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Kyoto University
Base change lift type spinor L-function of
$GSp_2(\mathbb{Q})$

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In [4], we happened to construct Siegel modular cuspform $F$ and non-cuspform $E$ of degree 2 having the same spinor L-function: $L^{\text{spin}}(s, E) = L^{\text{spin}}(s, F)$, which is equal to the Hasse-Weil zeta function of hyper-elliptic curve $y^2 = x^5 - x$. The CAP representation has a L-function of a non-cuspidal one, but, our phenomenon is not the case. In this article, we consider the problem 'What type of spinor L-function is related to cuspform and non-cuspform, simultaneously?'. To do it, we will classify the spinor $L$-functions of non-cuspsforms. The classical 'Zharkovskaya relation' describes $L$-function of Siegel non-cuspidal by that of the elliptic modular form obtained by the Siegel operator, as follows. If $E \in M_\kappa(Sp_2(\mathbb{Z}))$ is an eigenform, then it holds

$$L^{\text{spin}}(s, F) = L(s, \Phi(E))L(s - \kappa + 2, \Phi(E)).$$

where the elliptic modular eigenform $\Phi(E) \in M_\kappa(SL_2(\mathbb{Z}))$ is

$$\Phi(E)(z) = \lim_{t \to \infty} E([ \begin{array}{cc} z & 0 \\ 0 & it \end{array} ]), \quad z \in \mathbb{H}.$$  \quad (1)

We will generalize her relation for non-holomorphic and non-full modular cases. Let $N_1, N_2$ be the unipotent radicals of the two parabolic subgroups, such as

$$N_1(\mathbb{Q}) = \begin{bmatrix} 1 & * & * \\ 1 & * & * \\ 1 & 1 \end{bmatrix}, \quad N_2(\mathbb{A}) = \begin{bmatrix} * & 1 & * \\ 1 & * & 1 \\ 1 & 1 \end{bmatrix} \subset Sp_2(\mathbb{Q}).$$

If $E$ is not cuspidal, then

$$\int_{U_i(\mathbb{Q}) \backslash U_i(\mathbb{A})} E(ug)du \neq 0$$

for $i = 1$ or 2 where $dh$ is a suitable Haar measure $du$. We label the former case as (CASE 1), and the latter as (CASE 2). In the both cases, we obtain automorphic forms on $GL_2(\mathbb{A})$ by

$$\int_{U_i(\mathbb{Q}) \backslash U_i(\mathbb{A})} E(ue_i(g)h)du,$$

for some $h \in Sp_2(\mathbb{A})$. Here we write

$$e_1(g) = \begin{bmatrix} t^{-1}g & 0 \\ g & 1 \end{bmatrix}, e_2(g) = \begin{bmatrix} a & b \\ \text{det}(g) & d \\ c & 1 \end{bmatrix} \in GSp_2(\mathbb{A})$$

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for $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{A})$. So, after the original Siegel operator (1),

**Definition 1** We define ‘Siegel operator along $N_i$’ at $h \in Sp_2(\mathbb{A})$

$$\Phi_i(E)(g; h) = \int_{N_i(\mathbb{Q}) \backslash N_i(\mathbb{A})} E(ue_i(g)h)du,$$

where $du$ is the Haar measure so that $vol(N_1(\mathbb{Q}) \backslash N_1(\mathbb{A})) = 1$.

Remark that the Siegel operator (1) is equal to $\Phi_2$ at $h = 1$, and that holomorphic $E$ is cuspidal iff $\Phi_2(E) = 0$. Let $\psi$ be the standard additive character on $\mathbb{Q} \backslash \mathbb{A}$ so that $\psi_\infty(x) = \exp(2\pi ix), x \in \mathbb{R}$. For automorphic form $F$ and $T = {}^tT \in M_2(\mathbb{Q})$, $F_T$ denote the fourier coefficient;

$$F_T(g) = \int_{U_1(\mathbb{Q}) \backslash U_1(\mathbb{A})} \psi(-tr(S \cdot T))F(\begin{bmatrix} 1 & 0 \\ T & 1 \end{bmatrix})dS.$$

(CASE 1) Suppose that irreducible $\pi \in \widehat{Sp_2(\mathbb{A})}$ is in the (CASE.1). Take $E \in \pi$, so that $f = \Phi_1(E)(*; 1) \neq 0$. If an eigenform $\widetilde{E} \in \mathcal{A}(GSp_2(\mathbb{A}))$ is an extension of $E$, then there exists $\delta \in (\mathbb{Q} \backslash \mathbb{A})^\times$ such as

$$F_0(e_1(g)[ \begin{bmatrix} 1 & 2 \\ t & 1 \end{bmatrix} ]) = \delta(t)F_0(e_1(g)).$$

Since $E_0, \widetilde{E}_0$ and $f$ have the informations of L-parameters of themselves, by comparing the action of Hecke operator on them, we can obtain the following.

**Proposition 1** Let $S$ be the collection of bad primes of $E$. With the assumption as above, $f$ is an eigenform at every $p \not\in S$, and the standard L-function $L^S_\pi(s, E)$ is written as

$$L^S_\pi(s, E) = \zeta_S(s)L_S(s - 1, f)L_S(s + 2, f, w_f).$$

If an eigenform $\widetilde{E} \in \mathcal{A}(GSp_2(\mathbb{A}))$ is an extension of $E$, then

$$L^\text{spin}_S(s, \widetilde{E}, \delta^{-1}) = \zeta_S(s)L_S(s - 3, w_f^{-1})L_S(s - 1, f).$$

Here $L(s, f, w_f)$ means the $w_f$-twist of $L(s, f, w_f)$, and so on.

(CASE 2) Suppose that irreducible $\pi \in \Pi(Sp_2(\mathbb{A}))$ is in the (CASE.2) and take $E \in \pi$ so that $\Phi_2(E)(*; 1) \neq 0$. Let $(\kappa_1, \kappa_2)$ with $\kappa_1 \geq \kappa_2$ be the highest weight of $E$, and $E$ is an eigenvector with respect to $Z(\mathfrak{sp}(2, \mathbb{R}))$, the center of the Lie algebra $\mathfrak{sp}(2, \mathbb{R})$.

Then, for a certain $T_a = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \in M_2(\mathbb{Q}), E_{T_a}$ is not zero. $E_{T_a}$ has the property

$$E_{T_a}(\begin{bmatrix} 1 & x \\ * & 1 \end{bmatrix})e_2(g) = \psi(ax)E_{T_a}(e_2(g)), x \in \mathbb{A}.$$

There exists a unique $\xi \in Z(\mathfrak{sl}(2, \mathbb{R}))$ such as $E_{T_a}(e_2(z \star g)) = \xi(z)E_{T_a}(e_2(g))$ for $z \in Z(\mathfrak{sl}(2, \mathbb{R}))$. Indeed, $Z(\mathfrak{sp}(2, \mathbb{R}))$ is generated by two elements $L_1, L_2$ as in [3].
In particular, since $L_1$ acts on $E_{T_a}$ as an element of $Z(sl(2, \mathbb{R}))$, $E_{T_a}$ is an eigenvector with respect to $Z(sl(2, \mathbb{R}))$. Thus $f(g) = \Phi_2(E)(g; 1) = \sum_{b \in \mathbb{Q}} E_{T_b}^,(e_2(g)) \in A(SL_2(K_\mathbb{A}))$ is of weight $\kappa_1$ and corresponds to $\xi$. From $E_{T_a}$, for some $\chi \in \mathbb{Q}^x \backslash \mathbb{A}^x$, cut out a nonzero $\chi$-section, that is,

$$E_{T_a}^{(\chi)}(e_2(g) \begin{bmatrix} 1 & \nu \\ 1 & \nu^{-1} \end{bmatrix}) = E_{T_a}^{(\chi)} \begin{bmatrix} 1 & \nu \\ 1 & \nu^{-1} \end{bmatrix} e_2(g)) = \chi(y)E_{T_a}^{(\chi)}(e_2(g)).$$

Further, if an eigenform $\tilde{E} \in A(GSp_2(K_\mathbb{A}))$ is an extension of $E$, we cut out a nonzero $\omega$-section

$$\tilde{E}_{T_a}^{(\chi,\omega)}(e_2(zg)) = \omega(z)\tilde{E}_{T_a}^{(\chi,\omega)}(e_2(g)), \quad z \in \mathbb{A}^x.$$

Remark that $f^{(\chi)}(g) = \sum_{b \in \mathbb{Q}} E^{(\chi)}_{T_b}(e_2(g))$ belongs to $A(SL_2(K_\mathbb{A}))$, and that $f^{(\chi,\omega)}(g) = \sum_{b \in \mathbb{Q}} E^{(\chi,\omega)}_{T_b}(e_2(g))$ to $A(GL_2(\mathbb{A}))$.

**Proposition 2** With the assumptions as above, at $p \not\in S$, $f^{(\chi)}$ is an eigenform such as

$$L^g(s, E) = L_\mathbb{S}(s - 2, \chi)L_\mathbb{S}(s + 2, \chi^{-1})L^g(s, f^{(\chi)}).$$

If an eigenform $\tilde{E} \in A(GSp_2(\mathbb{A}))$ is an extension of $E$, then $f^{(\chi,\omega)}$ has the following properties:

i) the central character of $f^{(\chi,\omega)}$ is $| \cdot |^2 \chi \omega$.

ii) If $\chi_p(p) \neq -p^{-2}$, then $f^{(\chi,\omega)}$ is also an eigenform at $p$, such as

$$L^{spin}(s, \tilde{E})_p = L(s, f^{(\chi,\omega)})_pL(s - 2, f^{(\chi,\omega)}, \chi)_p.$$  \hspace{1cm} (7)

iii) Otherwise, $f^{(\chi,\omega)}$ is not an eigenform in general.

However, in the case iii), instead of $f^{(\chi,\omega)}$, we can take an eigenform $\tilde{f}' \in A_{\kappa_1}(GL_2(\mathbb{A}))$ having the same central character and satisfies (7) at every $p \not\in S$.

**Remark 1** The above $\chi$ is determined uniquely by $\pi$, and $\omega$ is by extended $\tilde{\pi}$ which contains $\tilde{E}$, indeed.

Summing up the above results, we can give the following answer to the first problem.

**Theorem 1** Suppose that a non CAP-type spinor $L$-function of $\Pi(GSp_2(\mathbb{A}))$ is related to a cuspidal form and a non-cuspidal form, simultaneously. Then it is a Base change lift type, i.e., $L(s, \sigma)L(s, \tau, \chi_\mathbb{E}/\mathbb{Q})$ for $\sigma \in \Pi(GL_2(\mathbb{A}))$ and quadratic character $\chi_\mathbb{E}/\mathbb{Q}$ associated to an extension $E/\mathbb{Q}$.

**Proof.** Suppose that cuspidal $\pi$ and non-cuspidal $\tau$ have an identical spinor $L$-function up to finitely many primes. We can assume $\pi$ and $\tau$ are unitalized. In the (CASE.1) of $\tau$, by Proposition 1, we can write

$$L^\text{spin}_s(s, \pi, \omega^{-1}_\tau) = L^\text{spin}_s(s, \tau, \omega^{-1}_\tau) = \zeta_s(s - \zeta_0)L_\mathbb{S}(s + \zeta_0, \omega^{-1}_\sigma)L_\mathbb{S}(s, \sigma).$$  \hspace{1cm} (8)
for a certain $z_0 \in \mathbb{C}$, where $\sigma \in \Pi(GL_2(A))$ is related to $\tau$ and unitarized. According to Jacquet, Shalika [1], Shahidi [7], $L_S(s, \sigma)$ and $L_S(s, \sigma, \omega_\tau)$ does not vanish in the region $\text{Re}(s) \geq 1$. Hence the right hand side of (8) has a pole at $1 + z_0$, or its $\omega_\tau$-twist has a pole at $1 - z_0$. By lemma 3.1 of Piatetski-Shapiro [6], $\pi$ is written as $\pi_1 \otimes (\mu \circ \nu)$ by the similitude norm $\nu$ of $GSp(2)$, certain $\mu \in Q^x \backslash \mathbb{A}^x$ and $\pi_1 \in \Pi(GSp_2(A))$ with $\omega_\pi = 1$. And by Theorem 2.2 of [6], we conclude the spinor $L$-function is related to some CAP representation associated to Siegel parabolic subgroup.

In the (CASE.2) of $\tau$, $L_S^{\text{spin}}(s, \tau)$ is written in the form (7), and $\omega_\tau = \omega_\pi \chi$. From (7), the character $\xi := \omega_\pi (\omega_\sigma \chi)^{-1}$ satisfies $\xi^2 = 1$. In the case of $\xi = 1$ (i.e., $\omega_\pi = \omega_\sigma$), $\pi_v$ is equivalent to $\tau_v$ at almost all $v$ since they have identical Satake parameters. Hence $\pi$ is a CAP representation associated to Klingen parabolic subgroup. In the case that $\xi \neq 1$, calculating $L_S(s, \pi, \wedge^2)$, we see

\[
L_S(s, \omega_\pi)L_S^\pi(s, \pi, \omega_\pi) = L_S(s, \omega_\pi \xi^{-1})L_S(s - 2, \chi \omega_\pi \xi^{-1})L_S(s + 2, \chi^{-1} \omega_\pi \xi^{-1})L_S^\pi(s, \omega_\pi \xi^{-1}).
\]

Twisting both sides by $\omega_\pi^{-1}\xi$,

\[
L_S(s, \xi)L_S^\pi(s, \pi, \xi) = \zeta_S(s)L_S(s - 2, \chi)L_S(s + 2, \chi^{-1})L_S^\pi(s, \sigma) = \zeta_S(s)L_S(s + t, \chi_1)L_S(s - t, \chi_1^{-1})L_S^\pi(s, \sigma).
\]

Here $\chi_1$ is the unitalization of $\chi$ and we write $\chi_\infty = | \cdot |^{2+\text{sign}(t)} \zeta_\ell(s)$. Applying lemma 1 to (9), we find that $L_S^\pi(s, \pi, \chi \chi_1)$ (resp. $L_S^\pi(s, \pi, \chi \chi_1^{-1})$) has a simple pole at $s = 1 + t$ (resp. $s = 1 - t$), if $\text{Re}(t) > 0$ (resp. $\text{Re}(t) < 0$). However, $\pi$ is cuspidal, so $t$ is allowed to be $\pm 1$ and $\pi$ is a CAP representation along Klingen parabolic subgroup. If $\text{Re}(t) = 0$, we can also conclude $t = 0$ by considering the possibility of the location of the poles. In this case, if $\chi_1^2 \neq 1$, then $(\xi \chi_1)^2 \neq 1$ and we find that $L_S^\pi(s, \pi, \xi \chi_1)$ has a simple pole at $s = 1$, twisting (9) by $\chi_1$. This conflicts to [2]. If $\xi^2 = 1$ but $\chi_1 \xi \neq 1$, we find that $L_S^\pi(s, \pi, \xi \chi_1, st)$ has at least double pole at $s = 1$, which conflicts to [2], too. Thus, the remained possibility of $\chi_1$ is only $\chi_1 = \xi$, i.e., some quadratic character. This is just the Base Change lift type. 

**Remark 2** Conversely, for given spinor $L$-function $L(s, \sigma)L(s, \sigma, \chi_K/Q)$ of base change type, [5] gives generic non-cusform and cusform which is fixed by paramodular groups, if $\sigma$ is holomorphic.

The next lemma used in the proof of previous theorem follows from the results of Jacquet, Shalika [1] and Shahidi [7].

**Lemma 1** Let $\pi \in \Pi(GL_2(A))$ be cuspidal. Then,

i) $L_S(s, \pi, \eta, st) \neq 0$ for every unitary $\eta \in Q^x \backslash \mathbb{A}^x$ at $\text{Re}s \geq 1$.

ii) if $\pi$ comes from a größencharacter of a quadratic extension $K$ over $Q$, then

\[
\text{ord}_{s=1} L_S(s, \pi, \eta, st) = -1 \quad \text{if} \quad \eta = \chi_K/Q,
\]

\[
\text{ord}_{s=1} L_S(s, \pi, \eta, st) = 0 \quad \text{otherwise}.
\]

iii) if $\pi$ does not come from größencharacters, $\text{ord}_{s=1} L_S(s, \pi, \eta, st) = 0$ for every unitary $\eta \in Q^x \backslash \mathbb{A}^x$. 

Complementing Kudla-Rallis [2] by Proposition 2, we can give the following characterization of cuspidality of $\Pi(Sp_{2}(A))$ by standard $L$-functions:

**Theorem 2** Non CAP $\pi \in \Pi(Sp_{2}(A))$ is cuspidal, iff all the $i) \sim iii)$ are satisfied: For unitary $\eta \in \mathbb{Q}^{x}\backslash \mathbb{A}^{x}$, we can give the following characterization of cuspidality of $\Pi(Sp_{2}(A))$

1) $L_S(s, \pi, \eta, st)$ is entire at $\Re s > 1$;

2) if $\eta^2 = 1$, $\text{ord}_{s=1} L_S(s, \pi, \eta, st) \geq -1$;

3) if $\eta^2 \neq 1$, $\text{ord}_{s=1} L_S(s, \pi, \eta, st) \geq 0$.

**Proof.** By corollary 7.2.3, Theorem 7.2.5 of [2], and Soudry [8], cuspidal $\pi$ satisfies $i), ii)$. (If the standard $L$-function has a simple pole at $s = 2$, then $\pi$ is liftable to $O(2)$, and is a CAP representation.) Hence, our task is to show that both of $i), ii)$ are not satisfied by non-cuspidal $\pi'$ which is induced from cuspidal $\sigma \in \Pi(GL_{2}(\mathbb{Q}_{A}))$. Put $\chi_{1} = \chi/|\chi|$ and let $\chi_{\infty} = |\cdot|^t\cdot \text{sign}^a$ with $t, s \in \mathbb{R}$ and $a = 0$ or 1. In the case of $|\chi_{\infty}| \neq |\cdot|$, it holds

\[
\begin{cases}
L_S(s, \pi', \chi_{1}, st) \text{ has a double pole at a point in the region } \Re s \geq 1 & \text{if } \chi_{1}^2 = 1 \\
L_S(s, \pi', \chi_{1}, st) \text{ is not entire in } \Re s \geq 1 & \text{otherwise.}
\end{cases}
\]

Indeed, from (6), $L_S(s, \pi', \chi_{1}, st) = \zeta_{S}(s-t)\zeta_{S}(s+t)L_S(s, \sigma, \chi_{1}, st)$, if $\chi_{1}^2 = 1$. Obviously, this $L$-function has a simple pole at $1+|t|$, if $t \neq 0$. In the case of $|\chi_{\infty}| = |\cdot|$ or $|\cdot|^3$, we can say

\[
\begin{cases}
L_S(s, \pi, \chi_{1}, st) \text{ has a simple pole at } s = 2 & \text{if } \chi_{1}^2 = 1 \\
L_S(s, \pi, \chi_{1}, st) \text{ is not entire in } \Re s \geq 1 & \text{otherwise.}
\end{cases}
\]

We are going to see that $L_S(s, \pi, \chi_{1}, st)$ has a double pole at a point in the region $\Re s \geq 1$ if $\chi_{1}^2 = 1$, and that $L_S(s, \pi, \chi_{1}, st)$ is not entire in $\Re s \geq 1$ otherwise. If $\chi_{1}^2 = 1$, then

$L_S(s, \pi, \chi_{1}, st) = \zeta_{S}(s)^2 L(s, \sigma, \chi_{1}, st)$,

which has a double (at least) pole at $s = 1$. If $\chi_{1}^2 \neq 1$, then

$L_S(s, \pi, \chi_{1}, st) = \zeta_{S}(s)L(s, \chi_{1}^2) L(s, \sigma, \chi_{1}, st)$,

which has a simple (at least) pole at $s = 1$. This completes the proof. \qed

**Remark 3** If $\pi$ satisfies the generalized Ramanujan conjecture, we only need to see $ii), iii)$ for the cuspidality of $\pi$.

**References**


