Coefficient conditions for certain classes concerning starlike functions of complex order (Study on Non-Analytic and Univalent Functions and Applications)

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Coefficient conditions for certain classes concerning starlike functions of complex order

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Abstract
For functions $f(z)$ which are starlike of complex order $b$ ($b \neq 0$) in the open unit disk $U$, some interesting sufficient conditions for coefficient inequalities of $f(z)$ are discussed.

1 Introduction and Preliminaries

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

\[(1.1)\quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (a_0 = 0, \; a_1 = 1)\]

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

Furthermore, let $\mathcal{P}$ denote the class of functions $p(z)$ of the form

\[(1.2)\quad p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n\]

which are analytic in $U$. If $p(z) \in \mathcal{P}$ satisfies $\text{Re} \; p(z) > 0 \; (z \in U)$, then we say that $p(z)$ is the Carathéodory function (cf. [1]).

If $f(z) \in \mathcal{A}$ satisfies the following inequality

\[\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in U)\]

for some $\alpha$ ($0 \leq \alpha < 1$), then $f(z)$ is said to be starlike of order $\alpha$ in $U$. We denote by $S^*(\alpha)$ the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which are starlike of order $\alpha$ in $U$. Similarly, we say that $f(z)$ is a member of the class $\mathcal{K}(\alpha)$ of convex functions of order $\alpha$ in $U$ if $f(z) \in \mathcal{A}$ satisfies the following inequality

\[\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (z \in U)\]

for some $\alpha$ ($0 \leq \alpha < 1$).

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As usual, in the present investigation, we write

\[ S^* \equiv S^*(0) \quad \text{and} \quad K \equiv K(0). \]

Classes \( S^*(\alpha) \) and \( K(\alpha) \) were introduced by Robertson [5].

Next, a function \( f(z) \in A \) is called \( \lambda \)-spiral like of order \( \alpha \) in \( U \) if and only if

\[
\text{Re} \left[ e^{i\lambda} \left( \frac{zf'(z)}{f(z)} - \alpha \right) \right] > 0 \quad (z \in U)
\]

for some real \( \lambda \left( -\frac{\pi}{2} < \lambda < \frac{\pi}{2} \right) \) and \( \alpha \left( 0 \leq \alpha < 1 \right) \). We denote this class by \( SP(\lambda, \alpha) \).

Moreover, for some non-zero complex number \( b \), we consider the subclasses \( S_b^* \) and \( K_b \) of \( A \) as follows:

\[
S_b^* = \left\{ f(z) \in A : \text{Re} \left[ 1 + \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right] > 0 \ (b \neq 0; \ z \in U) \right\}
\]

and

\[
K_b = \left\{ f(z) \in A : \text{Re} \left[ 1 + \frac{1}{b} \left( \frac{zf''(z)}{f'(z)} \right) \right] > 0 \ (b \neq 0; \ z \in U) \right\}.
\]

If a function \( f(z) \) belongs to the class \( S_b^* \) or \( K_b \), we say that \( f(z) \) is starlike or convex of complex order \( b \ (b \neq 0) \), respectively. In [3], Nasr and Aouf introduced the class \( S_b^* \).

Then, we can see that

\[
S_{1-\alpha}^* = S^*(\alpha), \quad K_{1-\alpha} = K(\alpha) \quad \text{and} \quad S_{(1-\alpha)e^{-i\lambda}c\infty\lambda}^* = SP(\lambda, \alpha).
\]

**Example 1.1**

\[
f(z) = \frac{z}{(1-z)^{2b}} = z + \sum_{n=2}^{\infty} \frac{\prod_{j=2}^{n}(j+2(b-1))}{(n-1)!} z^n \in S_b^* \quad (b \neq 0)
\]

and

\[
f(z) = \begin{cases} 
\frac{1 - (1-z)^{1-2b}}{1 - 2b} = z + \sum_{n=2}^{\infty} \frac{\prod_{j=2}^{n}(j+2(b-1))}{n!} z^n \in K_b \quad (b \neq \frac{1}{2}) \\
\log \left( \frac{1}{1-z} \right) = z + \sum_{n=2}^{\infty} \frac{1}{n} z^n \in K_{\frac{1}{2}} = K \left( \frac{1}{2} \right)
\end{cases}
\]

We apply the following lemma to obtain our results.

**Lemma 1.2** A function \( p(z) \in P \) satisfies \( \text{Re} \ p(z) > 0 \ (z \in U) \) if and only if

\[
p(z) \neq \frac{x - 1}{x + 1} \quad (z \in U)
\]

for all \( |x| = 1 \).
Then, by using Lemma 1.2, various conditions for starlike functions are studied. The following results are enumerated as the same examples.

**Lemma 1.3** A function \( f(z) \in \mathcal{A} \) is in \( S^*(\alpha) \) if and only if

\[
1 + \sum_{n=2}^{\infty} A_n z^{n-1} \neq 0 \quad (z \in \mathbb{U}; \ |x| = 1)
\]

where

\[
A_n = \frac{n+1-2\alpha + (n-1)x}{2-2\alpha} a_n.
\]

Silverman, Silvia, and Telage [6] have given

**Remark 1.4** The relation (1.3) of Lemma 1.3 is equivalent to

\[
\frac{1}{z} \left( f(z) * \frac{z + \frac{x + 2\alpha - 1}{2 - 2\alpha}}{(1 - z)^2} \right) \neq 0 \quad (z \in \mathbb{U}, \ |x| = 1)
\]

where \(*\) means the convolution or Hadamard product of two functions.

Furthermore, letting \( \alpha = 0 \) in Lemma 1.3, Nezhmetdinov and Ponnusamy [4] have given the sufficient conditions for coefficients of \( f(z) \) to be in the class \( S^* \).

Hayami, Owa and Sirivastava [2] have shown the following results.

**Theorem 1.5** If \( f(z) \in \mathcal{A} \) satisfies the following condition

\[
\sum_{n=2}^{\infty} \left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} (j+1-2\alpha)(-1)^{k-j} \binom{\beta k-j}{k-j} a_j \right\} \binom{\gamma n-k}{n-k} \right| \leq 2(1 - \alpha)
\]

for some \( \alpha (0 \leq \alpha < 1) \), \( \beta \in \mathbb{R} \), and \( \gamma \in \mathbb{R} \), then \( f(z) \in S^*(\alpha) \).

**Theorem 1.6** If \( f(z) \in \mathcal{A} \) satisfies the following condition

\[
\sum_{n=2}^{\infty} \left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} j(j+1-2\alpha)(-1)^{k-j} \binom{\beta k-j}{k-j} a_j \right\} \binom{\gamma n-k}{n-k} \right| \leq 2(1 - \alpha)
\]

for some \( \alpha (0 \leq \alpha < 1) \), \( \beta \in \mathbb{R} \), and \( \gamma \in \mathbb{R} \), then \( f(z) \in \mathcal{K}(\alpha) \).
Theorem 1.7 If \( f(z) \in \mathcal{A} \) satisfies the following condition
\[
\sum_{n=2}^{\infty} \left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} (j - \alpha + (1 - \alpha)e^{-2i\lambda})(-1)^{k-j} \binom{\beta}{k-j} a_j \right\} \binom{\gamma}{n-k} \right| + \left| \sum_{k=1}^{\infty} \left\{ \sum_{j=1}^{k} (j-1)(-1)^{k-j} \binom{\beta}{k-j} a_j \right\} \binom{\gamma}{n-k} \right| \leq 2(1 - \alpha) \cos \lambda
\]
for some \( \alpha \) (\( 0 \leq \alpha < 1 \)), \( \lambda \) (\( -\frac{\pi}{2} < \lambda < \frac{\pi}{2} \)), \( \beta \in \mathbb{R} \) and \( \gamma \in \mathbb{R}_l \) then \( f(z) \in \mathcal{S}\mathcal{P}(\lambda, \alpha) \). 

2 Main results

Main result for starlike of complex order \( b \) is contained in

Theorem 2.1 If \( f(z) \in \mathcal{A} \) satisfies the following condition
\[
\sum_{n=2}^{\infty} \left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} (j - 1 + 2b)(-1)^{k-j} \binom{\beta}{k-j} a_j \right\} \binom{\gamma}{n-k} \right| + \left| \sum_{k=1}^{\infty} \left\{ \sum_{j=1}^{k} (j-1)(-1)^{k-j} \binom{\beta}{k-j} a_j \right\} \binom{\gamma}{n-k} \right| \leq 2|b|
\]
for some \( b \in \mathbb{C} (b \neq 0), \beta \in \mathbb{R}, \) and \( \gamma \in \mathbb{R}_2 \), then \( f(z) \in \mathcal{S}_b^* \).

Proof. Let us define the function \( p(z) \) by \( p(z) = 1 + \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - 1 \right) \) for \( f(z) \in \mathcal{A} \).

Applying Lemma 1.2, \( f(z) \in \mathcal{S}_b^* \) if and only if
\[
p(z) = 1 + \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - 1 \right) \neq \frac{x-1}{x+1} \quad (z \in \mathbb{U})
\]
for all \( |x| = 1 \).

Then, we need not consider Lemma 1.2 for \( z = 0 \), because it follows that
\[
p(0) = 1 \neq \frac{x-1}{x+1} \quad (|x| = 1).
\]

Hence, the relation (2.1) is equivalent to
\[
2bz + \sum_{n=2}^{\infty} \left\{ (n - 1 + 2b) + x(n - 1) \right\} n^2 a_n z^n \neq 0.
\]

Dividing the both sides of (2.2) by \( 2bz \) \( (z \neq 0) \), we obtain that
\[
1 + \sum_{n=2}^{\infty} B_n z^{n-1} \neq 0
\]
where
\[
B_n = \frac{(n - 1 + 2b) + x(n - 1)}{2b} n^2 a_n \quad (n \geq 2).
\]
Therefore, it is sufficient that we prove
\[
\left( 1 + \sum_{n=2}^{\infty} B_n z^{n-1} \right) (1 - z)^{\beta} (1 + z)^{\gamma} = 1 + \sum_{n=2}^{\infty} \left[ \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} B_j (-1)^{k-j} \binom{\gamma}{k-j} \right\} \binom{\delta}{n-k} \right] z^{n-1} \neq 0
\]
where \( \beta, \gamma \in \mathbb{R} \) and \( B_1 = 1 \). Thus, if \( f(z) \) satisfies
\[
\sum_{n=2}^{\infty} \left\{ \sum_{j=1}^{n} \left( j - 1 + 2b \right) (-1)^{k-j} \binom{\beta}{k-j} a_j \right\} \binom{\gamma}{n-k} z^{n-1} \leq 2|b|
\]
then \( f(z) \in S_b^* \). The proof of Theorem 2.1 is completed.

We next derive the coefficient condition for functions \( f(z) \) to be in the class \( \mathcal{K}_b \).

**Theorem 2.2** If \( f(z) \in \mathcal{A} \) satisfies the following condition
\[
\sum_{n=2}^{\infty} \left\{ \sum_{k=1}^{n} \left( j - 1 + 2b \right) (-1)^{k-j} \binom{\beta}{k-j} a_j \right\} \binom{\gamma}{n-k} z^{n-1} \leq 2|b|
\]
for some \( b \in \mathbb{C} (b \neq 0) \), \( \beta \in \mathbb{R} \), and \( \gamma \in \mathbb{R} \), then \( f(z) \in \mathcal{K}_b \).

**Proof.** Since \( zf'(z) \in S_b^* \) if and only if \( f(z) \in \mathcal{K}_b \) and since
\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad zf'(z) = z + \sum_{n=2}^{\infty} n a_n z^n,
\]
replacing \( a_j \) in Theorem 2.1 by \( ja_j \), we easily prove Theorem 2.2.

Putting \( \beta = \gamma = 0 \) in Theorem 2.1 and Theorem 2.2, we have

**Corollary 2.3** If \( f(z) \in \mathcal{A} \) satisfies the following inequality
\[
\sum_{n=2}^{\infty} \left\{ |n - 1 + 2b| + (n - 1) \right\} |a_n| \leq 2|b|
\]
for some \( b \in \mathbb{C} (b \neq 0) \), then \( f(z) \in S_b^* \).
Corollary 2.4  If \( f(z) \in A \) satisfies the following inequality

\[
\sum_{n=2}^{\infty} n\left\{|n - 1 + 2b| + (n - 1)\right\}|a_n| \leq 2|b|
\]

for some \( b \in \mathbb{C} \) \((b \neq 0)\), then \( f(z) \in K_b \).

Finally, taking \( b = 1 - \alpha \) in Theorem 2.1 and Theorem 2.2, or \( b = (1 - \alpha)e^{-\lambda}\cos \lambda \) in Theorem 2.1, we arrive Theorem 1.5, Theorem 1.6 and Theorem 1.7.

References


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