Spherical functions on $U(n, n)/(U(n) \times U(n))$ and hermitian Siegel series

Yumiko Hironaka

§0 Introduction

Let $k'$ be an unramified quadratic extension over a non-archimedean local field $k$ of characteristic 0. We fix a prime element $\pi$ of $k$, and the additive value $v_\pi(\ )$ and the normalized absolute value $|\ |$ on $k^\times$, where $|\pi|^{-1} = q$ is the cardinality of the residue class field of $k$. We consider hermitian matrices with respect to the involution $*$ on $k'$ which is identity on $k$, and set

$$\mathcal{H}_m = \{ A \in M_m(k') \mid A^* = A \}, \quad \mathcal{H}_m^{nd} = \mathcal{H}_m \cap GL_m(k'), \quad (0.1)$$

where, for a matrix $A = (a_{ij}) \in M_{mn}(k')$, we denote by $A^*$ the matrix $(a_{ji^*}) \in M_{nm}(k')$.

For $T \in \mathcal{H}_n^{nd}$, we define the spaces

$$\mathcal{X}_T = \{ x \in M_{2n}(k') \mid x^* H_n x = T \}, \quad X_T = \mathcal{X}_T / U(T),$$

where $H_n = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix} \in \mathcal{H}_{2n}$ and $U(T) = \{ g \in GL_n(k') \mid g^* T g = T \}$. We consider spherical functions on $X_T$, which is isomorphic to $U(n, n)/(U(T) \times U(T))$ over $k$, where $U(n, n) = U(H_n)$ (cf. Lemma 1.1). We consider the following integral

$$\omega_T(\overline{x}; s) = \int_K |f_T(kx)|^{s+\varepsilon} dk, \quad (\overline{x} \in X_T, \ s \in \mathbb{C}^n). \quad (0.2)$$

Here $dk$ is the normalized Haar measure on $K = U(n, n) \cap GL_{2n}(O_{k'})$, 

$$\varepsilon = (-1, \ldots, -1, -\frac{1}{2}) + \left( \frac{\pi \sqrt{-1}}{\log q}, \ldots, \frac{\pi \sqrt{-1}}{\log q} \right) \in \mathbb{C}^n,$$

$$|f_T(x)|^s = \prod_{i=1}^n |d_i(x_2 T^{-1} x_2^*)|^{s_i},$$

where $x_2$ is the lower half $n$ by $n$ block of $x \in \mathcal{X}_T$ and $d_i(y)$ is the determinant of the upper left $i$ by $i$ block of $y$. The right hand side of (0.2) is absolutely convergent.
if $\text{Re}(s_i) \geq 1$ ($1 \leq i \leq n - 1$) and $\text{Re}(s_n) \geq \frac{1}{2}$, continued to a rational function of $q^{s_1}, \ldots, q^{s_n}$, and becomes a common eigen function with respect to the action of Hecke algebra $\mathcal{H}(G, K)$ with $G = U(n, n)$; thus we have a spherical function on $X_T$. It is convenient to introduce the new variable $z$ which is related to $s$ by

$$s_i = -z_i + z_{i+1} \quad (1 \leq i \leq n-1), \quad s_n = -z_n,$$

(0.3)

and we write $\omega_T(\overline{x}; z) = \omega_T(\overline{x}; s)$. We denote by $W$ the Weyl group of $G$ with respect to the maximal $k$-split torus in $G$, which is isomorphic to $S_n \ltimes (C_2)^n$, $S_n$ acts on $z_i$ by permutation of indices. We denote by $\Sigma^+$ the set of positive roots of $G$ with respect to the Borel group, and regard it a subset of $\mathbb{Z}^n$ and write $\langle \alpha, z \rangle = \sum_{i=1}^{n} \alpha_i z_i$ for $\alpha \in \Sigma^+$ (for details, see §2.2).

Our main results in §1 and §2 are the following.

**Theorem 1** (i) For any $T \in \mathcal{H}_n^{nd}$, the function

$$\prod_{1 \leq i \leq j \leq n} \frac{(1 + q^{z_i - z_j})}{(1 - q^{z_i - z_j - 1})} \times \omega_T(\overline{x}; z)$$

is holomorphic for all $z$ in $\mathbb{C}^n$ and $S_n$-invariant, and the function

$$|2|^{-z_1 - z_n} \prod_{1 \leq i \leq j \leq n} \frac{(1 + q^{z_i - z_j})(1 + q^{z_i + z_j})}{(1 - q^{z_i - z_j - 1})(1 - q^{z_i + z_j - 1})} \times \omega_T(\overline{x}; z)$$

is also holomorphic for all $z$ in $\mathbb{C}^n$ and $W$-invariant. In particular the latter is an element in $\mathbb{C}[q^{\pm z_1}, \ldots, q^{\pm z_n}]^W$.

(ii) For any $T \in \mathcal{H}_n^{nd}$ and $\sigma \in W$, the following functional equation holds

$$\omega_T(x; z) = \Gamma_{\sigma}(z) \cdot \omega_T(x; \sigma(z)),$$

(0.4)

where

$$\Gamma_{\sigma}(z) = \prod_{\alpha \in \Sigma^+, \sigma(\alpha) < 0} f_{\alpha}(\langle \alpha, z \rangle),$$

$$f_{\alpha}(t) = \begin{cases} 1 - q^{t-1} & \text{if } \alpha \text{ is short} \\ \frac{q^t - q^{-1}}{2} & \text{if } \alpha \text{ is long} \end{cases}$$

In §3, we give an explicit expression for $\omega_T(x_T; s)$.

As an application, we consider the hermitian Siegel series in §4. For each $T \in \mathcal{H}_n$, the hermitian Siegel series $b_\pi(T; s)$ is defined by

$$b_\pi(T; s) = \int_{\mathcal{H}_n(k')} \nu_\pi(R)^{-s} \psi(\text{tr}(TR)) dR,$$

(0.5)

where $\psi$ is an additive character on $k$ of conductor $\mathcal{O}_k$, $\text{tr}( )$ is the trace of matrix and $\nu_\pi(R)$ is the "denominator" of $R$, which is certain non-negative powers of $q$ (cf. (4.1)) As for Siegel series (for symmetric matrices), F. Sato and the author have given
a new integral expression and related it to a spherical function on the symmetric space $O(2n)/(O(n) \times O(n))$ (cf. [HS]). In the present paper we develop the similar argument for hermitian Siegel series. Since we know well about the functional equations of spherical functions $\omega_T(\overline{x};s)$ with respect to $W$ as above, we can bring out the functional equation of $b_\pi(T;s)$ as an application; thus we obtain an integral expression of $b_\pi(T;s)$ and its functional equation.

**Theorem 2**

(i) If $\Re(s) > 2n$, one has

$$b_\pi(T;s) = \zeta_n(k'; \frac{s}{2})^{-1} \cdot \int_{X_T(O_{k'})} |N_{k'/k}(\det x_2)|^{\frac{s}{2}-n} |\Theta_T| (x),$$

(0.6)

where $X_T(O_{k'}) = X_T \cap M_{2n,n}(O_{k'})$, $\zeta_n(k'; )$ is the zeta function of the matrix algebra $M_n(k')$, and $|\Theta_T| (x)$ is a certain normalized measure on $X_T$.

(ii) For any $T \in H_{m}^{nd}$, one has

$$\frac{b_\pi(T;s)}{\prod_{i=0}^{n-1}(1-(-1)^i q^{-s+i})} = \chi_\pi(\det(T)^{n-1} |\det(T/2)|^{s-n} \times \frac{b_\pi(T;2n-s)}{\prod_{i=0}^{n-1}(1-(-1)^i q^{-(2n-s)+i})},$$

where $\chi_\pi$ is the character on $k^\times$ determined by

$$\chi_\pi(a) = (-1)^{v_\pi(a)} = |a|^{\frac{\pi \sqrt{-1}}{\log q}}, \quad a \in k^\times.$$

We note here that the above functional equation is related to an element of the Weyl group of $U(n, n)$, which was not the case for symmetric case when $n$ is odd. The existence of functional equation of $b_\pi(T;s)$ was known in an abstract form as functional equations of Whittaker functions of a $p$-adic group by Karel [Kr](cf. also Kudla-Sweet [KS], Ikeda [Ik]).

§1

We follow the notations in the introduction. For $A \in H_m$ and $X \in M_{mn}(k')$, we write

$$A[X] = X^*AX = X^* \cdot A \in H_n,$$

then our spaces are given for each $T \in H_m^{nd}$ by

$$X_T = \{ x \in M_{2n,n}(k') \mid H_n[x] = T \}, \quad X_T = X_T/U(T),$$

(1.1)

$$x_T = \left( \begin{array}{l} \frac{1}{2}T1_n \\ 1_n \end{array} \right) \in X_T.$$

The group $G = U(n, n)$ acts on $X_T$, as well as on $X_T$, through left multiplication, which is transitive by Witt’s theorem for hermitian matrices (cf. [Sch], Ch.7, §9). Our first observation is the following.
Lemma 1.1 The stabilizer subgroup of $G = U(n, n)$ at $x_T U(T) \in X_T$ is given as
\[
\{ \tilde{T}^{-1} \begin{pmatrix} h_1^* & 0 \\ 0 & h_2^* \end{pmatrix} \tilde{T} \mid h_1, h_2 \in U(T) \}, \quad \tilde{T} = \begin{pmatrix} 1_n & \frac{1}{2}T \\ 1_n & -\frac{1}{2}T \end{pmatrix} \in GL_{2n}(k').
\]
In particular, the space $X_T$ is isomorphic to $G/(U(T) \times U(T))$.

We fix the Borel subgroup $B$ of $G$ as
\[
B = \left\{ \begin{pmatrix} b & 0 \\ 0 & b^{*-1} \end{pmatrix} \begin{pmatrix} 1_n & a \\ 0 & 1_n \end{pmatrix} \mid b \text{ is upper triangular of size } n, \quad a + a^* = 0 \right\}, \quad (1.2)
\]
and introduce the $B$-relative invariants on $X_T$
\[
f_{T,i}(x) = d_i(x_{T}^{-1}x_{T}^{*}) \quad 1 \leq i \leq n, \quad (1.3)
\]
associated with $k$-rational characters $\psi_i$ of $B$ by
\[
f_{T,i}(bx) = \psi_i(b)f_{T,i}(x), \quad \psi_i(b) = N(d_i(b))^{-1}, \quad (1.4)
\]
where $x_T$ is the lower half $n$ by $n$ block of $x \in X_T$, $d_i(y)$ is the determinant of upper left $i$ by $i$ block of $y$ and $N = N_{k'/k}$. Since $f_{T,i}(xh) = f_{T,i}(x)$ for any $h \in U(T)$, we understand $f_{T,i}(x)$ as $B$-relative invariants on $X_T$, $1 \leq i \leq n$.

Remark 1.2 It is possible to realize above objects as the sets of $k$-rational points of algebraic sets defined over $k$ and develop the arguments, but we take down to earth way for simplicity of notations. We only note here that $X_T$ is isomorphic to $U(n, n)/(U(n) \times U(n))$ over the algebraic closure $\bar{k}$ of $k$ and $\{ x \in X_T | f_{T,i}(x) \neq 0, 1 \leq i \leq n \}$ is a Zariski open $B$-orbit over $\bar{k}$, where $U(n) = U(1, n)$.

Hereafter, we write an element $x = xU(T)$ in $X_T$ by its representative $x$ in $X_T$ for simplicity of notations. We set $|0| = 0$ for the absolute value on $k^{\times}$ for convenience.

The modulus character $\delta$ on $B$ (which is characterized by $d_i(bb') = \delta(b')^{-1}d_i(b)$ for the left invariant measure $d_i(b)$ on $B$) is given by
\[
\delta_b(b) = \prod_{i=1}^{n-1} |\psi_i(b)|^{-1} \times |\psi_n(b)|^{-\frac{1}{2}}.
\]

Now we introduce the spherical function $\omega(x; s)$ on $X_T = X_T/U(T)$
\[
\omega_T(x; s) = \omega_T^{(n)}(x; s) = \int_K |f_T(kx)|^{s+\epsilon} dk, \quad (1.5)
\]
where $dk$ is the normalized Haar measure on $K = G \cap GL_{2n}(\mathcal{O}_{k'})$, $s \in \mathbb{C}^n$
\[
\epsilon = (-1, \ldots, -1, -\frac{1}{2}) + \left( \frac{\pi \sqrt{-1}}{\log q}, \ldots, \frac{\pi \sqrt{-1}}{\log q} \right) \in \mathbb{C}^n;
\]
\[
f_T(x) = \prod_{i=1}^{n} f_{T,i}(x), \quad |f_T(x)|^s = \prod_{i=1}^{n} |f_{T,i}(x)|^{s_i}.
\]
The right hand side of (1.5) is absolutely convergent if \( \text{Re}(s_i) \geq 1 \) (1 \( \leq i \leq n - 1 \)) and \( \text{Re}(s_n) \geq \frac{1}{2} \), continued to a rational function of \( q^{s_1}, \ldots, q^{s_n} \), and becomes a common eigenfunction with respect to the action of the Hecke algebra \( \mathcal{H}(G, K) \) (cf. [H2], §1).

Since we see
\[
\omega_{T[h]}(x; s) = \omega_{T}(xh^{-1}; s), \quad h \in GL_n(k'), \ x \in X_{T[h]},
\]
it suffices to consider only for diagonal \( T \)'s for the study of functional properties of \( \omega_{T}(x; s) \) (e.g., Theorem 1 in the introduction).

We write \( \omega_{T}(x; z) = \omega_{T}(x; s) \) for the new variable \( z \) introduced by (1.7). The Weyl group \( W \) of \( G \) relative to the maximal \( k \)-split torus in \( B \) acts on rational characters of \( B \) as usual (i.e., \( \sigma(\psi)(b) = \psi(n^{-1}_\sigma bn_\sigma) \) by taking a representative \( n_\sigma \) of \( \sigma \), so \( W \) acts on \( z \in \mathbb{C}^n \) and on \( s \in \mathbb{C}^n \) as well. We will determine the functional equations of \( \omega_{T}(x; s) \) with respect to this Weyl group action. The group \( W \) is isomorphic to \( S_n \ltimes C_2^n \), \( S_n \) acts on \( z \) by permutation of indices and \( W \) is generated by \( S_n \) and \( \tau : (z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_{n-1}, -z_n) \).

By using a result on spherical functions on the space of hermitian forms ((cf. [H1]-§2 or [H3]-§4.2)), we obtain the following.

**Theorem 1.3** For any \( T \in \mathcal{H}^n \), the function
\[
\prod_{1 \leq i < j \leq n} \frac{q^{z_j} + q^{-z_i}}{q^{z_j} - q^{-z_i-1}} \times \omega_{T}(x; z)
\]
is holomorphic for any \( z \) in \( \mathbb{C}^n \) and \( S_n \)-invariant. In particular it is an element in \( \mathbb{C}[q^{\pm z_1}, \ldots, q^{\pm z_n}]^{S_n} \).

**Remark 1.4** For the transposition \( \tau_i = (i \ i+1) \in W, \ 1 \leq i \leq n - 1 \), the following functional equation holds by Theorem 1.3
\[
\omega_{T}(x; z) = \frac{1 - q^{z_i-z_{i+1}-1}}{q^{z_i-z_{i+1}} - q^{-1}} \times \omega_{T}(x; \tau_i(z)), \quad 1 \leq i \leq n - 1.
\]

On the other hand, one can obtain (1.8) directly in the similar way to the case of \( \tau \) in § 3, then Theorem 1.3 follows from (1.8).

**§2**

2.1. We fix a unit \( \epsilon \in \mathcal{O}_k^\times \) for which \( k' = k(\sqrt{\epsilon}) \) and \( \epsilon \in 1 + 4\mathcal{O}_k^\times \) if \( k \) is dyadic (cf. [Om]-63.3 and 63.4).

**Theorem 2.1** For any \( T \in \mathcal{H}^n \), the spherical function satisfies the following functional equation:
\[
\omega_T(x; z) = |2|^{2z_n} \omega_T(x; \tau(z)).
\]
The case \( n = 1 \) is easy; we calculate spherical functions explicitly for representatives of \( K_1 \)-orbits in \( X_T \), where \( K_1 = U(H_1) \cap GL_2(\mathcal{O}_k') \), and obtain the functional equation. For \( n \geq 2 \) we take a representative \( w_\tau \) of \( \tau \in W \) by

\[
w_\tau = \begin{pmatrix} 1_{n-1} & 0 & 1 \\ 0 & 1_{n-1} & 0 \\ 1 & 0 & 1_{n-1} \end{pmatrix} \in G,
\]

and take the parabolic subgroup \( P = P_\tau \) attached to \( \tau \) (cf. [Bo], 21.11)

\[
P = B \cup Bw_\tau B
\]

where each empty place in the above expression means zero-entry. Hereafter we fix a diagonal \( T \in H_n^{ad} \), and write \( f_i(x) = f_{T,i}(x) \) by abbreviating the suffix \( T \). The \( B \)-relative invariants \( f_i(x) \) become \( P \)-relative invariants associated with \( \psi_i \) except \( i = n \). We consider the following action of \( \tilde{P} = P \times GL_1 \) on \( \tilde{X}_T = X_T \times V \) with \( V = M_{21}(k') \):

\[
(p, r) \cdot (x, v) = (px, \rho(p)vr^{-1}),
\]

where \( \rho(p) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) for the decomposition of \( p \in P \) as in (2.1). For \((x, v) \in \tilde{X}_T \), set

\[
g(x, v) = \det \left[ \begin{pmatrix} 1_{n-1} & 0 \\ 0 & 1_{n-1} \end{pmatrix} \begin{pmatrix} x_2 - y \end{pmatrix} \right],
\]

where \( x_2 \) is the lower half \( n \) by \( n \) block of \( x \) (the same before) and \( y \) is the \( n \)-th row of \( x \). Then we obtain

**Lemma 2.2** \( g(x, v) \) is a relative \( \tilde{P} \)-invariant on \( \tilde{X}_T \) associated with character

\[
\tilde{\psi}(p, r) = N(d_{n-1}(p))^{-1}N(r)^{-1} = \psi_{n-1}(p)N(r)^{-1}, \quad (p, r) \in \tilde{P} = P \times GL_1,
\]

satisfies

\[
g(x, v_0) = f_n(x), \quad v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},
\]

and is expressed as

\[
g(x, v) = D(x)[v],
\]

with some hermitian matrix

\[
D(x) = \begin{pmatrix} a & b + c\sqrt{\epsilon} \\ b - c\sqrt{\epsilon} & d \end{pmatrix}, \quad a, c, d \in k, \quad b = -\frac{1}{2} f_{n-1}(x), \quad ad = b^2 - c^2\epsilon. \quad (2.2)
\]
In order to prove Theorem 2.2, we need the functional equation of the following function

$$\zeta_{K_1}(A; s) = \int_{K_1} |d_1(h \cdot A)|^{s-\frac{1}{2}} dh,$$

where $d_1$ is the normalized Haar measure on $K_1$.

**Lemma 2.3** Let $x \in X_T$ such that $f_T(x) \neq 0$ and $D(x)$ be given by (2.4). Then one has

$$\zeta_{K_1}(D(x), s) = |2|^{-2s} |f_{n-1}(x)|^{2s} \zeta_{K_1}(D(x), -s).$$

Now Theorem 2.2 is proved as follows. By the embedding

$$K_1 \rightarrow K = K_n, \quad h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \tilde{h} = \begin{pmatrix} 1_{n-1} & a & b \\ c & 1_{n-1} & d \end{pmatrix},$$

we have

$$\omega_T(x; s) = \int_{K_1} dh \int_K |f(\tilde{h}kx)|^{s+\epsilon} dk$$

$$= \int_K \chi_{\pi} \left( \prod_{i<n} f_i(kx) \right) \prod_{i<n} |f_i(kx)|^{s_i-1} \left( \int_{K_1} \chi_{\pi} (f_n(\tilde{h}kx)) |f_n(\tilde{h}kx)|^{s_n-\frac{1}{2}} dh \right) dk.$$

By definition of $f_n(x)$ and $g(x, v)$ and Lemma 2.3, we see

$$f_n(\tilde{h}x) = g(x, (d-c)) = D(x) \left[ \begin{pmatrix} d \\ -c \end{pmatrix} \right] = d_1(h^{-1} \cdot D(x)), \quad (h \in K_1),$$

hence we have

$$\omega_T(x; s) = \int_K \chi_{\pi} \left( \prod_{i<n} f_i(kx) \right) \prod_{i<n} |f_i(kx)|^{s_i-1} \zeta_{K_1}(D(kx); s_n + \frac{\pi\sqrt{-1}}{\log q}) dk.$$

Then the functional equation of $\omega_T(x; s)$ follows from Lemma 2.4.

2.2. We denote by $\Sigma$ the set of roots of $G$ with respect to the $k$-split torus of $G$ contained in $B$ and by $\Sigma^+$ the set of positive roots with respect to $B$. We may understand

$$\Sigma^+ = \{ e_i - e_j, e_i + e_j \mid 1 \leq i < j \leq n \} \cup \{ 2e_i \mid 1 \leq i \leq n \},$$

where $e_i \in \mathbb{Z}^n$ whose $j$-th component is given by the Kronecker delta $\delta_{ij}$, and the set

$$\Sigma_0 = \{ e_i - e_{i+1} \mid 1 \leq i \leq n - 1 \} \cup \{ 2e_n \}$$

forms the set of simple roots. We denote by $\Delta$ the subset of $W$ consisting of the reflections associated to elements in $\Sigma_0$. Then $\Delta = \{ \tau_i \mid 1 \leq i \leq n - 1 \} \cup \{ \tau \}$ generates $W$. We write $\alpha < 0$ if $\alpha \in \Sigma$ is negative. We see the pairing $\langle , \rangle$ on $\Sigma \times \mathbb{C}^n$ given by

$$\langle \alpha, z \rangle = \sum_{i=1}^n \alpha_i z_i, \quad (\alpha \in \Sigma, \ z \in \mathbb{C}^n).$$

is $W$-invariant. Then we obtain
Theorem 2.4  For $T \in \mathcal{H}_n^{nd}$ and $\sigma \in W$, the spherical function $\omega_T(x;z)$ satisfies the following functional equation

$$\omega_T(x;z) = \Gamma_\sigma(z) \cdot \omega_T(x;\sigma(z)),$$

(2.3)

where

$$\Gamma_\sigma(z) = \prod_{\alpha \in \Sigma^+, \sigma(\alpha) < 0} f_\alpha(\langle\alpha, z\rangle),$$

$$f_\alpha(t) = \begin{cases} |2|^t & \text{if } \alpha = 2e_i \text{ for some } i \\ 1 - q^{t-1} & \text{otherwise}, \\ q^t - q^{-1} & \text{otherwise}, \end{cases}$$

and $q = (\varphi, \varphi)^{1/2}$. In particular, the Gamma factor $\Gamma_\sigma(z)$ does not depend on $T$ nor $x$.

Proof. For an element of $\Delta$, we know the Gamma factor by (1.8) and Theorem 2.2. In general, assume that $\sigma \in W$ has the shortest expression

$$\sigma = \sigma_\ell \cdots \sigma_1,$$

with $\sigma_i \in \Delta$ associated by some $\alpha_i \in \Sigma_0$. Since the Gamma factors satisfy cocycle relations and $\langle , \rangle$ is $W$-invariant, we have

$$\Gamma_\sigma(z) = \Gamma_{\sigma_\ell}(\sigma_{\ell-1} \cdots \sigma_1(z)) \cdots \Gamma_{\sigma_2}(\sigma_1(z)) \cdot \Gamma_{\sigma_1}(z)$$

$$= f_{\alpha_\ell}(\langle\alpha_\ell, \sigma_{\ell-1} \cdots \sigma_1(z)\rangle) \cdots f_{\alpha_2}(\langle\alpha_2, \sigma_1(z)\rangle) \cdot f_{\alpha_1}(\langle\alpha_1, z\rangle).$$

Hence $\Gamma_\sigma(z)$ has the required form, since we have

$$\{ \alpha \in \Sigma^+ | \sigma(\alpha) < 0 \} = \{ \sigma_1 \cdots \sigma_{k-1}(\alpha_k) | 1 \leq k \leq \ell \}.$$

Corollary 2.5  Set $\rho \in W$ by

$$\rho(z_1, \ldots, z_n) = (-z_n, -z_{n-1}, \ldots, -z_1).$$

(2.4)

Then

$$\Gamma_\rho(z) = |2|^{2(z_1 + \cdots + z_n)} \prod_{1 \leq i < j \leq n} \frac{1 - q^{z_i + z_j - 1}}{q^{z_i + z_j} - q^{-1}}.$$  

(2.5)

Remark 2.6  The above $\rho$ gives the functional equation of the hermitian Siegel series (cf. §4), and it is interesting that such $\rho$ corresponds to the unique automorphism of the extended Dynkin diagram of the root system of type $(C_n)$, which was pointed out by Y. Komori.

By Theorem 1.3 and Theorem 2.5, we obtain the following.
Theorem 2.7 Set
\[ F(z) = \prod_{\alpha \in \Sigma^+} g_{\alpha}(z), \]
where, for \( \alpha \in \Sigma \),
\[ g_{\alpha}(z) = \begin{cases} 
|2|^{-(\langle \alpha, z \rangle / 2)} & \text{if } \alpha = \pm 2e_i \text{ for some } i \\
\frac{1 + q^{(\alpha, z)}}{1 - q^{(\alpha, z) - 1}} & \text{otherwise}
\end{cases} \]

Then, for any \( T \in \mathcal{H}^n_{nd} \), the function \( F(z) \omega_T(x; z) \) is holomorphic for all \( z \) in \( \mathbb{C}^n \) and \( W \)-invariant. In particular it is an element in \( \mathbb{C}[q^{\pm z_1}, \ldots, q^{\pm z_n}]^W \).

Proof. Take any \( \sigma \in \Delta \) associated by \( \alpha \in \Sigma_0 \). Then \( g_{\alpha}(\sigma z) = g_{\sigma\alpha}(z) = g_{-\alpha}(z) \) and \( \Gamma_{\sigma}(z) = g_{-\alpha}(z)/g_{\alpha}(z) \). Thus, \( F(z) \omega_T(x, z) \) is \( W \)-invariant, since \( \triangle \) generates \( W \).

Set \( F_1(z) = \prod_{1 \leq i < j \leq n} \frac{1 + q^{z_i - z_j}}{1 - q^{z_i - z_j - 1}} \) and \( F_2(z) = |2|^{-z-z_n} \prod_{1 \leq i < j \leq n} \frac{1 + q^{z_i + z_j}}{1 - q^{z_i + z_j - 1}} \). Then \( F(z) = F_1(z)F_2(z) \) and \( F_1(z) \omega_T(x; z) \) is holomorphic in \( z \in \mathbb{C}^n \) and \( S_n \)-invariant by Theorem 1.3. Hence \( F(z) \omega_T(x; z) \) is holomorphic in \( z \in \mathbb{C}^n \), since it is \( W \)-invariant and holomorphic for certain region e.g., \( \{ z \in \mathbb{C}^n \mid \text{Re}(z_i) \leq 0 \} \).

\[ \square \]

§3

3.1. In this section we give an explicit formula of \( \omega_T(x; s) \) at \( x_T \) by using the general formula of Proposition 1.9 in [H2] (or Theorem 2.6 in [H4]). In order to apply it, we have to check several conditions ((A1) – (A4) in [H4]-§1), and it is obvious our \( (B, X_T) \) satisfies them except (A3), which is the same as (C) below.

Proposition 3.1 The following condition \((C)\) is satisfied.
\((C): \) For \( y \in X_T \) such that \( f_T(y) = 0 \), there exists a character \( \psi \in \langle \psi_i | 1 \leq i \leq n \rangle \) whose restriction to the identity component of the stabilizer of \( B \) at \( y \) is not trivial.

Theorem 3.2 Let \( T = \text{Diag}(\pi^{\lambda_1}, \ldots, \pi^{\lambda_n}) \) with \( \lambda_1 \geq \lambda_2 \cdots \geq \lambda_n \geq v_{\pi}(2) \). Then
\[ \omega_T(x_T; z) = (-1)^{\sum_i \lambda_i(n-i+1)} q^{\sum_i \lambda_i(n-i+1)/2} \frac{1 - q^{-2}}{\prod_{i=1}^{2n} (1 - (-1)q^{-i})} \sum_{\sigma \in \Gamma} q^{<\lambda, \sigma(z)>} \gamma(z) \Gamma_{\sigma}(z), \]
(3.1)
where \(<\lambda, z> = \sum_{i=1}^{n} \lambda_i z_i\), \( \Gamma_{\sigma}(z) \) is defined in Theorem 2.5, and
\[ \gamma(z) = \prod_{1 \leq i < j \leq n} \frac{(1 - q^{2z_i - 2z_j})(1 - q^{2z_i + 2z_j})}{(1 - q^{2z_i - 2z_j})(1 - q^{2z_i + 2z_j})} \cdot \prod_{i=1}^{n} \frac{1 - q^{2z_i - 1}}{1 - q^{2z_i}}. \]
We admit Proposition 3.1 for the moment and prove Theorem 3.2.

The set $X^*_{T} = \{ x \in X_T \mid f_T(x) \neq 0 \}$ becomes a disjoint union of $B$-orbits as follows.

\[ X^*_{T} = \bigcup_{u \in \mathcal{U}} X_{T,u}, \quad \mathcal{U} = (\mathbb{Z}/2\mathbb{Z})^{n-1}, \]

\[ X_{T,u} = \{ x \in X_T \mid v_\pi(f_{T,i}(x)) \equiv u_1 + \cdots + u_i \pmod{2}, \quad 1 \leq i \leq n - 1 \} . \]

We set

\[ \omega_{T,u}(x;s) = \int_{K} |f_{T}(kx)|_{u}^{s+\epsilon} dk, \]

where

\[ |f_T(y)|_{u}^{s+\epsilon} = \begin{cases} |f_T(y)|^{s+\epsilon} & \text{if } y \in X_{T,u}, \\ 0 & \text{otherwise} . \end{cases} \]

For a character $\chi = (\chi_1, \ldots, \chi_{n-1})$ of $\mathcal{U}$, we set

\[ L_T(x; \chi; z) = \int_{K} \chi(f_T(kx)) |f_T(kx)|^{s+\epsilon} dk = \sum_{u \in \mathcal{U}} \chi(u) \omega_{T,u}(x;z), \]

where $\chi(u) = \prod_{i=1}^{n-1} \chi_i(u_1 + \cdots + u_i)$. Adjusting $z$ according to $\chi$, by adding $\frac{\pi\sqrt{-1}}{\log q}$ to $z_i$ if necessary, we may write

\[ L_T(x; \chi; z) = \omega_T(x;z\chi). \]

Then, by the functional equations of $\omega_T(x;z)$ (Theorem 2.5), we have

\[ L_T(x; \chi; z) = \Gamma_{\sigma}(z_{\chi}) L_T(x; \sigma(\chi); \sigma(z)), \quad \sigma \in W \tag{3.2} \]

by taking suitable character $\sigma(\chi)$ of $\mathcal{U}$. If $\chi$ is the trivial character $1$, then (3.2) coincides with the original functional equation of $\omega_T(x;z)$. We obtain

\[ (\omega_{T,u}(x_T;z))_{u} = \chi(u) \omega_{T,u}(x_T;\sigma(z))_{u} , \]

where

\[ A = (\chi(u))_{x,u}, \quad \sigma A = (\sigma(\chi)(u))_{x,u} \in GL_{2^n}(\mathbb{Z}), \]

and $G(\sigma, z)$ is the diagonal matrix of size $2^n$ whose $(\chi, \chi)$-component is $\Gamma_{\sigma}(z_{\chi})$. For $T$ given as in Theorem 3.2, we obtain

\[ \int_{U} |f_T(ux_T)|^{s+\epsilon} du = |f_T(x_T)|^{s+\epsilon} = (-1)^{\sum_i \lambda_i(n-i+1)} q^{\sum_i \lambda_i(n-i+\frac{1}{2})} q^{<\lambda,z>}, \]

where $U$ is the Iwahori subgroup of $K$ compatible with $B$ and $du$ is the normalized Haar measure on $U$. Setting

\[ \delta_u(x_T, z) = \begin{cases} (-1)^{\sum_i \lambda_i(n-i+1)} q^{\sum_i \lambda_i(n-i+\frac{1}{2})} q^{<\lambda,z>} & \text{if } x_T U(T) \in X_{T,u} \\ 0 & \text{otherwise}, \end{cases} \]
we have, by Proposition 1.9 in [H2] (or its generalization Theorem 2.6 in [H4]),

$$(\omega_{T,u}(x_{T};z))_{u} = \frac{1}{Q} \sum_{\sigma \in W} \gamma(\sigma(z)) (A^{-1} \cdot G(\sigma, z) \cdot \sigma A) (\delta_{u}(x_{T}, \sigma(z)))_{u},$$

where

$$Q = \sum_{\sigma \in W} [U\sigma U : U^{-1}] = \prod_{i=1}^{2n} (1 - (-1)^{i}q^{-i})/(1 - q^{-2})^{n}.$$

Hence we obtain

$$\omega_{T}(x_{T};z) = \sum_{u \in \mathcal{U}} 1(u)\omega_{u}(x_{T};z) = \frac{(-1)^{\sum_{i}\lambda_{i}(n-i+1)}q^{\sum_{i}\lambda_{i}(n-i+\frac{1}{2})}}{Q} \sum_{\sigma \in W} \gamma(\sigma(z))\Gamma_{\sigma}(z)q^{<\lambda,\sigma(z)>}.$$  

3.2. In order to prove Proposition 3.1, we consider the action of $G \times U(T)$ on $\mathfrak{X}_T$ by $(g, h) \circ x = gxh^{-1}$. Then, the stabilizer $B_y$ of $B$ at $yU(T) \in X_T$ coincides with the image $B_{(y)}$ of the projection to $B$ of the stabilizer $(B \times U(T))_{y}$ at $y \in \mathfrak{X}_T$ to $B$. Hence the condition (C) is equivalent to the following:

(C'): For $y \in \mathfrak{X}_T$ such that $f_{T}(y) = 0$ there exists $\psi \in \langle \psi_i | 1 \leq i \leq n \rangle$ whose restriction to the identity component of $B_{(y)}$ is not trivial.

It is sufficient to prove the condition (C) (equivalently, (C')) over the algebraic closure $\overline{k}$, since, for a connected linear algebraic group $\mathbb{H}$, $\mathbb{H}(k)$ is dense in $\mathbb{H}(\overline{k})$. In the rest of this section, we consider algebraic sets over $\overline{k}$, extend the involution $*$ on $k'$ to $\overline{k}$ and denote it by $-$, and write $\overline{x} = (\overline{x}_{ij})$ for any matrix $x = (x_{ij})$. Since $\mathfrak{X}_T$ is isomorphic to $\mathfrak{X}_{T[h]}$ by $x \mapsto xh$ and $B(x) = B_{(xh)}$ for $h \in GL_n$, we may assume that $T = 1_n$. Then, our situation is the following:

$$\mathfrak{X} = \mathfrak{X}_{1_n} = \{ x \in M_{2n,n} \ | \ H_n[x] = 1_n \},$$

$$(U(n, n) \times U(n)) \times \mathfrak{X} \rightarrow \mathfrak{X}, \quad ((g, h), x) \mapsto (g, h) \circ x = gxh^{-1}.$$  

We consider the set

$$\tilde{\mathfrak{X}} = \{ (x, y) \in M_{2n,n} \oplus M_{2n,n} \ | \ 'yH_nx = 1_n \}$$

together with $GL_{2n} \times GL_n$-action defined by

$$(g, h) \ast (x, y) = (gxh^{-1}, gy'h), \quad \dot{g} = H_n^t g^{-1} H_n,$$ (3.3)

and take the Borel subgroup $P$ of $GL_{2n}$ by

$$P = \left\{ \begin{pmatrix} p_1 & r \\ 0 & p_2 \end{pmatrix} \in GL_{2n} \ | \ p_1, 'p_2 \in B_n, \ r \in M_n \right\},$$

where $B_n$ is the Borel subgroup of $GL_n$ consisting of upper triangular matrices.
Then, the embedding \( \iota : \mathfrak{X} \hookrightarrow \widetilde{\mathfrak{X}} \), \( x \mapsto (x, \overline{x}) \) is compatible with the actions, i.e., we have the commutative diagram

\[
\begin{array}{ccc}
(U(n, n) \times U(n)) & \times & \mathfrak{X} \\
\downarrow id & & \downarrow \iota \\
(GL_{2n} \times GL_{n}) & \times & \widetilde{\mathfrak{X}} \\
\end{array}
\]

For \( (x, y) \in \widetilde{\mathfrak{X}} \) and \( p \in P \), set

\[
\tilde{f}_{i}(x, y) = d_{i}(x_{2}, y_{2}), \quad \tilde{\psi}_{i}(p) = \prod_{1 \leq j \leq i} p_{j}^{-1} p_{n+j}, \quad (1 \leq i \leq n),
\]

where \( x_{2} \) (resp. \( y_{2} \)) is the lower half \( n \) by \( n \) block of \( x \) (resp. \( y \)), and \( p_{j} \) is the \( j \)-th diagonal entry of \( p \). Then for each \( i \), we see

\[
\tilde{f}_{i}((p, r) \star (x, y)) = \tilde{\psi}_{i}(p) \tilde{f}_{i}(x, y), \quad (p, r) \in P \times GL_{n},
\]

\[
\tilde{f}_{i}(x, \overline{x}) = f_{i}(x), \quad (x \in \mathfrak{X}), \quad \tilde{\psi}_{i}|_{B} = \psi_{i}.
\]

We set

\[
S = \left\{ (x, y) \in \widetilde{\mathfrak{X}} \mid \prod_{i=1}^{n} \tilde{f}_{i}(x, y) = 0, \quad (P \times GL_{n}) \star (x, y) \cap \mathfrak{X} \neq \emptyset \right\}.
\]

For \( \alpha = (x, y) \in \widetilde{\mathfrak{X}} \), we denote by \( H_{\alpha} \) the stabilizer of \( P \times GL_{n} \) at \( \alpha \), and by \( P_{\alpha} \) its image of the projection to \( P \). In order to prove the condition \((C)\), it is sufficient to show the following:

\((\widetilde{C})\) : For each \( \alpha \in S \), there exists some \( \psi \in \langle \tilde{\psi}_{i} \mid 1 \leq i \leq n \rangle \) whose restriction to the identity component of \( P_{\alpha} \) is not trivial.

We show the condition \((\widetilde{C})\) by taking suitable representatives by \( P \times GL_{n} \)-action.

(i) Assume \( \alpha = (x, y) \in S \) satisfies \( \det(x_{2}) \neq 0 \). Then, in the \( P \times GL_{n} \)-orbit containing \( \alpha \), there is \( \beta = \left( \begin{array}{cc} 0 & 1_{n} \\ 1_{n} & h \end{array} \right) \) with some hermitian matrix \( h \), further we may assume

\[
h = 1_{r} \perp (0) \perp h_{1} \quad \text{or} \quad h = 1_{r} \perp h_{2},
\]

where \( 0 \leq r \leq n - 1 \), and for \( h_{2} \), there is some \( i, \ (1 < i \leq n - r) \) such that each entry in the first row and column or in the \( i \)-th row and column is 0 except at \( (1, i) \) or \( (i, 1) \) which are 1.

Then \( H_{\beta} \) contains the following elements, according to the above type of \( h \),

\[
\left( \delta_{r+1}(a) \begin{array}{c} 1_{n} \\ 1_{n} \end{array} \right), 1_{n} \quad \text{or} \quad \left( \delta_{r+1}(a) \begin{array}{c} 1_{n} \\ 1_{n} \end{array} \right) \delta_{r+1}(a),
\]

where \( \delta_{j}(a) \) is the diagonal matrix in \( GL_{n} \) whose diagonal entries are 1 except the \( j \)-th which is \( a \in GL_{1} \). Hence we see \( \tilde{\psi}_{r+1} \neq 1 \) on the identity component of \( P_{\beta} \).

(ii) The case \( \alpha = (x, y) \in S \) with \( \det(y_{2}) \neq 0 \) is reduced to the case \( \det(x_{2}) \neq 0 \), since \( \beta = (y, x) \in S \) and \( H_{\beta} = \{(p, r^{-1}) \mid (p, r) \in H_{\alpha} \} \) and \( \tilde{\psi}_{i}(p) = \tilde{\psi}_{i}(p)^{-1} \).
(iii) Assume \( \alpha = (x', y') \in \mathcal{S} \) satisfies \( \det x'_2 = \det y'_2 = 0 \). Then, in the \( P \times GL_n \)-orbit containing \( \alpha \), there is some \( \beta = (x, y) \) of the following type: for some integers \( r_i, e_j \) satisfying

\[
\begin{align*}
1 & \leq r_1 < r_2 < \cdots < r_\ell \leq n \quad (1 \leq \ell < n), \\
1 & \leq e_1 < e_2 < \cdots < e_k \leq n \quad (k = n - \ell),
\end{align*}
\]

\[
x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}
\]

with \( x_i, y_i \in M_n \) is given by

\[
x_1 : 1 \text{ at } (r_i, k + i)\text{-entry for } 1 \leq i \leq \ell \text{ and } 0 \text{ at any other entry};
\]

\[
x_2 : 1 \text{ at } (e_i, i)\text{-entry for } 1 \leq i \leq k \text{ and } 0 \text{ at any other entry};
\]

\[
y_1 : \text{the } e_i\text{-th row is the same as in } x_2 \text{ for } 1 \leq i \leq k, \text{ and the } j\text{-the column is } 0 \text{ if } j > k;
\]

\[
y_2 : \text{the } r_i\text{-th row is the same as in } x_1 \text{ for } 1 \leq i \leq \ell, \text{ and for each } i, \text{ any } (i, j)\text{-entry is } 0 \text{ for } j > k \text{ if some } (i, j')\text{-entry is non-zero with } j' \leq k.
\]

Let \( D(a) \) be the diagonal matrix in \( GL_n \) whose \( i\)-th diagonal entry is \( a \in GL_1 \) (resp. 1) if every \( (i, j)\)-entry of \( y_2 \) is 0 for \( j \leq k \) (resp. otherwise), where the \( r_i\)-th diagonal entry of \( D(a) \) is \( a \) by this choice. Then \( H_\beta \) contains

\[
\left( \begin{array}{l|l}
D(a) & 1_n \\
\hline
1_n & a1_\ell
\end{array} \right),
\]

and \( \tilde{\psi}_{r_i} \neq 1 \) on the identity component of \( P_\beta, 1 \leq i \leq \ell \).

\[\Box\]

\section{§4}

We recall the hermitian Siegel series, and give its integral representation and functional equation. Let \( \psi \) be an additive character of \( k \) of conductor \( \mathcal{O}_k \). For \( T \in \mathcal{H}_n(k') \), the hermitian Siegel series \( b_\pi(T; s) \) is defined by

\[
b_\pi(T; s) = \int_{\mathcal{H}_n(k')} \nu_\pi(R)^{-s} \psi(\text{tr}(TR)) dR,
\]

where \( \text{tr}(\cdot) \) is the trace of matrix and \( \nu_\pi(R) \) is defined as follows: if the elementary divisors of \( R \) with negative \( \pi \)-powers are \( \pi^{-e_1}, \ldots, \pi^{-e_r} \), then \( \nu_\pi(R) = q^{e_1 + \cdots + e_r} \), and \( \nu_\pi(R) = 1 \) otherwise (cf. [Sh]-§13).

In the following we assume that \( T \) is nondegenerate, since the properties of \( b_\pi(T; s) \) can be reduced to the nondegenerate case. We recall the set \( \mathcal{X}_T \) for \( T \in \mathcal{H}_n^{nd}(k') \)

\[
\mathcal{X}_T = \mathcal{X}_T(k') = \{ x \in M_{2n,n}(k') \mid H_n[x] = T \},
\]

which is the fibre space \( g^{-1}(T) \) for the polynomial map \( g : M_{2n,n}(k') \to \mathcal{H}_n(k'), g(x) = H_n[x] \) defined over \( k \). We may take the measure \( |\Theta_T| \) on \( \mathcal{X}_T \) induced by a \( k \)-rational differential form \( \omega \) on \( M_{2n,n}(k') \) satisfying \( \omega \wedge g^*(dT) = dx \) where \( dT \) is the canonical
gauge form on $\mathcal{H}_n(k')$, $dx$ is the canonical gauge form on $M_{2n,n}(k')$. Then the following identity holds (cf. [Ym], [HS]-§2):

$$\int_{X_T(k')} \phi(x) \Theta_T(x) = \lim_{\varepsilon \to \infty} \int_{\mathcal{H}_n(\pi^{-\varepsilon})} \psi(-\text{tr}(Ty)) \int_{M_{2n,n}(k')} \phi(x) \psi(\text{tr}(H_n[x]y)) dx dy,$$

where $\phi \in \mathcal{S}(M_{2n,n}(k'))$, a locally constant compactly supported function on $M_{2n,n}(k')$ and $\mathcal{H}_n(\pi^{-\varepsilon}) = \mathcal{H}_n(k') \cap M_{n}(\pi^{-\varepsilon}\mathcal{O}_{k'})$.

The following lemma can be proved in the similar line to the case of symmetric matrices (cf. [HS]-§2).

**Lemma 4.1** If $\text{Re}(s) > n$, one has

$$\int_{X_T(\mathcal{O}_{k'})} |N_{k'/k}(\det x_2)|^{s-n} |\Theta_T|(x) = \lim_{\varepsilon \to \infty} \int_{\mathcal{H}_n(\pi^{-\varepsilon}\mathcal{O}_{k'})} \psi(-\text{tr}(Ty))dy \int_{M_{2n,n}(\mathcal{O}_{k'})} |N_{k'/k}(\det x_2)|^{s-n} \psi(\text{tr}(H_n[x]y)) dx.$$

Let us recall the zeta function of the matrix algebra $M_n(k')$ and its explicit formula:

$$\zeta(k'; s) = \int_{M_n(\mathcal{O}_{k'})} |\det x|_{k'}^{s-n} dx = \int_{M_n(\mathcal{O}_{k'})} |N_{k'/k}(\det x)|^{s-n} dx = \prod_{i=1}^{n} \frac{1-q^{-2i}}{1-q^{-2(s-i+1)}}.$$

Then we obtain the following integral expression of hermitian Siegel series, which can be proved in a similar line to the case of Siegel series.

**Theorem 4.2** If $\text{Re}(s) > 2n$, we have

$$b_{\pi}(T; s) = \zeta_n(k'; \frac{s}{2})^{-1} \times \int_{X_T(\mathcal{O}_{k'})} |N_{k'/k}(\det x_2)|^{\frac{s}{2}-n} |\Theta_T|(x).$$

We introduce the spherical function on $X_T$ with respect to the Siegel parabolic subgroup $P = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G \right\ | a, b, d \in M_n(k') \}$ by

$$\tilde{\omega}_T(x; s) = \int_{K} |N_{k'/k}(\det(kx)_2)|^{s-n} dk.$$
Proposition 4.3 Denote the $K$-orbit decomposition of $X_T(O_k')$ as

$$X_T(O_k') = \sqcup_{i=1}^{r} Kx_i.$$

Then one has

$$b_\pi(T; s) = \zeta_n(k'; \frac{s}{2})^{-1} \cdot \sum_{i=1}^{r} c_i \tilde{\omega}_T(x_i; \frac{s}{2}), \quad c_i = \int_{Kx_i} |\Theta_T(y)|.$$

By Proposition 4.3 and Corollary 2.6, we obtain the following functional equation of hermitian Siegel series.

Theorem 4.4

$$\frac{b_\pi(T; s)}{\prod_{i=0}^{n-1} (1 - (-1)^i q^{-s+i})} = \chi_\pi(\det T)^{n-1} |\det(T/2)|^{s-n} \times \frac{b_\pi(T; 2n - s)}{\prod_{i=0}^{n-1} (1 - (-1)^i q^{-(2n-s)+i})}. $$

References


Yumiko Hironaka
Department of Mathematics, Faculty of Education and Integrated Sciences, Waseda University, Nishi-Waseda, Tokyo 169-8050, JAPAN.