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QUICK REVIEW ON PROPERTY (X)

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ABSTRACT. We will review some materials that are useful to prove the uniqueness of preduals. Those were used crucially in our recent work on the uniqueness of predual of any ‘finite’ non-commutative $H^\infty$.

1. INTRODUCTION

In [12] we established, among other things, the uniqueness of predual of any ‘finite’ non-commutative $H^\infty$-algebra $H^\infty(M, \tau)$, which was introduced by Bill Arveson modeled after the usual pair $H^\infty(\mathbb{D}) \hookrightarrow L^\infty(\mathbb{T})$ with the aid of operator algebra theory. The class of finite non-commutative $H^\infty$-algebras contains $H^\infty(\mathbb{D})$ as well as its abstract generalizations. Thus [12, Theorem 2] covers any existing generalization of the famous result due to Tsuyoshi Ando [3].

The most key ingredient of our proof of the uniqueness of predual of $H^\infty(M, \tau)$ is to provide a non-commutative analog of Amar–Lederer’s peak set result [2] (also see [4]), which we fully explained in [12]. However, our proof of the uniqueness of predual also uses two purely Banach space theoretic techniques – Property (X) due to Godefroy and Talagrand and a very clever trick, both of which we just borrowed from some references without any detailed explanation. Here we will give detailed accounts (for non-experts like us) on those techniques as supplements to [12, Theorem 2].

In closing, we should mention our sincere thanks to Professor Kichi-Suke Saito for giving this opportunity.

2. GODEFROY–TALAGRAND’S PROPERTY (X)

This section mainly follows Godefroy and Talagrand’s elegant work [6]. The key ingredient behind Godefroy–Talagrand’s property (X) is the next proposition.

Proposition 2.1. Let $E$ and $G$ be Banach spaces with $E^* = G^*$. If a sequence $\{x_n\} \subset E^*$ satisfies

(i) $x_n \rightarrow 0$ in $\sigma(E^*, E)$; and

(ii) $\sum_{n=1}^\infty |\psi(x_{n+1} - x_n)| < +\infty$ for all $\psi \in E^{**}$,

then $x_n \rightarrow 0$ in $\sigma(E^*, G)$.

Proof. Set $u_0 := x_1$, $u_1 := x_2 - x_1$, and $u_n := x_{n+1} - x_n$, and then by (i)

$$\sum_{k=0}^{n} u_k = x_{n+1} \rightarrow 0 \text{ in } \sigma(E^*, E).$$ (1)
For each $n \in \mathbb{N}_0 := \{0, 1, 2, \ldots \}$ we consider the map $T_n : \alpha = (\alpha_k) \in \ell^\infty(\mathbb{N}_0) \mapsto \sum_{k=0}^{n} \alpha_k u_k \in E^*$ ($\hookrightarrow E^{***}$ via the canonical embedding). Then one has, by (ii),

$$
\sup \{ \|T_n \alpha(\phi)\| : \|\alpha\|_\infty \leq 1, n \in \mathbb{N}_0 \} \leq \sum_{k=0}^{\infty} |\phi(u_k)| < +\infty
$$

for all $\phi \in E^{**}$, and hence the uniform boundedness principle shows that there is $K > 0$ such that

$$
\left\| \sum_{k=0}^{n} \alpha_k u_k \right\|_{E^{**}} \leq K
$$

(2)

for all $n \in \mathbb{N}_0$ and for all $\alpha_k \in \mathbb{C}$ with $|\alpha_k| \leq 1$.

Choose an arbitrary free ultrafilter $\omega \in \beta(\mathbb{N}_0) \setminus \mathbb{N}_0$ and put $\xi_\omega := \lim_{n \rightarrow \omega} \sum_{k=0}^{n} u_k$ in $\sigma(E^*, G)$. Let us choose arbitrary $n_1 < n_2 < \cdots < n_{2l-1} < n_{2l}$. Then, using (2) with

$$
\alpha_k = \begin{cases} 
1 & n_{2j-1} \leq k \leq n_{2j}, j = 1, \ldots, l, \\
0 & \text{otherwise}
\end{cases}
$$

we get

$$
\left\| \sum_{k=n_1}^{n_2} u_k + \sum_{k=n_3}^{n_4} u_k + \cdots + \sum_{k=n_{2l-1}}^{n_{2l}} u_k \right\| \leq K.
$$

Here we have

$$
\sum_{k=n_1}^{n_2} u_k + \sum_{k=n_3}^{n_4} u_k + \cdots + \sum_{k=n_{2l-1}}^{n_{2l}} u_k = \sum_{k=n_1}^{n_2} u_k + \sum_{k=n_3}^{n_4} u_k + \cdots + \left( \sum_{k=0}^{n_{2l-1}} u_k - \sum_{k=0}^{n_{2l-1}} u_k \right)
$$

$$
\rightarrow \sum_{k=n_1}^{n_2} u_k + \sum_{k=n_3}^{n_4} u_k + \cdots + \sum_{k=0}^{n_{2l-1}} u_k + \xi_\omega - \sum_{k=0}^{n_{2l-1}} u_k
$$

as $n_{2l} \rightarrow \omega$ but $n_1, \ldots, n_{2l-1}$ are fixed. Then it follows that

$$
\left\| \sum_{k=n_1}^{n_2} u_k + \sum_{k=n_3}^{n_4} u_k + \cdots + \sum_{k=0}^{n_{2l-1}} u_k - \sum_{k=0}^{n_{2l-1}} u_k \right\| \leq K
$$

for any fixed $n_1 < n_2 < \cdots < n_{2l-1}$. We also have, by (1),

$$
\sum_{k=n_1}^{n_2} u_k + \sum_{k=n_3}^{n_4} u_k + \cdots + \xi_\omega - \sum_{k=0}^{n_{2l-1}} u_k
$$

$$
\rightarrow \sum_{k=n_1}^{n_2} u_k + \sum_{k=n_3}^{n_4} u_k + \cdots + \xi_\omega - 0
$$

in $\sigma(E^*, E)$

as $n_{2l-1} \rightarrow \infty$ but $n_1, \ldots, n_{2l-2}$ are fixed. Therefore, we get

$$
\left\| \sum_{k=n_1}^{n_2} u_k + \sum_{k=n_3}^{n_4} u_k + \cdots + \sum_{k=n_{2l-2}}^{n_{2l-3}} u_k + \xi_\omega \right\| \leq K
$$

for any fixed $n_1 < n_2 < \cdots < n_{2l-2}$. Clearly, this procedure can be continued for $n_{2l-2}, n_{2l-4}$ and so on, and we finally get $l \cdot \|\xi_\omega\| = \|\xi_\omega\| \leq K$. Since $l$ can be arbitrarily large, $\xi_\omega$ must be zero for any $\omega \in \beta(\mathbb{N}_0) \setminus \mathbb{N}_0$, which means that $\lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} \sum_{k=0}^{n} u_k = 0$ in $\sigma(E^*, G)$. □

Based on the lemma, Godefroy and Talagrand introduced property (X).
Quick Review on Property (X)

Definition 2.1. A Banach space $E$ has property (X) if for any $\psi \in E^{**}$ the following conditions are equivalent:

(a) $\psi \in E$ with the canonical embedding $E \hookrightarrow E^{**}$.
(b) For any sequence $\{x_n\} \subset E^{*}$ with the properties
   - $x_n \rightarrow 0$ in $\sigma(E^{*}, E)$,
   - $\sum_{n=1}^{\infty} |\phi(x_{n+1} - x_n)| < +\infty$ for all $\phi \in E^{**}$,
   one has $\psi(x_n) \rightarrow 0$.

This definition gives, in some sense, a criterion of $w^{*}$-continuity for bounded linear functionals on the dual $E^{*}$ of a Banach space $E$ with property (X).

Definition 2.2. A Banach space $E$ is said to be the unique predual of its dual $E^{*}$ if another Banach space $G$ with $G^{*} = E^{*}$ must coincide with $E$ inside the dual $E^{**}$ of $E^{*}$ ($= G^{*}$) via the canonical embedding.

Corollary 2.2. If a Banach space $E$ has property (X), then $E$ must be the unique predual of its dual $E^{*}$.

Proof. Assume another Banach space $G$ satisfies $G^{*} = E^{*}$. Embed $G \hookrightarrow (E^{*})^{*} = E^{**}$ by $g(x) := x(g)$ for $x \in E^{*} = G^{*}$ and $g \in G$. Let $\{x_n\} \subset E^{*}$ be chosen in such a way that $x_n \rightarrow 0$ in $\sigma(E^{*}, E)$ and $\sum_{n=1}^{\infty} |\phi(x_{n+1} - x_n)| < +\infty$ for all $\phi \in E^{**}$. By Proposition 2.1 we get $x_n \rightarrow 0$ in $\sigma(E^{*}, G)$, which shows that $g(x_n) = x_n(g) \rightarrow 0$ for all $g \in G$. Thus, Property (X) ensures that any $g$ must fall in $E \hookrightarrow E^{**}$, that is, $G \subseteq E$ inside $E^{**}$. If $G \not\subset E$ inside $E^{**}$, then by the Hahn–Banach extension theorem there is $x \in E^{*}$ such that $x \neq 0$ but $x|_{G} = 0$. (Indeed, there is $e \in E \setminus G$ by the assumption, and thus $[e] \in E/G$ with $[e] \neq 0$. Then by the Hahn–Banach extension theorem there is $\varphi \in (E/G)^{*}$ sending $[e]$ to $\|\varphi\| = \inf\{\|e - g\| : g \in G\} \neq 0$. Hence the $x := \varphi \circ Q \in E^{*}$ with the quotient map $Q : E \rightarrow E/G$ becomes a desired element.) This $x$ is a non-zero element in $G^{*} = E^{*}$ but it is identically zero on $G$, a contradiction. Hence $G = E$ inside $E^{**}$.

The next proposition has been known, but we do give one proof, which is a prototype of our proof of the uniqueness of predual of $H^{\infty}(M, \tau)$.

Proposition 2.3. Let $M$ be a $\sigma$-finite von Neumann algebra and $M_{*}$ be its predual. Then, $M_{*}$ has property (X).

Proof. It suffices to show that, if $\varphi \in M^{*}$ satisfies $\varphi(x_n) \rightarrow 0$ for any $\{x_n\} \subset M$ with the properties

- $x_n \rightarrow 0$ in $\sigma(M, M_{*})$ and
- $\sum_{n=1}^{\infty} |\phi(x_{n+1} - x_n)| < +\infty$ for all $\phi \in M^{*}$,

then $\varphi$ must fall in $M_{*} \hookrightarrow M^{*}$. Here we need the following standard facts on von Neumann algebras (see e.g. [9] and [11] for their proofs):

1. Any $\psi \in M^{*}$ can be decomposed into $\psi = \psi_{nor} + \psi_{sing}$ with $\psi_{nor} \in M_{*}$ and $\psi_{sing} \in M^{*} \ominus M_{*}$, and $\|\psi\| = \|\psi_{nor}\| + \|\psi_{sing}\|$ holds. (This is the so-called non-commutative Lebesgue decomposition due to Takesaki.) We call $M_{*}$ the normal part and $M^{*} \ominus M_{*}$ the singular part. Remark that the notation here is a little bit different from that in [12].

2. For any $\psi \in M^{*}$ (or $\psi \in M_{*}$) there are a unique positive linear functional $|\psi| \in M_{*}$ (resp. $\psi \in M_{*}$) and a unique partial isometry $v \in M^{**}$ (resp. $v \in M_{*}$) such that $\langle \psi, x \rangle = \langle |\psi|, xv \rangle$ as well as $\langle \psi, x \rangle = \langle |\psi|, xv \rangle$ for $x \in M^{**}$, where $\langle \cdot, \cdot \rangle : M^{*} \times M^{**} \rightarrow \mathbb{C}$ stands for the canonical pairing. (This is the so-called polar decomposition...
of linear functionals due to Sakai and also Tomita.) Remark here that the second dual $M^{**}$ becomes a von Neumann algebra, which naturally contains the original $M$ as a subalgebra via the canonical embedding $M \hookrightarrow M^{**}$.

(3) Both the closed subspaces $M_{*}$ and $M^{*} \ominus M_{*}$ of $M^{*}$ are closed under the operation $\psi \in M^{*} \mapsto |\psi| \in M^{*}$. (This follows from the construction of the decomposition in (1) together with (2).)

(4) For a positive linear functional $\psi \in M^{*}$ the following are equivalent:

- $\psi \in M^{*} \ominus M_{*}$.
- For every nonzero projection $e \in M$ there is a non-zero projection $e_{0} \in M$ such that $e_{0} \leq e$ and $\psi(e_{0}) = 0$. 

(This is Takesaki’s criterion for ‘singularity’ of linear functionals.)

(5) Any $\psi \in M^{*}$ (or $M_{*}$) can be written as a linear combination of four positive linear functionals in $M^{*}$ (resp. $M_{*}$).

Let us decompose the given $\varphi$ into $\varphi = \varphi_{\text{nor}} + \varphi_{\text{sing}}$ as in (1), and what we have to show is $\varphi_{\text{sing}} = 0$, i.e., $\varphi = \varphi_{\text{nor}} \in M_{*}$. For contrary we suppose $\varphi_{\text{sing}} \neq 0$. Then, by (2) and (3), $|\varphi_{\text{sing}}| \neq 0$ and $|\varphi_{\text{sing}}| \in M^{*} \ominus M_{*}$ still holds. Clearly, the orthogonal families of non-zero projections in $\text{Ker} |\varphi_{\text{sing}}|$ forms an inductive set by inclusion, and Zorn’s lemma ensures the existence of a maximal family $\{q_{k}\}$, which is at most countable since $M$ is $\sigma$-finite. Put $q_{0} := \sum_{k} q_{k}$ in $M$, and then $q_{0} = 1$ since $q_{0} \neq 1$ clearly contradicts to the above (4). Also, if $\{q_{k}\}$ is a finite family, then $|\varphi_{\text{sing}}|(1) = \sum_{k} |\varphi_{\text{sing}}|(q_{k}) = 0$, a contradiction. Therefore, $\{q_{k}\}$ must be a countably infinite family with $\sum_{k} q_{k} = 1$ in $M$. Letting $p_{n} := 1 - \sum_{k \leq n} q_{k}$ we have $p_{n} \searrow 0$ in $\sigma(M, M_{*})$ but $|\varphi_{\text{sing}}|(p_{n}) = |\varphi_{\text{nor}}|(1)$ for all $n$. The latter says that $p_{n}$ converges a non-zero projection $p \in M^{**}$ in $\sigma(M^{**}, M^{*})$ with $|\varphi_{\text{sing}}|(p) = |\varphi_{\text{sing}}|(1)$ since $p_{n}$ is a decreasing sequence. Let $\mu \in M$ and $v \in M^{*}$ be the partial isometries for the polar decompositions of $\varphi_{\text{nor}}$ and $\varphi_{\text{sing}}$, respectively. Then, for $x \in M^{**}$ one has $|\langle \varphi_{\text{sing}}, (1 - p)x \rangle| = \langle |\varphi_{\text{sing}}|, (1 - p)x \rangle \leq \langle |\varphi_{\text{sing}}|, 1 - p \rangle^{1/2} \langle |\varphi_{\text{sing}}|, v^{*}x^{*}xv \rangle^{1/2} = 0$ so that $\langle \varphi_{\text{sing}}, x \rangle = \langle \varphi_{\text{sing}}, px \rangle$ since $\langle \psi, px \rangle = \langle \psi, 1 \rangle$ for all $x \in M^{*}$. One has $|\varphi_{\text{nor}}| = |\varphi_{\text{nor}}|, pxu \rangle^{1/2} \langle |\varphi_{\text{nor}}|, u^{*}x^{*}xu \rangle^{1/2}$. Since $|\varphi_{\text{nor}}|$ still falls in $M_{*}$, $|\varphi_{\text{nor}}|, p \rangle = \lim_{n \to \infty} |\varphi_{\text{nor}}|, p_{n} \rangle = 0$ so that $\langle \varphi_{\text{nor}}, px \rangle = 0$. Consequently, we get $\langle \varphi, px \rangle = \langle \varphi_{\text{nor}} + \varphi_{\text{sing}}, px \rangle = \varphi_{\text{sing}}(x)$ for $x \in M$.

Let $x \in M$ be arbitrary. Clearly, $p_{n}x \to 0$ in $\sigma(M, M_{*})$. Let $\phi \in M^{*}$ be arbitrary, and decompose $y \in M \mapsto \phi(yx)$ into a linear combination of four positive linear functionals $\psi_{i} \in M^{*}$, $i = 1, 2, 3, 4$, thanks to the above (5). Since $\sum_{n=1}^{N} |\phi_{i}(p_{n+1} - p_{n})| = \sum_{n=1}^{N} |\phi_{i}(q_{n+1})| \leq \phi_{i}(1) + \infty$ for all $N \in \mathbb{N}$, it follows that $\sum_{n=1}^{\infty} |\phi_{i}(p_{n+1} - p_{n})x| < \infty$. Therefore, by the assumption above one has $\phi(p_{n}x) \to 0$. On the other hand, $\langle \phi, px \rangle = \langle \varphi, px \rangle = \varphi_{\text{sing}}(x)$ so that $\varphi_{\text{sing}} = 0$, a contradiction. \hfill \Box

The heart of the above proof is as follows. Although $\varphi_{\text{nor}}$ and $\varphi_{\text{sing}}$ are ‘orthogonal’, we cannot find a projection in $M$ that distinguishes those. (Of course, we can find such a projection in $M^{**}$ since both functionals can be regarded as ‘normal’ ones on $M^{**}$.) Thus we first construct a projection $p \in M^{**}$ in such a way that it can be ‘nicely’ approximated by projections in $M$ and $p$ is greater than ‘the support of $\varphi_{\text{sing}}$’ but ‘disjoint’ from ‘the support of $\varphi_{\text{nor}}$’. This essentially means $M$ ‘remembers’ the decomposition $M^{*} = M_{*} \oplus (M_{*} \ominus M_{*})$ of $M^{*}$ (the second dual of $M_{*}$). This suggests us that such a decomposition of the second dual should be related to property (X) of a Banach space in question. This was quite recently answered affirmatively by Hermann Pfitzner when a Banach space in question is separable, see [8].

Further accounts on the present topics can be found in [5].
3. ADDENDUM – A CLEVER TRICK DUE TO PELCZYŃSKI

The essential idea of our proof of the uniqueness of predual of $H^\infty(M, \tau)$ is similar to that of Proposition 2.3. However, the luck of self-adjointness of our algebra $H^\infty(M, \tau)$ (thus we cannot use the order structure) makes some trouble, which we overcame with a clever trick borrowed from the proof of [7, Proposition 1.c.3]. (The trick is due to Aleksander Pelczyński, see [10, p.637] for this credit, and it was originally used for proving that if a Banach space has Pelczyński’s property (u) then so does any closed subspace, see [7] or more recent [1].) Here we will explain it. The situation we deal with is as follows. Let $M$ be a von Neumann algebra and $A$ be its $\sigma$-weakly closed (possibly non-self-adjoint) unital subalgebra. Assume that we have two sequences $\{a_n\} \subset A$ and $\{b_n\} \subset M$ such that

(i) both $a_n$ and $b_n$ converge to the same $p \in M^{**}$ in $\sigma(M^{**}, M^*)$, and

(ii) $\sum_{n=1}^{\infty} |\phi(b_{n+1} - b_n)| < +\infty$ for all $\phi \in M^*$.

What we want to do is to replace $a_n$ by a new one with keeping (i) and further satisfying (ii). This can be done by utilizing the above-mentioned clever trick in Banach space theory.

Proposition 3.1. There is another $\{a'_n\} \subset A$ such that

(i') $a'_n \longrightarrow p$ in $\sigma(M^{**}, M^*)$, and

(ii') $\sum_{n=1}^{\infty} |\phi(a'_n - a'_n)| < +\infty$ for all $\phi \in M^*$.

We need one elementary lemma due to Stanislaw Mazur.

Lemma 3.2. Let $E$ be a normed space and $\{x_n\} \subset E$ be such that $x_n \longrightarrow 0$ in $\sigma(E, E^*)$. Then, for each $\varepsilon > 0$ and each $m \in \mathbb{N}$ there is a convex combination $y = \sum_{n \geq m} \lambda_n x_n$ with $\|y\| < \varepsilon$.

Proof. Let $C_m$ be the closed convex hull of $\{x_n\}_{n \geq m}$ in $E$. It suffice to show $0 \in C_m$. Thus, for contrary, suppose $0 \notin C_m$. Then there is a small open ball $B$ centered at $0$ with $C_m \cap B = \emptyset$. The Hahn–Banach separation theorem ensures that there are $\varphi \in E^*$ and $t \in \mathbb{R}$ such that $\text{Re} \varphi(b) \leq t \leq \text{Re} \varphi(c)$ for all $b \in B$ and $c \in C_m$. This is impossible since $x_n \longrightarrow 0$ in $\sigma(E, E^*)$ (implying $t \leq 0$) and $0 \in B$ (implying $t \geq 0$). Thus $0 \notin C_m$, which means the desired assertion.

Proof. (Proposition 3.1) Putting $b_0 := 0$ we have $\sum_{n=1}^{\infty} |\phi(b_n - b_{n-1})| < +\infty$ for all $\phi \in M^*$. Set $u_n := a_n - \sum_{k=1}^{n} b_k - b_{k-1}$, and then $u_n = a_n - b_n \longrightarrow 0$ in $\sigma(M, M^*)$ by (i). By Lemma 3.2 there are convex combinations $u'_j = \sum_{n=p_{j-1}+1}^{p_j} \lambda_n^{(j)} u_n$ such that $0 = p_0 < p_1 < p_2 < \cdots$ and $\|u'_j\| \leq 2^{-j}$. Then We define $a'_j := \sum_{n=p_{j-1}+1}^{p_j} \lambda_n^{(j)} a_n \in A$ and put $a'_0 := 0$ for convenience. Let us prove that this $\{a'_j\}$ gives a desired sequence.

Since $a_n \longrightarrow p$ in $\sigma(M^{**}, M^*)$, for any $\varepsilon > 0$ and any $\phi \in M^*$ there is $n_0 \in \mathbb{N}$ such that $|\langle a_n, \phi \rangle - \langle p, \phi \rangle| < \varepsilon$ for all $n \geq n_0$, where $\langle \cdot, \cdot \rangle : M^{**} \times M^* \rightarrow C$ is the canonical pairing. If $j_0$ is chosen so that $p_{j_0} + 1 \geq n_0$, then one has $|\langle a'_j, \phi \rangle - \langle p, \phi \rangle| \leq \sum_{n=p_{j-1}+1}^{p_j} \lambda_n^{(j)} |\langle a_n, \phi \rangle - \langle p, \phi \rangle| < \varepsilon$ for all $j \geq j_0$. Thus $a'_j \longrightarrow p$ in $\sigma(M^{**}, M^*)$ as $j \rightarrow \infty$.

One has

$$a'_{j+1} - a'_j = u'_{j+1} + \sum_{n=p_{j+1}}^{p_{j+1}+1} \lambda_n^{(j+1)} (a_n - u_n) - u'_j - \sum_{n=p_{j-1}+1}^{p_{j}} \lambda_n^{(j)} (a_n - u_n)$$

$$= u'_{j+1} - u'_j + \sum_{n=p_{j-1}+1}^{p_{j+1}} \lambda_n^{(j+1)} (\sum_{k=1}^{n} b_k - b_{k-1}) - \sum_{n=p_{j-1}+1}^{p_{j}} \lambda_n^{(j)} (\sum_{k=1}^{n} b_k - b_{k-1})$$

$$= u'_{j+1} - u'_j + \sum_{n=p_{j-1}+1}^{p_{j+1}} \mu_n^{(j)} (b_n - b_{n-1})$$

where $\mu_n^{(j)} := \lambda_n^{(j+1)} - \lambda_n^{(j)}$. The desired result follows from this.
with some $0 \leq \mu_{n}^{(j)} \leq 1$. Hence,
\[
\sum_{j=0}^{\infty} |\phi(a_{j+1}'-a_{j}')| \\
\leq \sum_{j=0}^{\infty} \|\phi\| \|u_{j+1}'\| + \sum_{j=1}^{\infty} \sum_{n=p_{j-1}+1}^{p_{j+1}} \mu_{n}^{(j)} |\phi(b_{n}-b_{n-1})| \\
\leq 2 \sum_{j=0}^{\infty} \|\phi\| \|u_{j}'\| + \sum_{n=1}^{\infty} |\phi(b_{n}-b_{n-1})| \\
\leq 4 \|\phi\| + \sum_{n=1}^{\infty} |\phi(b_{n}-b_{n-1})| < +\infty
\]
by $\|u_{j}'\| \leq 2^{-j}$ and (ii).

Remark here that the argument presented above uses only the linear structure; hence clearly it can be applied to more general situations.

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