<table>
<thead>
<tr>
<th>Title</th>
<th>Predual of Campanato spaces and Riesz potentials (Potential Theory and its related Fields)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Nakai, Eiichi</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2009), 1669: 122-131</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2009-11</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/141126">http://hdl.handle.net/2433/141126</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Predual of Campanato spaces and Riesz potentials

Eiichi Nakai
Department of Mathematics
Osaka Kyoiku University

1. INTRODUCTION

This is an announcement of my recent work.

Let $X = (X, \delta, \mu)$ be a space of homogeneous type (SHT), i.e. $X$ is a topological space endowed with a quasi-distance $\delta$ and a nonnegative measure $\mu$ such that

$$
\delta(x, y) \geq 0 \quad \text{and} \quad \delta(x, y) = 0 \text{ if and only if } x = y,
\delta(x, y) = \delta(y, x),
$$

(1.1) $$
\delta(x, y) \leq K_1 (\delta(x, z) + \delta(z, y)),
$$

the balls $B(x, r) = \{y \in X : \delta(x, y) < r\}$, $r > 0$, form a basis of neighborhoods of the point $x$, $\mu$ is defined on a $\sigma$-algebra of subsets of $X$ which contains all balls, and

$$
0 < \mu(B(x, 2r)) \leq K_2 \mu(B(x, r)) < \infty,
$$

(1.2) where $K_i \geq 1$ ($i = 1, 2$) are constants independent of $x, y, z \in X$ and $r > 0$.

If there are constants $\theta$ ($0 < \theta \leq 1$) and $K_3 \geq 1$ such that

$$
|\delta(x, z) - \delta(y, z)| \leq K_3 (\delta(x, z) + \delta(y, z))^{1-\theta} \delta(x, y)^{\theta}, \quad x, y, z \in X,
$$

(1.3) then the balls are open sets. The number $\theta$ is called the order of the SHT.

We shall say that a SHT is normal if there are constants $K_4 > 0$ and $K_5 > 0$

$$
K_4 r \leq \mu(B(x, r)) \leq K_5 r \quad \text{for } x \in X \text{ and } \mu(\{x\}) < r < \mu(X).
$$

(1.4) We note that, for any SHT $(X, d, \mu)$, there exists a quasi-distance $\delta$ such that $(X, \delta, \mu)$ is normal and of some order $\theta$, and that the topologies induced on $X$ by $d$ and $\delta$ coincide (Macías and Segovia (1979)).

Let $X = \mathbb{R}^n$, $d(x, y) = |x - y|$ and $\mu$ be the Lebesgue measure. If $\delta(x, y) = |x - y|^n$, then $(\mathbb{R}^n, \delta, \mu)$ is normal and of order $1/n$.


Key words and phrases. Riesz potential, fractional integral, Hardy space, variable exponent, Campanato space, Lipschitz spaces, space of homogeneous type.
In this talk we always assume that \((X, \delta, \mu)\) is normal and of order \(\theta\) and that 
\(\mu(\{x\}) = 0\) for all \(x\) in \(X\).

We consider Riesz potentials
\[
I_\alpha f(x) = \int_X \frac{f(y)}{\delta(x, y)^{1-\alpha}} \, d\mu(y),
\]
for \(0 < \alpha < \theta\). It is known that the operator \(I_\alpha\) is bounded from \(L^p(X)\) to \(L^q(X)\) if 
\(1 < p < q < \infty\) and \(-1/p + \alpha = -1/q\) (Gatto and Vagi(1990)). This boundedness is well known as the Hardy-Littlewood-Sobolev theorem in \(\mathbb{R}^n\) case.

In this report, we define a generalized Hardy space \(H_U^{[\phi, q]}(X)\) and investigate continuity of \(I_\alpha\) on \(H_U^{[\phi, q]}(X)\). We show
\[
\left(H_U^{[\phi, q]}(X)\right)^* = \mathcal{L}_{q', \phi}(X),
\]
where \(\mathcal{L}_{q', \phi}(X)\) is a Campanato space. Campanato spaces are Banach spaces modulo constants, which include BMO\((X)\) and \(\text{Lip}_\alpha(X)\) as special cases.

We first define \(I_\alpha\) for functions \(f \in \mathcal{L}_{q', \phi}(X)\). To do this we define the modified version of \(I_\alpha\) as follows;
\[
\tilde{I}_\alpha f(x) = \int_X f(y) \left( \frac{1}{\delta(x, y)^{1-\alpha}} - \frac{1 - \chi_{B_0}(y)}{\delta(x_0, y)^{1-\alpha}} \right) \, dy,
\]
where \(B_0 = B(x_0, r_0)\) is a fixed ball. We can show that \(\tilde{I}_\alpha f(x)\) converges absolutely for all \(x\) and therefore changing \(B_0\) in the definition above results in adding a constant. We assume that \(\delta\) satisfies the cancellation property;
\[
\int_X \left( \frac{1}{\delta(x, y)^{1-\alpha}} - \frac{1}{\delta(x', y)^{1-\alpha}} \right) \, d\mu(y) = 0 \quad \text{for any } x, x' \in X.
\]
(1.5)

In case of \(X = \mathbb{R}^n\) or \(\mathbb{T}^n\), (1.5) holds for \(\delta(x, y) = |x - y|^n\) and for \(0 < \alpha < 1\). For other examples of spaces of homogeneous type with the property (1.5), see [3]. We note that, for all normal spaces \((X, \delta, \mu)\) with \(\mu(X) = \infty\) and \(\mu(\{x\}) = 0\) for all \(x \in X\), we can fined a quasi-distance \(\delta_\alpha\) equivalent to \(\delta\), such that (1.5) holds (see [2]).

2. Campanato spaces \(\mathcal{L}_{p, \phi}(X)\) and Hölder spaces \(\Lambda_\phi(X)\)

Let \(1 \leq p < \infty\) and \(\phi : X \times (0, \infty) \to (0, \infty)\). For a ball \(B = B(x, r)\), we shall write \(\phi(B)\) in place of \(\phi(x, r)\). The function spaces \(\mathcal{L}_{p, \phi}(X)\) and \(\Lambda_\phi(X)\) are defined
to be the sets of all $f$ such that $\|f\|_{\mathcal{L}_{p,\phi}} < \infty$ and $\|f\|_{\Lambda_{\phi}} < \infty$, respectively, where

$$
\|f\|_{\mathcal{L}_{p,\phi}} = \sup_{B} \frac{1}{\phi(B)} \left( \frac{1}{\mu(B)} \int_{B} |f(x) - f_{B}|^{p} d\mu(x) \right)^{1/p},
$$

$$
\|f\|_{\Lambda_{\phi}} = \sup_{x, y \in X, x \neq y} \frac{2|f(x) - f(y)|}{\phi(x, \delta(x, y)) + \phi(y, \delta(x, y))},
$$

and

$$
f_{B} = \mu(B)^{-1} \int_{B} f(x) d\mu(x).
$$

Then $\mathcal{L}_{p,\phi}(X)$ and $\Lambda_{\phi}(X)$ are Banach spaces modulo constants with the norms $\|f\|_{\mathcal{L}_{p,\phi}}$ and $\|f\|_{\Lambda_{\phi}}$, respectively. If $p = 1$ and $\phi \equiv 1$, then $\mathcal{L}_{1,\phi}(X) = \text{BMO}(X)$.

Let $\mathcal{G} \ast$ be the set of all functions $\phi : X \times (0, \infty) \to (0, \infty)$ such that

\begin{align*}
(2.1) & \quad \frac{1}{A_{1}} \leq \frac{\phi(x, s)}{\phi(x, r)} \leq A_{1}, \quad \frac{1}{2} \leq \frac{s}{r} \leq 2, \\
(2.2) & \quad \phi(x, r) \leq A_{2}\phi(y, s), \quad B(x, r) \subset B(y, s),
\end{align*}

where $A_{1}$ and $A_{2} > 0$ are independent of $r, s > 0, x, y \in X$.

**Theorem 2.1.** Let $\phi \in \mathcal{G} \ast$. Then

$$
\mathcal{L}_{p,\phi}(X) = \mathcal{L}_{1,\phi}(X)
$$

with equivalent norms for every $1 \leq p < \infty$.

**Theorem 2.2.** Let $\phi \in \mathcal{G} \ast$ and there exists $C > 0$ such that

$$
\int_{0}^{\delta(x, y)} \frac{\phi(x, t)}{t} dt \leq C\phi(x, \delta(x, y)), \quad x, y \in X.
$$

Then

$$
\Lambda_{\phi}(X) = \mathcal{L}_{p,\phi}(X)
$$

with equivalent norms for every $1 \leq p < \infty$.

We say that $\alpha(\cdot) : X \to [0, \infty)$ is log-Hölder continuous if there exists $C_{0} > 0$ such that

$$
|\alpha(x) - \alpha(y)| \leq \frac{C_{0}}{\log(1/\delta(x, y))} \quad \text{for} \quad \delta(x, y) < 1/2.
$$

Let $\alpha_{-} = \inf_{x \in X} \alpha(x)$ and $\alpha_{+} = \sup_{x \in X} \alpha(x)$.

**Example 2.1.** Let $\alpha(\cdot)$ be log-Hölder continuous and

$$
\phi(x, r) = r^{\alpha(x)} \quad \text{with} \quad 0 < \alpha_{-} \leq \alpha_{+} \leq \theta.
$$
Then $\phi \in \mathcal{G}_*$ and satisfies (2.3). In this case we denote $\Lambda_\phi(X)$ by $\text{Lip}_{\alpha()}(X)$ and

$$\|f\|_{\text{Lip}_{\alpha()}} = \sup_{x,y \in X, x \neq y} \frac{2|f(x) - f(y)|}{\delta(x, y)^{\alpha(x)} + \delta(y, x)^{\alpha(y)}}.$$ 

If $\alpha(x) \equiv \alpha$, then $\text{Lip}_{\alpha()}(X) = \text{Lip}_\alpha(X)$.

3. Generalized Hardy spaces $H^{[\phi, q]}_{U}(X)$

Let $\phi : X \times (0, \infty) \to (0, \infty)$, $1 < q \leq \infty$ and $1/q + 1/q' = 1$.

**Definition 3.1** ([\phi, q]-atom). A function $a$ on $X$ is called a $[\phi, q]$-atom if there exists a ball $B$ such that

(i) $\text{supp } a \subset B$,

(ii) $\|a\|_q \leq \frac{1}{\mu(B)^{1/q'} \phi(B)}$,

(iii) $\int_X a(x) \, d\mu(x) = 0$,

where $\|a\|_q$ is the $L^q$ norm of $a$. We denote by $A[\phi, q]$ the set of all $[\phi, q]$-atoms.

Let $\mathcal{F}$ be the set of all continuous, increasing and bijective functions $\Phi : [0, \infty) \to [0, \infty)$. Then $\Phi(0) = 0$ and $\lim_{r \to \infty} \Phi(r) = \infty$ for all $\Phi \in \mathcal{F}$.

Let $\mathcal{F}_X$ be the set of all functions $\Phi : X \times [0, \infty) \to [0, \infty)$ such that

(i) $\Phi(x, \cdot) \in \mathcal{F}$ for every $x \in X$, and

(ii) $\Phi(\cdot, r)$ is measurable on $X$ for all $r \in [0, \infty)$.

We denote by $\Phi^{-1}(x, \cdot)$ the inverse of $\Phi(x, \cdot)$ with respect to $r \in [0, \infty)$.

For $\Phi \in \mathcal{F}_X$ and $B = B(x, r)$, let

\[
(3.1) \quad \phi(x, r) = \phi(B) = \frac{1}{\mu(B) \Phi^{-1}(x, 1/\mu(B))}.
\]

Then

$$\frac{1}{\mu(B)^{1/q'} \phi(B)} = \mu(B)^{1/q} \Phi^{-1} \left( \frac{x}{\mu(B)} \right).$$

If $\Phi(x, r) = r^{p(x)}$, $p(\cdot) : X \to (0, 1]$, then

$$\frac{1}{\mu(B)^{1/q'} \phi(B)} = \mu(B)^{1/q - 1/p(x)}.$$ 

If $\Phi(x, r) = r^p$, $0 < p \leq 1$, then

$$\frac{1}{\mu(B)^{1/q'} \phi(B)} = \mu(B)^{1/q - 1/p}.$$ 

In this case, $[\phi, q]$-atoms are the usual $(p, q)$-atoms.
We define $H_{U}^{[\phi,q]}(X)$ as a subspace of the dual of $L_{q',\phi}(X)$. We can see $A[\phi, q] \subset (L_{q',\phi}(X))^*$ as follows. If $a$ is a $[\phi, q]$-atom and a ball $B$ satisfies (i)–(iii), then

\begin{align}
\left| \int_{X} a(x)g(x) \, d\mu(x) \right| &= \left| \int_{B} a(x)(g(x) - g_{B}) \, d\mu(x) \right| \\
&\leq \|a\|_{q} \left( \int_{B} |g(x) - g_{B}|^{q'} \, d\mu(x) \right)^{1/q'} \\
&\leq \frac{1}{\phi(B)} \left( \frac{1}{\mu(B)} \int_{B} |g(x) - g_{B}|^{q'} \, d\mu(x) \right)^{1/q'} \\
&\leq \|g\|_{L_{q',\phi}(X)}.
\end{align}

That is, the mapping $g \mapsto \int_{X} ag \, d\mu$ is a bounded linear functional on $L_{q',\phi}(X)$ with norm not exceeding 1.

**Definition 3.2 ($H_{U}^{[\phi,q]}(X)$).** Let $\phi : X \times (0, \infty) \to (0, \infty)$, $1 < q \leq \infty$, $1/q + 1/q' = 1$ and $U \in \mathcal{F}$ be concave. We define the space $H_{U}^{[\phi,q]}(X) \subset (L_{q',\phi}(X))^*$ as follows:

$f \in H_{U}^{[\phi,q]}(X)$ if and only if there exist sequences $\{a_{j}\} \subset A[\phi, q]$ and positive numbers $\{\lambda_{j}\}$ such that

\begin{equation}
(3.3) \quad f = \sum_{j} \lambda_{j}a_{j} \quad \text{in} \quad (L_{q',\phi}(X))^* \quad \text{and} \quad \sum_{j} U(\lambda_{j}) < \infty.
\end{equation}

From $U(0) = 0$ and the concavity of $U$ it follows that

\begin{align}
(3.4) \quad U(Cr) &\leq CU(r), \quad 1 \leq C < \infty, \quad 0 \leq r < \infty, \\
(3.5) \quad U(r + s) &\leq U(r) + U(s), \quad 0 \leq r, s < \infty.
\end{align}

Then $H_{U}^{[\phi,q]}(X)$ is a linear space.

In general, the expression (3.3) is not unique. We define

\[ \|f\|_{H_{U}^{[\phi,q]}} = \inf \left\{ U^{-1} \left( \sum_{j} U(\lambda_{j}) \right) \right\}, \]

where the infimum is taken over all expressions as in (3.3). We note that $\|f\|_{H_{U}^{[\phi,q]}}$ is not a norm in general. Let $m(f, g) = U(\|f - g\|_{H_{U}^{[\phi,q]}})$ for $f, g \in H_{U}^{[\phi,q]}(X)$. Then $m(f, g)$ is a metric and $H_{U}^{[\phi,q]}(X)$ is complete with respect to this metric.

If $\phi(B) = \mu(B)^{1/p-1}$ and $U(r) = r^{p}$, then $H_{U}^{[\phi,q]}(X)$ coincides $H^{p,q}(X)$ defined by Coifman and Weiss (1977). They showed $H^{p,q}(X) = H^{p,\infty}(X)$ with equivalent metrics when $0 < p \leq 1 < q \leq \infty$ and denoted this space by $H^{p}(X)$. We extend this result to $H_{U}^{[\phi,q]}(X) = H_{U}^{[\phi,\infty]}(X)$ in the next section.
Let $I(r) = r$. Then $\|f\|_{H_{I}^{[\phi,q]}}$ is a norm and $H_{I}^{[\phi,q]}(X)$ is a Banach space, which was defined by Zorko (1986) in the case $X = \mathbb{R}^n$. Therefore, our definition is a generalization of both definitions.

From the definition we have the following relations.

**Proposition 3.1.**

(i) If $1 < q_1 < q_2 \leq \infty$, then

$$H_{U}^{[\phi,q_2]}(X) \subset H_{U}^{[\phi,q_1]}(X).$$

(ii) If $\psi(B) \leq C\phi(B)$ for all balls $B$, then

$$H_{U}^{[\phi,q]}(X) \subset H_{U}^{[\psi,q]}(X).$$

(iii) If $V(r) \leq CU(r)$ for $0 \leq r \leq 1$, then

$$H_{U}^{[\phi,q]}(X) \subset H_{V}^{[\phi,q]}(X).$$

(iv) For any concave function $U \in \mathcal{F}$,

$$H_{U}^{[\phi,q]}(X) \subset H_{I}^{[\phi,q]}(X).$$

In the above, the inclusion mapping are continuous.

4. **Equivalence $H_{U}^{[\phi,q]}(X) = H_{U}^{[\phi,\infty]}(X)$**

**Theorem 4.1.** Let $\phi \in \mathcal{G}_*$. If there exists $C_* > 0$ such that

\begin{align}
(4.1) & \quad U(rs) \leq C_* U(r) U(s) \quad \text{for} \quad 0 < r, s \leq 1, \\
(4.2) & \quad U \left( \frac{\mu(B_1)\phi(B_1)}{\mu(B_2)\phi(B_2)} \right) \leq C_* \frac{\mu(B_1)}{\mu(B_2)} \quad \text{for} \ B_1 \subset B_2,
\end{align}

then

$$H_{U}^{[\phi,q]}(X) = H_{U}^{[\phi,\infty]}(X),$$

with equivalent topologies.

For $\Phi(x, r) \in \mathcal{F}_X$, let

$$\phi(x, r) = \phi(B) = \frac{1}{\mu(B)\Phi^{-1}(x, 1/\mu(B))}.$$ 

**Example 4.1.** Assume that $\mu(X) < \infty$. Let $p(\cdot)$ be log-Hölder continuous and

$$\Phi(x, r) = r^{p(x)}, \quad U(r) = r^{p+} \quad \text{with} \quad 0 < p_- \leq p_+ \leq 1.$$ 

Then the assumption of Theorem 4.1 holds. Therefore

$$H_{U}^{[\phi,q]}(X) = H_{U}^{[\phi,\infty]}(X).$$
In this case we denote $H_{U}^{[\phi,q]}(X)$ by $H^{p(\cdot)}(X)$. If $p(\cdot) \equiv p$, then $H^{p(\cdot)}(X) = H^{p}(X)$, the usual Hardy space.

5. **Duality**

Let $L_{c}^{q}(X)$ be the set of all $L^{q}$-functions with bounded support, and let

$$L_{c}^{q,0}(X) = \left\{ f \in L_{c}^{q}(X) : \int_{X} f \, d\mu = 0 \right\}.$$  

Then, for $1 < q \leq \infty$, $L_{c}^{q,0}(X)$ is dense in $H_{U}^{[\phi,q]}(X)$.

If $g \in L_{q',\phi}(X)$ and $f \in L_{c}^{q,0}(X)$, then $f(g + c)$ is integrable for all constants $c$ and $\int_{X} f(g + c) \, d\mu$ is independent of $c$.

**Theorem 5.1.** If $U$ satisfies

\[ \sup_{0<s \leq 1} \frac{U(rs)}{U(s)} \to 0 \quad (r \to 0), \]

then

\[ \left( H_{U}^{[\phi,q]}(X) \right)^{*} = L_{q',\phi}(X). \]

More precisely, if $g \in L_{q',\phi}(X)$, then the mapping $\ell : f \mapsto \int_{X} f(g + c) \, d\mu$, for $f \in L_{c}^{q,0}(X)$, can be extended to a continuous linear functional on $H_{U}^{[\phi,q]}(X)$. Conversely, if $\ell$ is a continuous linear functional on $H_{U}^{[\phi,q]}(X)$, then there exists $g \in L_{q',\phi}(X)$ such that $\ell(f) = \int_{X} f(g + c) \, d\mu$ for $f \in L_{c}^{q,0}(X)$. The norm $\|\ell\|$ is equivalent to $\|g\|_{L_{q',\phi}}$.

**Corollary 5.2.** Let $\phi \in \mathcal{G}_{*}$. Then, for any $q \in (1, \infty]$ and for any concave function $U \in \mathcal{F}$ with (5.1),

\[ \left( H_{U}^{[\phi,q]}(X) \right)^{*} = L_{1,\phi}(X). \]

**Corollary 5.3.** Let $\phi \equiv 1$. Then, for any $q \in (1, \infty]$ and for any concave function $U \in \mathcal{F}$ with (5.1),

\[ \left( H_{U}^{[\phi,q]}(X) \right)^{*} = \text{BMO}(X). \]

**Corollary 5.4.** Let $\phi \in \mathcal{G}_{*}$ and there exists $C > 0$ such that

\[ \int_{0}^{\delta(x,y)} \frac{\phi(x,t)}{t} \, dt \leq C\phi(x, \delta(x,y)), \quad x, y \in X. \]

Then, for any $q \in (1, \infty]$ and for any concave function $U \in \mathcal{F}$ with (5.1),

\[ \left( H_{U}^{[\phi,q]}(X) \right)^{*} = \Lambda_{\phi}(X). \]
Example 5.1. Under the assumption of Example 4.1, let \( \alpha(x) = 1/p(x) - 1 \). Then
\[
(H^{p(.)}(X))^* = \text{Lip}_{\alpha(.)}(X).
\]

6. EQUIVALENCE \( H_{U}^{\phi,q}(X, d, \mu) = H_{U}^{\psi,q}(X, \delta, \mu) \)

For a space of homogeneous type \((X, d, \mu)\) such that the balls are open sets, let
\[
\delta(x, y) = \begin{cases} 
\inf \{ \mu(B^d) : B^d \ni x, y \} & \text{if } x \neq y, \\
0 & \text{if } x = y,
\end{cases}
\]
where \( B^d \) denotes a ball by the quasi-distance \( d \). Then \((X, \delta, \mu)\) is normal and the topologies induced on \( X \) by \( d \) and \( \delta \) coincide.

**Theorem 6.1.** Suppose that \( \psi : X \times (0, \infty) \to (0, \infty) \) satisfies (2.1). Let \( \tilde{\phi}(x, r) = \phi(x, \mu(B^d(x, r))) \). Then
\[
\mathcal{L}_{p,\phi}(X, d, \mu) = \mathcal{L}_{p,\overline{\phi}}(X, \delta, \mu),
\]
with equivalent topologies, respectively.

Example 6.1. Let \( X = \mathbb{R}^n \), \( d(x, y) = |x - y| \) and \( \mu \) be the Lebesgue measure. Then
\[
\delta(x, y) = \frac{v_n}{2^n} |x - y|^n,
\]
where \( v_n \) is the volume of the unit ball. Therefore, \((\mathbb{R}^n, \delta, \mu)\) is of order \( 1/n \) and, for \( 0 < \alpha < \theta = 1/n \),
\[
I_{\alpha}f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{\delta(x, y)^{1-\alpha}} d\mu(y) = \int_{\mathbb{R}^n} \frac{f(y)}{(\frac{v_n}{2^n} |x - y|^{n})^{1-\alpha}} d\mu(y).
\]

7. RIESZ POTENTIALS ON \( \mathcal{L}_{p,\phi}(X) \)

**Theorem 7.1.** Let \( 0 < \alpha < \theta \), \( 1 \leq p < \infty \) and \( \phi, \psi \in \mathcal{G}_{\ast} \). Assume that there exists a constant \( A > 0 \) such that, for all \( x \in X \) and \( r > 0 \),
\[
\int_{r}^{\infty} \frac{t^{\alpha} \phi(x, t)}{t^{1+\theta}} dt \leq A \psi(x, r).
\]
Then \( \tilde{I}_{\alpha} \) is bounded from \( \mathcal{L}_{p,\phi}(X) \) to \( \mathcal{L}_{p,\psi}(X) \).
Corollary 7.2. Let $\mu(X) < \infty$, $0 < \alpha < \theta$. Assume that $\beta(\cdot)$ and $\gamma(\cdot)$ are log-Hölder continuous and
\[
\alpha + \beta(x) = \gamma(x) \quad \text{with} \quad 0 < \beta^- < \gamma^+ < \theta.
\]
Then $I_\alpha$ is bounded from $\text{Lip}_{\beta(\cdot)}(X)$ to $\text{Lip}_{\gamma(\cdot)}(X)$.

8. Riesz potentials on $H^{[\phi,\infty]}_U(X)$

Theorem 8.1. Let $0 < \alpha < \theta$, $\phi, \psi \in G_*$ and $U, V \in \mathcal{F}$ be concave. Assume that there exist $0 < \epsilon < 1$, $0 < \tau \leq 1$ and $A > 0$ such that
\[
\begin{align*}
(8.1) & \quad \psi(x, r)r^\alpha \leq A\phi(x, r), \quad r > 0, \\
(8.2) & \quad s^{\alpha-\theta-1}(s\psi(x, s))^{1/\epsilon} \leq Ar^{\alpha-\theta-1}(r\psi(x, r))^{1/\epsilon}, \quad 0 < r \leq s, \\
(8.3) & \quad V(r) \leq Ar^\tau, \quad r \in (0, 1], \\
(8.4) & \quad V(rs) \leq AV(r)U(s), \quad 0 \leq r, s \leq 1.
\end{align*}
\]
Then there exists $C > 0$ such that
\[
\|I_\alpha a\|_{H^{[\psi,\infty]}_V} \leq C \quad \text{for all} \quad a \in A[\phi, \infty],
\]
and $I_\alpha$ extends to a continuous linear map from $H^{[\phi,\infty]}_U(X)$ to $H^{[\psi,\infty]}_V(X)$.

Corollary 8.2. Let $\mu(X) < \infty$, $0 < \alpha < \theta$. Assume that $p(\cdot)$ and $q(\cdot)$ are log-Hölder continuous and
\[
\begin{align*}
(8.5) & \quad -\frac{1}{p(x)} + \alpha = -\frac{1}{q(x)} \quad \text{with} \quad \frac{1}{1+\theta} < p^- < q^+ \leq 1.
\end{align*}
\]
Then there exists $C > 0$ such that
\[
\|I_\alpha a\|_{H^{p(\cdot)}} \leq C \quad \text{for all} \quad a \in A(p(\cdot), \infty),
\]
and $I_\alpha$ extends to a continuous linear map from $H^{p(\cdot)}(X)$ to $H^{q(\cdot)}(X)$.

In the above, $a \in A(p(\cdot), \infty)$ means that there exists $B = B(x, r)$ such that
\[
\begin{align*}
(i) & \quad \text{supp} \ a \subset B, \\
(ii) & \quad \|a\|_q \leq \mu(B)^{1/q-1/p(x)}, \\
(iii) & \quad \int_X a(x) \ d\mu(x) = 0.
\end{align*}
\]
REFERENCES


EIICHI NAKAI, DEPARTMENT OF MATHEMATICS, OSAKA KYOIKU UNIVERSITY, KASHIWARA, OSAKA 582-8582, JAPAN

E-mail address: enakai@cc.osaka-kyoiku.ac.jp