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Applications of subspace theorem to the fractional parts of geometric series

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1 Introduction

Weyl's criterion states that a sequence $x_n \ (n = 0, 1, \ldots)$ is uniformly distributed modulo 1 if and only if

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \exp(2\pi i h x_n) = 0 \quad (1.1)$$

for every nonzero integer $h$. As a corollary, an arithmetic progression $\xi n + \eta \ (n = 0, 1, \ldots)$ is uniformly distributed modulo 1 if and only if its common difference is an irrational number. On the other hand, it is generally difficult to check the criterion (1.1) in the case where the sequence $x_n \ (n = 0, 1, \ldots)$ is a geometric progression $\xi \alpha^n \ (n = 0, 1, \ldots)$.

In this paper we study the fractional parts of geometric sequences whose common ratio $\alpha > 1$ is an algebraic number. We now review the fractional parts of powers of Pisot and Salem numbers. Pisot numbers are algebraic integers greater than 1 whose conjugates different from themselves have absolute values strictly less than 1. Salem numbers are algebraic integers greater than 1 which have at least one conjugate with modulus 1 and exactly one conjugate outside the unit circle. Let $||x||$ denote the distance from the real number $x$ to the nearest integer. Moreover, we write $\{x\}$ and $[x]$ the fractional part of $x$ and the integral part of $x$, respectively. Take a Pisot number $\alpha$. Since the trace of $\alpha^n$ is a rational integer,

$$\lim_{n \to \infty} ||\alpha^n|| = 0.$$

Next, let $\alpha$ be a Salem number. Then for any positive $\varepsilon$ there exists a nonzero $\xi \in \mathbb{Q}(\alpha)$ satisfying

$$\limsup_{n \to \infty} ||\xi \alpha^n|| < \varepsilon$$
(see [4]). However, little is known about the fractional parts of the sequence \( \xi \alpha^n \) \((n = 0, 1, \ldots)\) in the case of \( \xi \notin \mathbb{Q}(\alpha) \). For example, suppose that \( \alpha > 1 \) is a natural number and that \( \xi \) is a positive number. Then \( \xi \alpha^n \) \((n = 0, 1, \ldots)\) is uniformly distributed modulo 1 if and only if \( \xi \) is normal in base \( \alpha \). However, we even do not know whether the numbers \( \sqrt{2}, \sqrt[3]{5}, \) and \( \pi \) are normal in base 10 or not. In section 2 we survey the normality of an algebraic irrational number \( \xi \). In particular, we give a lower bound of the number \( \lambda_N(\alpha, \xi) \) of nonzero digits among the first \( N \) digits of the \( \alpha \)-ary expansion of \( \xi \). In other words, we count the number of \( n \in \mathbb{N} \) such that

\[
\{ \xi \alpha^n \} \geq \frac{1}{\alpha}.
\]

In section 3 and 4, we estimate the number of \( n \in \mathbb{N} \) satisfying

\[
\{ \xi \alpha^n \} \geq c(\alpha)
\]

for an algebraic number \( \alpha \) and a positive constant \( c(\alpha) \) depending only on \( \alpha \). In this paper, we introduce results without proofs in this paper.

2 Borel conjecture

Borel [5] showed that almost all positive numbers are normal in every integral base \( \alpha \geq 2 \). He [6] also conjectured that all irrational numbers \( \xi \) are normal. However, there is no such an irrational \( \xi \) whose normality was proved. In the case of \( \alpha \geq 3 \), we even do not know whether all digits 0, 1, \ldots, \( \alpha - 1 \) occur infinitely many times in the \( \alpha \)-ary expansion of an irrational number. In this section we introduce some partial results.

Let \( \alpha \geq 2 \) be a natural number and \( \xi > 0 \) an irrational number. In what follows, we denote the \( \alpha \)-ary expansion of \( \xi \) by

\[
\xi = \sum_{i=-\infty}^{M} s_i(\xi) \alpha^i = s_M(\xi) \cdots s_0(\xi).s_{-1}(\xi)s_{-2}(\xi) \cdots.
\]

Define the infinite word \( s \) by

\[
s = s_{-1}(\xi)s_{-2}(\xi) \cdots.
\]

First, we measure the complexity of the \( \alpha \)-ary expansion of \( \xi \) by the number \( p(N) \) of distinct blocks of length \( N \) appearing in the words \( s \). If \( \xi \) is normal in base \( \alpha \), then \( p(N) = \alpha^N \) for any positive \( N \). Ferenczi and Mauduit [9] showed that

\[
\lim_{N \to \infty} (p(N) - N) = \infty.
\]
Adamczewski and Bugeaud [1] improved their results as follows:

$$\lim_{N \to \infty} \frac{p(N)}{N} = \infty.$$ 

Moreover, Bugeaud and Evertse [8] showed for any positive $\xi$ with $\eta < 1/11$ that

$$\limsup_{N \to \infty} \frac{p(N)}{N(\log N)^{\eta}} = \infty.$$ 

Next, we give an lower bound of $\lambda_N(\alpha, \xi)$ in the case of $\alpha = 2$ , which we define in the previous section. Put

$$\xi' = \frac{\xi}{2[\log_2 \xi]}.$$ 

Note that $1 < \xi' < 2$. Let $D(\geq 2)$ be the degree of $\xi'$ and $A_D$ the leading coefficient of the minimum integer polynomial of $\xi'$. Bailey, Borwein, Crandall, and Pomerance [3] showed for any positive $\epsilon$ that there exists a positive $c(\epsilon)$ satisfying

$$\lambda_N(2, \xi) > (1 - \epsilon)(2A_D)^{-1/D}N^{1/D}$$ 

for $N \geq c(\epsilon)$. Rivoal [15] improved the coefficient $(1 - \epsilon)(2A_D)^{-1/D}$ of (2.1) for certain classes of algebraic irrational numbers $\xi$. Namely, suppose that there exist two polynomials $P, Q$ with positive integral coefficients and two positive integers $a, b$ fulfilling $P(\xi) = a + bQ(\xi)^{-1}$. Let $\epsilon$ be an arbitrary positive number. Then we have for sufficiently large $N$ (with threshold depending on $\xi$ and $\epsilon$)

$$\lambda_N(2, \xi) \geq (1 - \epsilon)(B(p)B(q))^{-1/\delta}N^{1/\delta},$$ 

where $\delta = \text{deg}(PQ)$ and $p, q$ are the dominant coefficients of $P$ and $Q$, respectively.

For instance, let $\xi_0 = 0.558 \ldots$ be the unique real zero of the polynomial $8X^3 - 2X^2 + 4X - 3$. (2.1) implies

$$\lambda_N(2, \xi_0) \geq (1 - \epsilon)16^{-1/3}N^{1/3}.$$ 

On the other hand, since $4\xi_0 = 1 + 2(2\xi_0^2 + 1)^{-1}$, we can apply (2.2) to $\xi_0$. Thus,

$$\lambda_N(2, \xi_0) \geq (1 - \epsilon)N^{1/3}.$$
3 Limit points of the fractional parts of powers of geometric series

Koksma [14] proved that, if any common ratio $\alpha > 1$ is given, then for almost all initial values $\xi$ the geometric sequences $\xi \alpha^n$ ($n = 0, 1, \ldots$) are uniformly distributed modulo 1. Similarly, let $\xi$ be any nonzero initial value. Then $\alpha \xi \alpha^n$ ($n = 0, 1, \ldots$) are uniformly distributed modulo 1 for almost all common ratios.

Now we introduce the exceptional set of Koksma's theorem. In particular, we consider the maximal limit points $\limsup_{n \to \infty} \{\xi \alpha^n\}$. It is known for a fixed $\alpha > 1$ that there is a nonzero $\xi$ satisfying

$$\limsup_{n \to \infty} \{\xi \alpha^n\} < 1.$$  

Hence, the sequence $\xi \alpha^n$ ($n = 0, 1, \ldots$) isn't uniformly distributed modulo 1. More precisely, let $\alpha > 2$. Then Tijdeman [16] constructed a nonzero $\xi = \xi(\alpha)$ such that

$$\limsup_{n \to \infty} \{\xi \alpha^n\} \leq \frac{1}{\alpha - 1}. \quad (3.1)$$

Let $\alpha_0 = 2.025 \ldots$ be the unique solution of $34X^3 - 102X^2 + 75X - 16 = 0$. Dubickas [11] showed for $1 < \alpha < \alpha_0$ that there exists a nonzero $\xi = \xi(\alpha)$ such that

$$\limsup_{n \to \infty} \{\xi \alpha^n\} \leq 1 - \frac{2(\alpha - 1)^2}{9(2\alpha - 1)^2} \quad (3.2)$$

Note that if $2 < \alpha < \alpha_0$, then (3.2) is stronger than (3.1). In fact, it is easy to check

$$1 - \frac{2(\alpha - 1)^2}{9(2\alpha - 1)^2} < \frac{1}{\alpha - 1}$$

for such an $\alpha$. It is a interesting problem to estimate the value

$$\inf_{\xi \in \mathbb{R}, \xi \neq 0} \limsup_{n \to \infty} \{\xi \alpha^n\} \quad (3.3)$$

for a given $\alpha$. Let $\alpha > 1$ be an algebraic number with minimal polynomial $a_dX^d + a_{d-1}X^{d-1} + \cdots + a_0 \in \mathbb{Z}[X]$ ($a_d > 0$). Take a positive $\xi$. If $\alpha$ is a Pisot of Salem number, then suppose $\xi \notin \mathbb{Q}(\alpha)$. Then Dubickas [10] proved

$$\limsup_{n \to \infty} \{\xi \alpha^n\} \geq c(\alpha) := \min \left\{ \frac{1}{L_+(\alpha)}, \frac{1}{L_-(\alpha)} \right\},$$
where
\[ L_+(\alpha) = \sum_{a_i > 0} a_i, \quad L_-(\alpha) = \sum_{a_i \leq 0} a_i. \]

Moreover, let
\[ \lambda_N(\alpha, \xi) = \text{Card} \{n \in \mathbb{Z} | 0 \leq n < N, \{\xi \alpha^n\} \geq c(\alpha)\}, \]
where Card denotes the cardinality. Note that if \( \alpha > 1 \) is a natural number, then \( \lambda(\alpha, \xi) \) means the number of nonzero digits of \( \alpha \)-ary expansion of \( \xi \). For simplicity, suppose that \( \alpha \) is an algebraic integer and that \( \alpha \) has at least one conjugate different from itself which is outside the unit circle. Let \( \alpha_1 = \alpha, \alpha_2, \ldots, \alpha_p \) be the conjugates of \( \alpha \) whose absolute values are greater than 1. In the same way as that of Theorem 3 of [10], we can show that
\[
\liminf_{N \to \infty} \frac{\lambda_N(\alpha, \xi)}{\log N} \geq \left( \log \left( 1 + \frac{\log \alpha}{\log |\alpha_2| + \cdots + \log |\alpha_p|} \right) \right)^{-1}. \tag{3.4}
\]

In the section 4, we improve this inequality in the case where \( \xi \) is an algebraic number with \( \xi \notin \mathbb{Q}(\alpha) \).

In the last of this section, we consider geometric sequences \( \xi \alpha^n \) (\( n = 0, 1, \ldots \)) for a fixed initial value. The author [12] gave an algorithm to construct common ratios \( \alpha \) such that \( \|\xi \alpha^n\| \) is arbitrarily small for all \( n \). Let \( \xi \) be a nonzero real number. Then for any positive numbers \( \epsilon \) and \( M \), there exists a common ratio \( \alpha \) with \( \alpha > M \) such that
\[
\limsup_{n \to \infty} \|\xi \alpha^n\| \leq \frac{1 + \epsilon}{2\alpha}.
\]
Moreover, the set of \( \alpha \) satisfying
\[
\limsup_{n \to \infty} \|\xi \alpha^n\| \leq \frac{1 + \epsilon}{\alpha}. \tag{3.5}
\]
is uncountable. In particular, there is an \( \alpha \) transcendental over the field \( \mathbb{Q}(\xi) \) satisfying (3.5).

4 Main results

In what follows, we assume that \( \alpha > 1 \) is an algebraic number with minimal polynomial \( a_d X^d + a_{d-1} X^{d-1} + \cdots + a_0 \in \mathbb{Z}[X] \) \( (a_d > 0) \). Write the conjugates of \( \alpha \) by \( \alpha_1 = \alpha, \ldots, \alpha_d \). Take an algebraic irrational positive number \( \xi \) with \( \xi \notin \mathbb{Q}(\alpha) \). Then we have the following:
THEOREM 4.1. (1) If $\alpha$ is a Pisot or Salem number, then
\[
\lim_{N \to \infty} \frac{\lambda_N(\alpha, \xi)}{\log N} = \infty.
\]
(2) Otherwise,
\[
\lim_{N \to \infty} \inf \frac{\lambda_N(\alpha, \xi)}{\log N} \geq \left( \log \left( \frac{\log M(\alpha)}{\log \alpha} \right) \right)^{-1},
\]
where $M(\alpha)$ is the Mahler measure of $\alpha$ defined by
\[
M(\alpha) = a_d \prod_{i=1}^{d} \max\{1, |\alpha_i|\}.
\]

Theorem 4.1 gives a good estimation if $\log M(\alpha)/\log \alpha$ is small. Now we give a numerical example in the case of $\alpha = 4 + \sqrt{2}$. Let $\xi$ be a positive number. By (3.4), we get
\[
\lim_{N \to \infty} \inf \frac{\lambda_N(4 + \sqrt{2}, \xi)}{\log N} \geq \log \left( \frac{\log(14)}{\log(4 - \sqrt{2})} \right)^{-1} = 0.978 \ldots
\]
Moreover, if $\xi$ is an algebraic number with $\xi \notin \mathbb{Q}(\sqrt{2})$, then Theorem 4.1 implies
\[
\lim_{N \to \infty} \inf \frac{\lambda_N(4 + \sqrt{2}, \xi)}{\log N} \geq \log \left( \frac{\log(14)}{\log(4 + \sqrt{2})} \right)^{-1} = 2.24 \ldots
\]
If $\alpha = 2$, then there is a big gap between the estimation (2.1) and the first statement of Theorem 4.1. So we give a stronger lower bound for $\lambda_N(\alpha, \xi)$ than that of Theorem 4.1 in the case where $\alpha$ is a Pisot or Salem number.

THEOREM 4.2. Let $\alpha > 1$ be a Pisot or Salem number. Let $\xi$ be a positive algebraic number with $\xi \notin \mathbb{Q}(\alpha)$. Put
\[
D = [\mathbb{Q}(\alpha, \xi) : \mathbb{Q}(\alpha)].
\]
Then there exists an effectively computable absolute constant $c > 0$ such that
\[
\lambda_N(\alpha, \xi) \geq c \frac{(\log N)^{3/2}}{(\log(4D))^{1/2}(\log \log N)^{1/2}}
\]
for every sufficiently large $N$. 
References


[7] Y. Bugeaud, On the $\beta$-expansion of an algebraic number in algebraic base $\beta$, manuscript.


