Nonlinear problems with singular diffusivity
and inhomogeneous terms

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In this talk we consider a singular diffusion equation associated with total variation with inhomogeneous terms as follows

\[ u : [0,1] \times [0, T) \to \mathbb{R}^n \ (n \geq 1) : \text{unknown function} \]

\[
(P) \left\{ \begin{array}{ll}
  u_t - \frac{1}{b(x)} \text{div} \left( a(x) \frac{u_x}{|u_x|} \right) = 0, & (x, t) \in (0, 1) \times (0, T), \\
  u(x, 0) = u_0(x), & x \in (0, 1), \\
  u(0, t) = g_0, \quad u(1, t) = g_1, & t \in (0, T),
\end{array} \right.
\]

where \( a(x), b(x) \) are given positive, continuous functions on \([0, 1]\) and \( u_0 \) is an initial data and \( g_0, g_1 \in \mathbb{R}^n \) are boundary condition. This equation (1) is written as the gradient system by taking energy

\[ E(u) = \int_0^1 a(x)|u_x| \, dx \]

with respect to the norm \( ||f||^2 = \int_0^1 b(x)|f(x)|^2 \, dx \). The equation (1) describes the motion of multi-grain problem studied in [3].

In the scalar valued case with boundary condition \( u(0) = 0, u(1) = 1 \), if \( a(x) \) has a unique minimum point \( x_0 \), then

\[ E(u) = \int_0^1 a(x)|u_x| \, dx \geq a(x_0) \int_0^1 u_x \, dx = a(x_0)(u(1) - u(0)) = a(x_0). \]

If \( u \) is a step function and jumps only at \( x_0 \), then the equality holds. So global minimizer is unique [2]. In general case, a global minimizer quite naturally has a discontinuity since it makes the energy low by concentrating its variation at the point where \( a(x) \) is minimal. It follows that many global minimizers may be piecewise constant functions.

We consider stationary problem of \( P \) in the vector valued case. Suppose that inhomogeneous term \( a(x), b(x) \) satisfy "concave condition" (cf [1]). We characterize stationary piecewise constant solutions.

References

