}_{i}b_{1} \otimes b_{2}, & \text{if } \varphi_{i}(b_{1}) \geq \epsilon_{i}(b_{2}), \\
b_{1} \otimes \tilde{e}_{i}b_{2}, & \text{if } \varphi_{i}(b_{1}) < \epsilon_{i}(b_{2}),
\end{cases} \\
\tilde{f}_{i}(b_{1} \otimes b_{2}) &= \begin{cases} \\
\tilde{f}_{i}b_{1} \otimes b_{2}, & \text{if } \varphi_{i}(b_{1}) > \epsilon_{i}(b_{2}), \\
b_{1} \otimes \tilde{f}_{i}b_{2}, & \text{if } \varphi_{i}(b_{1}) \leq \epsilon_{i}(b_{2}),
\end{cases}
\end{align*}$$

for $b_{1} \otimes b_{2} \in B_{1} \otimes B_{2}$ and $i \geq 1$, where we assume that $0 \otimes b_{2} = b_{1} \otimes 0 = 0$. Then $B_{1} \otimes B_{2}$ is also a crystal.

A map $\psi : B_{1} \rightarrow B_{2}$ is called an isomorphism of crystals if it is a bijection, preserves wt, $\epsilon_{i}$ and $\varphi_{i}$ and commutes with $\tilde{e}_{i}$, $\tilde{f}_{i}$ ($i \geq 1$), where we assume that $\psi(0) = 0$. In this case, we say that $B_{1}$ is isomorphic to $B_{2}$ and write $B_{1} \simeq B_{2}$. For example, $(B_{1} \otimes B_{2})^{\vee} \simeq B_{2}^{\vee} \otimes B_{1}^{\vee}$, where $(b_{1} \otimes b_{2})^{\vee}$ is mapped to $b_{2}^{\vee} \otimes b_{1}^{\vee}$.

For $b_{i} \in B_{i}$ ($i = 1, 2$), we say that $b_{1}$ is equivalent to $b_{2}$, and write $b_{1} \equiv b_{2}$ if there exists an isomorphism of crystals $C(b_{1}) \rightarrow C(b_{2})$ sending $b_{1}$ to $b_{2}$, where $C(b_{i})$ denotes the connected component of $B_{i}$ including $b_{i}$ ($i = 1, 2$).

2.3. We identify a partition with a Young diagram as usual (see [11]), where we enumerate rows and columns from the top and the left, respectively. Let $\mathcal{A}$ be a linearly ordered set. A tableau $T$ obtained by filling a skew Young diagram $\lambda/\mu$ with entries in $\mathcal{A}$ is called a semistandard tableau of shape $\lambda/\mu$ if the entries in each row are weakly increasing from left to right, and the entries in each column are strictly increasing from top to bottom. We denote by $\text{SST}_{\mathcal{A}}(\lambda/\mu)$ the set of all semistandard tableaux of shape $\lambda/\mu$ with entries in $\mathcal{A}$ (cf.[3, 11]).
For $T \in SST_{A}(\lambda/\mu)$, let $w(T)_{\text{col}}$ (resp. $w(T)_{\text{row}}$) denote the word obtained by reading the entries of $T$ column by column (resp. row by row) from right to left (resp. top to bottom), and in each column (resp. row) from top to bottom (resp. right to left). For $a \in A$, we denote by $(a \rightarrow T)$ (resp. $(T \leftarrow a)$) the tableau obtained by the Schensted column (resp. row) insertion (see for example [3, Appendix A.2]). For a finite word $w = w_{1}\ldots w_{r}$ with letters in $A$, we let $(w \rightarrow T) = (w_{r} \rightarrow (\cdots (w_{1} \rightarrow T) \cdots))$ and $(T \leftarrow w) = ((\cdots (T \leftarrow w_{1}) \cdots) \leftarrow w_{r})$. For semistandard tableaux $S$ and $T$, we define $(T \rightarrow S)$ (resp. $(S \leftarrow T)$) to be $(w(T)_{\text{col}} \rightarrow S)$ (resp. $S \leftarrow (w(T)_{\text{row}})^{\text{rev}}$) where $w^{\text{rev}}$ is the reverse word of $w$.

We denote by $T^{\vee}$ the tableau obtained from $T$ by $180^\circ$-rotation and replacing each entry $t$ with $t^{\vee}$. Then $T^{\vee}$ is a semistandard tableau with entries in $A^{\vee}$, where $A^{\vee} = \{ a^{\vee} | a \in A \}$ and $a^{\vee} < b^{\vee}$ if and only if $b < a$ for $a, b \in A$. Here we use the convention $(t^{\vee})^{\vee} = t$ and hence $(T^{\vee})^{\vee} = T$.

Let $A$ be either $\mathbb{N}$ or $\mathbb{N}^{\vee}$ with the following regular crystal structures

$$
\begin{array}{ccccccc}
1 & 1 & 2 & 2 & 3 & 3 & \cdots,
\end{array}
$$

$$
\begin{array}{ccccccc}
\cdots & 3^{\vee} & 2^{\vee} & 1^{\vee} & & & ,
\end{array}
$$

where $\text{wt}(k) = \epsilon_{k}$ and $\text{wt}(k^{\vee}) = -\epsilon_{k}$ for $k \geq 1$. Then the set of all finite words with letters in $A$ is a regular crystal, where we identify each word of length $r$ with an element in $A^{\otimes r} = A \otimes \cdots \otimes A$ ($r$ times). Now, the injective image of $SST_{A}(\lambda/\mu)$ in the set of finite words under the map $T \mapsto w(T)_{\text{col}}$ (or $w(T)_{\text{row}}$) together with $\{0\}$ is invariant under $\bar{e}_{i}, \bar{f}_{i}$. Hence $SST_{A}(\lambda/\mu)$ is a regular crystal [8]. Also, the row or column insertion is compatible with the crystal structure on tableaux in the following sense [10];

$$(a \rightarrow T) \equiv T \otimes a, \quad (T \leftarrow a) \equiv a \otimes T,$$

for $a \in A$ and $T \in SST_{A}(\lambda)$, and hence $(T \rightarrow S) \equiv S \otimes T, (S \leftarrow T) \equiv T \otimes S$ for $S \in SST_{A}(\mu)$.

2.4. For $\Lambda \in P$, let $B(\Lambda)$ be the crystal base of the extremal weight $U_{q}(\mathfrak{gl}_{1>0})$-module with extremal weight $\Lambda$. Then $B(\Lambda)$ is a regular crystal, and $B(\Lambda) \simeq B(w\Lambda)$ for $w \in W$. Moreover, if $\Lambda \in P_{+}$ (resp. $-\Lambda \in P_{+}$), then $B(\Lambda)$ is isomorphic to the crystal base of the irreducible highest (resp. lowest) weight $U_{q}(\mathfrak{gl}_{1>0})$-module with highest (resp. lowest) weight $\Lambda$ (see [5, 7] for detailed exposition of extremal weight modules and their crystal bases).

Recall that for $\lambda \in \mathcal{P}$

$$
B(\omega_{\lambda}) \simeq SST_{N}(\lambda), \quad B(-\omega_{\lambda}) \simeq B(\omega_{\lambda})^{\vee} \simeq SST_{N^{\vee}}(\lambda^{\vee}),
$$

where $\lambda^{\vee}$ is the skew Young diagram obtained from $\lambda \in \mathcal{P}$ by $180^\circ$-rotation, and $SST_{N}(\lambda)$ is connected with a unique highest weight element $H_{\lambda}$, where each $i$-th row is filled with $i$ for $i \geq 1$ [8].
Now, for $\mu, \nu \in \mathcal{P}$, we define $\mathcal{B}_{\mu, \nu}$ to be the set of bitableaux $(S, T)$ such that

(E1) $S \in SST_{N}(\mu)$ and $T \in SST_{N'}(\nu')$,

(E2) for each $k \geq 1$, $s(k) + t(k) \leq k$

where $s(k)$ is the number of entries in the left-most column of $S$ no more than $k$, and $t(k)$ is the number of entries in the right-most column of $T$ no less than $k'$. Since $\mathcal{B}_{\mu, \nu} \subset SST_{N}(\mu) \otimes SST_{N'}(\nu')$, we can apply $\hat{e}_{i}, \hat{f}_{i}$ ($i \geq 1$) on $\mathcal{B}_{\mu, \nu}$. Then $\mathcal{B}_{\mu, \nu} \cup \{0\}$ is stable under $\hat{e}_{i}, \hat{f}_{i}$ ($i \geq 1$) and hence a regular crystal. Moreover, we have the following [9, Theorem 3.5].

**Theorem 2.1.** For $\mu, \nu \in \mathcal{P}$,

1. $\mathcal{B}_{\mu, \nu}$ is connected,
2. $\mathcal{B}_{\mu, \nu} \simeq B(\Lambda)$, where $W\Lambda \in P/W$ corresponds to $(\mu, \nu) \in \mathcal{P}^{2}$.

### 3. Insertion Algorithm

3.1. For $\lambda, \mu, \nu \in \mathcal{P}$, let $LR_{\mu \nu}^{\lambda}$ be the set of tableaux $U$ in $SST_{N}(\lambda/\mu)$ such that for $i \geq 1$

(LR1) the number of $i$'s in $U$ is $\nu_{i}$,
(LR2) the number of $i$'s in $w_{1} \ldots w_{k}$ is no less than that of $i+1$'s in $w_{1} \ldots w_{k}$ for $1 \leq k \leq r$, where $w(U)_{col} = w_{1} \ldots w_{r}$.

We call $LR_{\mu \nu}^{\lambda}$ the set of Littlewood-Richardson tableaux of shape $\lambda/\mu$ with content $\nu$ and put $c_{\mu \nu}^{\lambda} = |LR_{\mu \nu}^{\lambda}|$ [11].

Suppose that $\mathcal{A}$ is a linearly ordered set. For $S \in SST_{\mathcal{A}}(\mu)$ and $T \in SST_{\mathcal{A}}(\nu)$, let $\lambda$ be the shape of $(T \rightarrow S)$ and $w(T)_{col} = w_{1} \cdots w_{r}$. If $w_{i}$ is in the $k$th row of $T$ and inserted into $(w_{i-1} \rightarrow (\cdots (w_{1} \rightarrow T)))$ to create a node in $\lambda/\mu$, then let us fill the node with $k$. We denote the resulting tableau in $SST_{N}(\lambda/\mu)$ by $(T \rightarrow S)_{R}$ and call it the recording tableau of $(T \rightarrow S)$. Then we have a bijection

\[
SST_{\mathcal{A}}(\mu) \times SST_{\mathcal{A}}(\nu) \xleftarrow{\sim} \bigsqcup_{\lambda \in \mathcal{P}} SST_{\mathcal{A}}(\lambda) \times LR_{\mu \nu}^{\lambda},
\]

where $(S, T)$ corresponds to $((T \rightarrow S), (T \rightarrow S)_{R})$ [15]. Moreover, if we assume that $\mathcal{A}$ is either $\mathbb{N}$ or $\mathbb{N'}$, then the above bijection commutes with $\tilde{e}_{i}$ and $\tilde{f}_{i}$ ($i \geq 1$) (cf.[4, 10]), where $\tilde{e}_{i}$ and $\tilde{f}_{i}$ act on the first component of $SST_{\mathcal{A}}(\lambda) \times LR_{\mu \nu}^{\lambda}$. Summarizing, we have

**Proposition 3.1.** Let $\mu, \nu \in \mathcal{P}$ be given.

1. The map sending $S \otimes T$ to $((T \rightarrow S), (T \rightarrow S)_{R})$ is an isomorphism of crystals

$$SST_{N}(\mu) \otimes SST_{N}(\nu) \xrightarrow{\sim} \bigsqcup_{\lambda \in \mathcal{P}} SST_{N}(\lambda) \times LR_{\mu \nu}^{\lambda}.$$
(2) The map sending $S \otimes T$ to $((S^\vee \to T^\vee)^\vee, (S^\vee \to T^\vee)^R)$ is an isomorphism of crystals

$$SST_{N^\vee}(\mu^\vee) \otimes SST_{N^\vee}(\nu^\vee) \cong \bigsqcup_{\lambda \in \mathcal{P}} SST_{N^\vee}(\lambda^\vee) \times LR^\lambda_{\nu \mu}.$$

**Remark 3.2.** (1) Let $U \in SST_N(\lambda/\mu)$ be given. Then as a crystal element, $U \in LR^\lambda_{\mu \nu}$ if and only if $U \equiv H_\nu$.

(2) For $U \in LR^\lambda_{\mu \nu}$, one may identify $U$ with a unique $T \in SST_N(\nu)$, say $\iota(U)$, such that the number of $k$'s in the $i$-th row of $T$ is equal to the number of $i$'s in the $k$-th row of $\lambda/\mu$ for $i, k \geq 1$. Equivalently, $H_\mu \otimes \iota(U) \equiv H_\lambda$ [12].

3.2. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two linearly ordered sets. Let $U$ be a tableau of shape $\lambda/\mu$ with entries in $\mathcal{A} \cup \mathcal{B}$, satisfying the following conditions;

(S1) if $u, u' \in \mathcal{X}$ are entries of $U$ and $u$ is northwest of $u'$, then $u \leq u'$,

(S2) in each column of $U$, entries in $\mathcal{X}$ increase strictly from top to bottom,

where $\mathcal{X} = \mathcal{A}$ or $\mathcal{B}$, and we say that $u$ is northwest of $u'$ provided the row and column indices of $u$ are no more than those of $u'$. Suppose that $a \in \mathcal{A}$ and $b \in \mathcal{B}$ are two adjacent entries in $U$ such that $a$ is placed above or to the left of $b$. Interchanging $a$ and $b$ is called a switching if the resulting tableau still satisfies the conditions (S1) and (S2).

For $S \in SST_A(\mu)$ and $T \in SST_B(\lambda/\mu)$, we denote by $S \ast T$ be the tableau in $SST_{A \cup B}(\lambda)$ obtained by gluing $S$ and $T$. Let $U$ be a tableau obtained from $S \ast T$ by applying switching procedures as far as possible. Then it is shown in [1, Theorems 2.2 and 3.1] that

(1) $U = T' \ast S'$, where $T' \in SST_B(\nu)$ and $S' \in SST_A(\lambda/\nu)$ for some $\nu$,

(2) $U$ is uniquely determined by $S$ and $T$,

(3) when $\mathcal{A} = \mathcal{N}$, $S' \in LR^\lambda_{\nu \mu}$ if and only if $S = H_\mu$.

Suppose that $\mathcal{A} = \mathcal{N}$ and $S = H_\mu$. Put

$$j(T) = T', \quad j(T)^R = S'.$$

Then the map $T \mapsto (j(T), j(T)^R)$ gives a bijection [1]

$$(3.2) \quad SST_B(\lambda/\mu) \xrightarrow{\sim} \bigsqcup_{\nu \in \mathcal{P}} SST_B(\nu) \times LR^\lambda_{\nu \mu}.$$

If $\mathcal{B} = \mathcal{N}$, then the map $Q \mapsto j(Q)^R$ restricts to a bijection from $LR^\lambda_{\mu \nu}$ to $LR^\lambda_{\nu \mu}$. Moreover, if $\mathcal{B}$ is either $\mathcal{N}$ or $\mathcal{N}^\vee$, then we can check that $T \equiv j(T)$ and $j(T)^R$ is invariant under $\bar{e}_i$ and $\bar{f}_i$ $(i \geq 1)$. Hence we have the following.

**Proposition 3.3.** Suppose that $\mathcal{B}$ is either $\mathcal{N}$ or $\mathcal{N}^\vee$. For a skew Young diagram $\lambda/\mu$, we have an isomorphism of crystals

$$SST_B(\lambda/\mu) \xrightarrow{\sim} \bigsqcup_{\nu \in \mathcal{P}} SST_B(\nu) \times LR^\lambda_{\nu \mu},$$

where $T$ is mapped to $(j(T), j(T)^R)$. 
3.3. Let us review an insertion algorithm for extremal weight crystal elements [9].

3.3.1. Let $\mu, \nu \in \mathcal{P}$ be given. For $a \in \mathbb{N}$ and $(S, T) \in \mathcal{B}_{\mu, \nu}$, we define $(a \rightarrow (S, T))$ in the following way;

Suppose that $S$ is empty and $T$ is a single column tableau. Let $(T', a')$ be the pair obtained by the following process;

1. If $T$ contains $a^\vee, (a + 1)^\vee, \ldots, (b - 1)^\vee$ but not $b^\vee$, then $T'$ is the tableau obtained from $T$ by replacing $a^\vee, (a + 1)^\vee, \ldots, (b - 1)^\vee$ with $(a + 1)^\vee, (a + 2)^\vee, \ldots, b^\vee$, and put $a' = b$.
2. If $T$ does not contain $a^\vee$, then leave $T$ unchanged and put $a' = a$.

Now, we suppose that $S$ and $T$ are arbitrary.

1. Apply the above process to the leftmost column of $T$ with $a$.
2. Repeat (1) with $a'$ and the next column to the right.
3. Continue this process to the right-most column of $T$ to get a tableau $T'$ and $a''$.
4. Define

$$
(a \rightarrow (S, T)) = ((a'' \rightarrow S), T')
$$

Then $(a \rightarrow (S, T)) \in \mathcal{B}_{\sigma, \tau}$ for some $\sigma \in \mathcal{P}$ with $|\sigma/\mu| = 1$. For a finite word $w = w_1 \ldots w_r$ with letters in $\mathbb{N}$, we let $(w \rightarrow (S, T)) = (w_r \rightarrow (\cdots (w_1 \rightarrow (S, T)) \cdots))$.

3.3.2. For $a \in \mathbb{N}$ and $(S, T) \in \mathcal{B}_{\mu, \nu}$, we define $((S, T) \leftarrow a^\vee)$ to be the pair $(S', T')$ obtained in the following way;

1. If the pair $(S, (T^\vee \leftarrow a)^\vee)$ satisfies the condition (E2) in Section 2.4, then put $S' = S$ and $T' = (T^\vee \leftarrow a)^\vee$.
2. Otherwise, choose the smallest $k$ such that $a_k$ is bumped out of the $k$-th row in the row insertion of $a$ into $T^\vee$ and the insertion of $a_k$ into the $(k + 1)$-th row violates the condition (E2) in Section 2.4.
3. (a) Stop the row insertion of $a$ into $T^\vee$ when $a_k$ is bumped out and let $T'$ be the resulting tableau after taking $\vee$.
4. (b) Remove $a_k$ in the left-most column of $S$, which necessarily exists, and then apply the jeu de taquin (see for example [3, Section 1.2]) to obtain a tableau $S'$.

In this case, $((S, T) \leftarrow a^\vee) \in \mathcal{B}_{\sigma, \tau}$, where either (1) $|\mu/\sigma| = 1$ and $\tau = \nu$, or (2) $\sigma = \mu$ and $|\tau/\nu| = 1$. For a finite word $w = w_1 \ldots w_r$ with letters in $\mathbb{N}^\vee$, we let $((S, T) \leftarrow w) = (\cdots ((S, T) \leftarrow w_1) \cdots) \leftarrow w_r$.

3.3.3. Let $\mu, \nu, \sigma, \tau \in \mathcal{P}$ be given. For $(S, T) \in \mathcal{B}_{\mu, \nu}$ and $(S', T') \in \mathcal{B}_{\sigma, \tau}$, we define

$$
((S', T') \rightarrow (S, T)) = ((w(S')_{\text{col}} \rightarrow (S, T)) \leftarrow w(T')_{\text{col}}).
$$
Then \( ((S', T') \rightarrow (S, T)) \in \mathcal{B}_{\zeta, \eta} \) for some \( (\zeta, \eta) \in \mathcal{P}^{2} \). Assume that \( w(S'_{\text{col}}) = w_{1} \ldots w_{s} \) and \( w(T'_{\text{col}}) = w_{s+1} \ldots w_{s+t} \). For \( 1 \leq i \leq s+t \), let
\[
(S^{i}, T^{i}) = \begin{cases} 
  w_{i} \rightarrow (\cdots (w_{1} \rightarrow (S, T))), & \text{if } 1 \leq i \leq s, \\
  (((S^{s}, T^{s}) \leftarrow w_{s+1}) \cdots) \leftarrow w_{i}, & \text{if } s+1 \leq i \leq s+t,
\end{cases}
\]
and \( (S^{0}, T^{0}) = (S, T) \). We define
\[
((S', T') \rightarrow (S, T))_{R} = (U, V),
\]
where \( (U, V) \) is the pair of tableaux with entries in \( \mathbb{Z}^{x} = \mathbb{Z} \setminus \{0\} \) determined by the following process;

1. \( U \) is of shape \( \sigma \) and \( V \) is of shape \( \tau \).
2. Let \( 1 \leq i \leq s \). If \( w_{i} \) is inserted into \( (S^{i-1}, T^{i-1}) \) to create a dot (or box) in the \( k \)-th row of the shape of \( S^{i-1} \), then we fill the dot in \( \sigma \) corresponding to \( w_{i} \) with \( k \).
3. Let \( s+1 \leq i \leq s+t \). If \( w_{i} \) is inserted into \( (S^{i-1}, T^{i-1}) \) to create a dot in the \( k \)-th row (from the bottom) of the shape of \( T^{i-1} \), then we fill the dot in \( \tau \) corresponding to \( w_{i} \) with \( -k \). If \( w_{i} \) is inserted into \( (S^{i-1}, T^{i-1}) \) to remove a dot in the \( k \)-th row of the shape of \( S^{i-1} \), then we fill the corresponding dot in \( \tau \) with \( k \).

We call \( ((S', T') \rightarrow (S, T))_{R} \) the recording tableau of \( ((S', T') \rightarrow (S, T)) \). By [9, Theorem 4.10], we have the following.

**Proposition 3.4.** Under the above hypothesis, we have

1. \( ((S', T') \rightarrow (S, T)) \equiv (S, T) \otimes (S', T') \),
2. \( ((S', T') \rightarrow (S, T))_{R} \in SST_{N}(\sigma) \times SST_{\mathbb{Z}^{x}}(\tau) \),
3. the recording tableaux are constant on the connected component of \( \mathcal{B}_{\mu, \nu} \otimes \mathcal{B}_{\sigma, \tau} \) including \( (S, T) \otimes (S', T') \),

where the linear ordering on \( \mathbb{Z}^{x} \) is given by \( 1 \prec 2 \prec 3 \prec \cdots \prec -3 \prec -2 \prec -1 \).

**Example 3.5.** Consider
\[
(S, T) = \begin{pmatrix} 2 & 3 & 4 & 5^{\vee} & 5^{\vee} \\ 3 & 5 \end{pmatrix}, \quad (S', T') = \begin{pmatrix} 3 & 3 & 4^{\vee} \\ 5 & 3^{\vee} & 1^{\vee} \end{pmatrix}.
\]

Since \( w(S'_{\text{col}}) = 3 3 5 \) and \( w(T'_{\text{col}}) = 4^{\vee} 1^{\vee} 3^{\vee} \), we have
\[
(w(S')_{\text{col}} \rightarrow (S, T)) = \begin{pmatrix} 2 & 3 & 3 & 4 \\ 3 & 5 \\ 4 \\ 6 \end{pmatrix}, \quad \begin{pmatrix} 6^{\vee} & 5^{\vee} \\ 4^{\vee} & 2^{\vee} \end{pmatrix}.
\]
and
\[
((w(S')_{\text{col}} \rightarrow (S, T)) \leftarrow w(T')_{\text{col}}) = \begin{pmatrix}
2 & 3 & 3 & 4 & 5^\vee \\
3 & 5 & & 6^\vee & 4^\vee \\
4 & & 4^\vee & 3^\vee & 1^\vee
\end{pmatrix}.
\]

Hence,
\[
((S', T') \rightarrow (S, T)) = \begin{pmatrix}
3 & 3 & 3 & 4 & 5^\vee \\
4 & 5 & & 6^\vee & 4^\vee \\
6 & & 4^\vee & 3^\vee & 1^\vee
\end{pmatrix},
\]
\[
((S', T') \rightarrow (S, T))_{R} = \begin{pmatrix}
1 & 3 & 4 & -3 \\
4 & & -1
\end{pmatrix}.
\]

**Remark 3.6.** For \((U, V) \in SST_N(\sigma) \times SST_{Z^X}(\tau)\), an equivalent condition for \((U, V)\) to be a recording tableau, that is, \((U, V) = ((S', T') \rightarrow (S, T))_{R}\) for some \((S, T) \in \mathcal{B}_{\mu,\nu}\) and \((S', T') \in \mathcal{B}_{\sigma,\tau}\), can be found in [9, Section 4.3].

### 4. Main Theorem

To prove our main theorem, let us first describe the decompositions of \(SST_N(\nu^\vee) \otimes SST_N(\mu)\) and \(SST_N(\mu) \otimes SST_N(\nu^\vee)\) for \(\mu, \nu \in \mathcal{P}\).

**Proposition 4.1.** For \(\mu, \nu \in \mathcal{P}\), we have an isomorphism of crystals
\[
SST_N(\nu^\vee) \otimes SST_N(\mu) \xrightarrow{\sim} \mathcal{B}_{\mu,\nu},
\]
where \(T \otimes S\) is mapped to \(((S, \emptyset) \rightarrow (\emptyset, T))\).

**Proof.** For \(T \otimes S \in SST_N(\nu^\vee) \otimes SST_N(\mu)\), it follows from Proposition 3.4 (2) that
\[
1) \quad ((S, \emptyset) \rightarrow (\emptyset, T)) \in \mathcal{B}_{\mu,\nu},
\]
\[
2) \quad ((S, \emptyset) \rightarrow (\emptyset, T))_{R} = (H_{\mu}, \emptyset).
\]

Therefore, we have a map
\[
SST_N(\nu^\vee) \otimes SST_N(\mu) \longrightarrow \mathcal{B}_{\mu,\nu} \times \{(H_{\mu}, \emptyset)\}
\]
sending \(T \otimes S\) to \(((S, \emptyset) \rightarrow (\emptyset, T)), ((S, \emptyset) \rightarrow (\emptyset, T))_{R}\). Since the insertion algorithm is reversible [9, Proposition 4.9], the above map is indeed a bijection and hence an isomorphism of crystals by Proposition 3.4 (1).

Next, suppose that \(S \otimes T \in SST_N(\mu) \otimes SST_N(\nu^\vee)\) is given. Let \(U^{>0}\) (resp. \(U^{<0}\)) be the subtableau in \(((\emptyset, T) \rightarrow (S, \emptyset))_{R}\) consisting of positive (resp. negative) entries. We define
\[
\theta(S \otimes T) = (\nu^{-1}(U^{>0}), j(U^{<0}))_{R}
\]
(see Remark 3.2 (2) and Section 3.2 (3.2)).
Proposition 4.2. For $\mu, \nu \in \mathcal{P}$, we have an isomorphism of crystals

$$SST_N(\mu) \otimes SST_N(\nu') \sim \bigsqcup_{\ldots} \mathcal{B}_{\sigma, \tau} \times LR_{\sigma \lambda}^\mu \times LR_{\lambda \tau}^\nu,$$

where $S \otimes T$ is mapped to $(((\emptyset, T) \to (S, \emptyset)), \theta(S \otimes T))$.

Proof. For $S \otimes T \in SST_N(\mu) \otimes SST_N(\nu')$, suppose that $(((\emptyset, T) \to (S, \emptyset)) \in \mathcal{B}_{\sigma, \tau}$ for some $\sigma, \tau \in \mathcal{P}$. First, note that $U^{>0} \in SST_N(\lambda)$ for some $\lambda \subset \nu$. Then it is not difficult to check that $\nu^{-1}(U^{>0}) \in LR_{\sigma \lambda}^\mu$ (see Remark 3.2). Next, consider $U^{<0} \in SST_{\mathbb{N}^{<0}}(\nu \lambda)$. Then $(w(U^{<0})_{\text{col}})^{\text{rev}}$ satisfies (LR1) with respect to $\tau$ and (LR2), ignoring $-$ sign in each letter. Let $L_{\tau}$ be the tableau in $SST_{\mathbb{N}^{<0}}(\tau)$, where the $i$-th entry from the bottom in each column is $-i$. Considering the Knuth equivalence on the set of words with letters in $\mathbb{Z}_{<0}$ (cf. [3]), we have $j(U^{<0}) = L_{\tau}$ and $j(U^{<0})_{R} \in LR_{\tau \lambda}^\nu$ by (3.2).

Now, we get $j(j(U^{<0})_{R})_{R} \in LR_{\lambda \tau}^\nu$. So, we have a map $SST_N(\mu) \otimes SST_N(\nu') \to \bigsqcup_{\ldots} \mathcal{B}_{\sigma, \tau} \times LR_{\sigma \lambda}^\mu \times LR_{\lambda \tau}^\nu$, sending $S \otimes T$ to $(((\emptyset, T) \to (S, \emptyset)), \theta(S \otimes T))$. Since the insertion algorithm is reversible [9, Proposition 4.9], the above map is a bijection and therefore an isomorphism of crystals by Proposition 3.4 (1) and (3).

Now, we are in a position to state our main result in this note.

Theorem 4.3. For $(\mu, \nu), (\sigma, \tau) \in \mathcal{P}^2$, we have an isomorphism of crystals

$$\mathcal{B}_{\mu, \nu} \otimes \mathcal{B}_{\sigma, \tau} \simeq \bigsqcup_{\ldots} \mathcal{B}_{\zeta, \eta} \times LR_{\zeta \alpha}^\mu \times LR_{\alpha \beta}^\nu \times LR_{\beta \gamma}^\tau \times LR_{\gamma \nu}^\tau.$$

Proof. Note that $\mathcal{B}_{\mu, \emptyset} = SST_N(\mu)$ and $\mathcal{B}_{\emptyset, \nu} = SST_N(\nu')$. Then as a crystal, we have

$$\mathcal{B}_{\mu, \emptyset} \otimes \mathcal{B}_{\sigma, \tau} \simeq \mathcal{B}_{\emptyset, \nu} \otimes \mathcal{B}_{\sigma, \emptyset} \simeq \bigsqcup_{\ldots} \mathcal{B}_{\zeta, \eta} \times LR_{\zeta \alpha}^\mu \times LR_{\alpha \beta}^\tau \times LR_{\beta \gamma}^\nu.$$

(by Proposition 3.1)

*(Proof complete)*
Corollary 4.4. The multiplicity of $\mathcal{B}_{\zeta,\eta}$ in $\mathcal{B}_{\mu,\nu} \otimes \mathcal{B}_{\sigma,\tau}$ is given by

$$\sum_{\alpha,\beta,\gamma \in \mathcal{P}} c_{\alpha}^{\zeta} c_{\mu}^{\beta} c_{\tau}^{\gamma} c_{\nu}^{\eta}.$$ 

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