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SMITH PROBLEM FOR A FINITE OLIVER GROUP WITH NON-TRIVIAL CENTER

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1. INTRODUCTION

The Smith problem is that two tangential representations are isomorphic or not for a smooth action on a homotopy sphere with exactly two fixed points. Two real $G$-modules $U$ and $V$ are called Smith equivalent if there exists a smooth action of $G$ on a sphere $\Sigma$ such that $S^G = \{x, y\}$ for two points $x$ and $y$ at which $T_x(\Sigma) \cong U$ and $T_y(\Sigma) \cong V$ as real $G$-modules. We will consider a subset $Sm(G)$ of the real representation ring $RO(G)$ of $G$ consisting of the differences $U - V$ of real $G$-modules $U$ and $V$ which are Smith equivalent. We also define a subset $CSm(G)$ of $RO(G)$ consisting of the differences $U - V \in Sm(G)$ of real $G$-modules $U$ and $V$ such that for the sphere $\Sigma$ appearing in the notion of Smith equivalence of $U$ and $V$ satisfies that $\Sigma^P$ is connected for every $P \in \mathcal{P}(G)$. Moreover, we assume that $0 \in CSm(G)$ as definition.

In many groups, Smith equivalent modules are not isomorphic. In this paper we discuss the Smith problem for an Oliver group with non-trivial center. Throughout this paper we assume a group is finite.

2. TOPOLOGICAL VIEWPOINT

We denote by $\mathcal{P}(G)$ the family of subgroups of $G$ consisting of the trivial subgroup of $G$ and all subgroups of $G$ of prime power order, and by $\mathcal{L}(G)$ the family of large subgroups of $G$. Here, by a large subgroup of $G$ we mean a subgroup $H \leq G$ such that $O^p(G) \leq H$ for some prime $p$, where $O^p(G)$ is the smallest normal subgroup of $G$ such that $|G/O^p(G)| = p^k$ for some integer $k \geq 0$. A real $G$-module $V$ is called $\mathcal{L}(G)$-free if $\dim V^H = 0$ for each $H \in \mathcal{L}(G)$, which amounts to saying that $\dim V^{O^p(G)} = 0$ for each prime $p$ dividing $|G|$. Following [PSO], we denote by $LO(G)$ the subgroup of $RO(G)$ consisting of the differences $U - V$ of two real $\mathcal{L}(G)$-free $G$-modules $U$ and $V$ such that $\text{Res}^G_P(U) \cong \text{Res}^G_P(V)$ for every $P \in \mathcal{P}(G)$.

For two subgroups $P < H$ of $G$ with $P \in \mathcal{P}(G)$, and a smooth $G$-manifold $X$ or a real $G$-module $X$, we consider the number

$$d_X(P, H) = \dim X^P - 2 \dim X^H$$

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where \( \text{dim} \) means the dimension of the \( G \)-CW complex. Furthermore we define by 
\[ \text{dim} Z = \text{dim} X - \text{dim} Y \] 
for a virtual real \( G \)-module \( Z = X - Y \) of \( RO(G) \). A smooth \( G \)-manifold \( X \) satisfies the \textit{gap condition} (GC) if \( d_X(P,H) > 0 \) for every pair \( (P,H) \) of subgroups \( P < H \) of \( G \) with \( P \in \mathcal{P}(G) \).

The following theorem goes back to [PSo], the Realization Theorem.

**Theorem 2.1** ([PSo]). Let \( G \) be a finite Oliver gap group. Then \( L_0(G) \subseteq CSm(G) \).

We impose a number of restrictions on a smooth \( G \)-manifold, in particular, a real \( G \)-module \( X \). The restrictions are collected in the following conditions, where we consider series \( P \leq H \leq G \) of subgroups \( P \) and \( H \) of \( G \) always with \( P \in \mathcal{P}(G) \). We say that a smooth \( G \)-manifold \( X \) satisfies the weak gap condition (WGC) if the conditions (WGC1)–(WGC4) all hold (cf. [LM], [MP]), and we say that \( X \) satisfies the semi-weak gap condition (SWGC) if the conditions (WGC1) and (WGC2) both hold.

- (WGC1) \( d_X(P,H) \geq 0 \) for every \( P < H \leq G \), \( P \in \mathcal{P}(G) \).
- (WGC2) If \( d_X(P,H) = 0 \) for some \( P < H \leq G \), \( P \in \mathcal{P}(G) \), then \( [H : P] = 2 \), \( \dim X^H > \dim X^K + 1 \) for every \( H < K \leq G \), and \( X^H \) is connected.
- (WGC3) If \( d_X(P,H) = 0 \) for some \( P < H \leq G \), \( P \in \mathcal{P}(G) \), and \( [H : P] = 2 \), then \( X^H \) can be oriented in such a way that the map \( g: X^H \to X^H \) is orientation preserving for any \( g \in N_G(H) \).
- (WGC4) If \( d_X(P,H) = d_X(P,H') = 0 \) for some \( P < H, P < H' \), \( P \in \mathcal{P}(G) \), then the smallest subgroup \( \langle H, H' \rangle \) of \( G \) containing \( H \) and \( H' \) is not a large subgroup of \( G \).

Now, for a finite group \( G \), we define subgroups \( VLO(G), WLO(G) \) and \( MLO(G) \) of the free abelian group \( L_0(G) \) as follows.

\[
VLO(G) = \{ U - V \in L_0(G) \mid U \oplus W \text{ and } V \oplus W \text{ both satisfy the gap condition for some real } L(G)\text{-free } G\text{-module } W \}
\]

\[
WLO(G) = \{ U - V \in L_0(G) \mid U \oplus W \text{ and } V \oplus W \text{ both satisfy the weak gap condition for some real } L(G)\text{-free } G\text{-module } W \}
\]

\[
MLO(G) = \{ U - V \in L_0(G) \mid U \oplus W \text{ and } V \oplus W \text{ both satisfy the semi-weak gap condition for some real } L(G)\text{-free } G\text{-module } W \}
\]

Note that if \( \mathcal{P}(G) \cap L(G) = \emptyset \) then for an \( L(G)\)-free real \( G \)-modules \( U \) and \( V \) there is a real \( L(G)\)-free \( G \)-module \( W \) such that both \( U \oplus W \) and \( V \oplus W \) satisfy (WGC2), and if \( G \) is an Oliver group then for an \( L(G)\)-free real \( G \)-modules \( U \) and \( V \) there is a real \( L(G)\)-free \( G \)-module \( W \) such that both \( U \oplus W \) and \( V \oplus W \) satisfy (WGC2) and (WGC4).

In general, \( VLO(G) \subseteq WLO(G) \subseteq MLO(G) \subseteq L_0(G) \) by definitions. But if \( G \) is a gap group, then for every \( U - V \in L_0(G) \), there exists a real \( L(G)\)-free \( G \)-module \( W \) satisfying the gap condition, such that \( U \oplus W \) and \( V \oplus W \) also satisfy the gap condition, and thus \( U - V \in VLO(G) \), and hence

\[
VLO(G) = WLO(G) = MLO(G) = L_0(G).
\]
Therefore, the following theorem extends the result in Theorem 2.1 by using Theorem in [MP].

**Theorem 2.2.** Let $G$ be a finite Oliver group. Then $WLO(G) \subseteq CSm(G)$.

3. ALGEBRAIC VIEWPOINT

We denote by $PO(G)$ the subgroup of $RO(G)$ of $G$ consisting of the differences $U - V$ of representations $U$ and $V$ such that $\dim U^G = \dim V^G$ and $\text{Res}_P^G(U) \cong \text{Res}_P^G(V)$ for any subgroup $P$ of $G$ of prime power order. We note that in [PSo], $PO(G)$ is denoted by $IO(G, G)$. Similarly, we denote by $\overline{PO}(G)$ the subgroup of $RO(G)$ of $G$ consisting of the differences $U - V$ of representations $U$ and $V$ such that $\dim U^G = \dim V^G$ and $\text{Res}_P^G(U) \cong \text{Res}_P^G(V)$ for any subgroup $P$ of $G$ of odd prime power order and order 2.

By a theorem of Sanchez [Sa], the difference of two Smith equivalent representations lies in $\overline{PO}(G)$ and the difference of two $P$-matched Smith equivalent representations lies in $PO(G)$.

We define the Laitinen number $a_G$ as the number of real conjugacy classes in $G$ represented by elements of $G$ not of prime power order. The rank of $PO(G)$ is equal to the maximum of 0 and $a_G - 1$. Moreover the rank of $\overline{PO}(G)$ is equal to the rank of $PO(G)$ plus the number of all real conjugacy classes represented by 2-elements of order $\geq 8$. Now, let $H$ be a normal subgroup of $G$. We denote by $PO(G, H)$ the subgroup of $RO(G)$ consisting of the differences $U - V$ of representations $U$ and $V$ such that $U^H \cong V^H$ as representations over $G/H$, and $\text{Res}_P^G(U) \cong \text{Res}_P^G(V)$ for any subgroup $P$ of $G$ of prime power order. Again, we note that in [PSo], $PO(G, H)$ is denoted by $IO(G, H)$. It holds that $PO(G) = PO(G, G)$. Let $b_{G/H}$ be the number of all real conjugacy classes in $G/H$ which are images from real conjugacy classes of $G$ represented by elements not of prime power order by the surjection $G \rightarrow G/H$. Then the rank of $PO(G, H)$ is equal to $a_G - b_{G/H}$ (see [PSo]).

**Proposition 3.1** (cf. [PSo]). It holds that

$$PO(G, G^{nil}) \leq LO(G) \leq PO(G) \leq \overline{PO}(G) \leq RO(G).$$

Note that $G^{nil} = \bigcap_p O^p(G)$. Also it is known that

$$LO(G) \subseteq CSm(G) \subseteq Sm(G)$$

if $G$ is an Oliver gap group.

4. UPPER RESTRICTION

Let $S$ be a set of primes dividing $|G|$ and 1, and let denote by $G^{\cap S}$ the normal subgroup of $G$ defined as

$$G^{\cap S} = \bigcap_{L \triangleleft G; [G:L] \in S} L.$$
Theorem 4.1 ([M07a, KMK]). Let $G$ be a finite Oliver group. We set $S = \{2, 3\}$ if a Sylow 2-subgroup of $G$ is normal and set $S = \{2\}$ otherwise. Then it holds that
\[ CSm(G) \subseteq PO(G, G^{n;l}) \quad \text{and} \quad Sm(G) \subseteq \overline{PO}(G, G^{n;l}). \]
In addition if $G$ is a gap group and $G^{nil} = G^{n;l}$, then it holds that
\[ LO(G) = CSm(G) = PO(G, G^{nil}). \]

Here $G^{nil}$ is the minimal subgroup among normal subgroups $N$ of $G$ such that $G/N$ is nilpotent.

In particular, $a_G = b_{G/G^{n;l}}$ yields that $CSm(G) = 0$.

Proposition 4.2 (cf. [PSu07]). $G/G^{n;l}$ is an elementary abelian group.

5. Known results

In this section we summarize several known results ([Ju, M07a, M07b, PSo, PSu07, Su]). First we treat a non-solvable group. Pawalowski and Solomon [PSo] showed that $0 \neq PO(G, G^{nil}) \subseteq CSm(G)$ if $G$ is a non-solvable gap group with $a_G \geq 2$, Pawalowski and Sumi [PSu07] showed that $0 \neq LO(G) \cap CSm(G)$ if $G$ is a non-solvable group with $a_G \geq 2$, except $Aut(A_6)$, $PSL(2, 27)$, and Morimoto [M07a, M07b] showed that $Sm(Aut(A_6)) = 0$ and $CSm(PSL(2, 27)) \neq 0$. Combining these results we can state that

Theorem 5.1. For a finite non-solvable group $G$, $Sm(G) \neq 0$ if and only if $a_G \leq 1$ or $G \equiv Aut(A_6)$.

We say that an element not of prime power order is an NPP element. Morimoto showed the following theorem to get $CSm(PSL(2, 27)) \neq 0$.

Theorem 5.2 (Morimoto). Let $G$ be an Oliver gap group. Suppose that $O^2(G)$ has a dihedral subgroup $D_{2pq}$ of order $2pq$ with distinct primes $p$ and $q$ and $G$ has two real conjugacy classes of NPP elements contained in $O^2(G)$. Then $CSm(G) \neq 0$.

To show $LO(G) \cap CSm(G) \neq 0$ for a non-solvable group with $LO(G) \neq 0$, Pawalowski and Sumi introduced a basic pair (cf. [PSu07, Su]). Let $f: G \rightarrow G/G^{nil}$ be a natural homomorphism. For two NPP elements $x$ and $y$ of an finite Oliver group $G$, we call $(x, y)$ a basic pair, if $f(x) = f(y)$, $x$ is not real conjugate to $y$, and one of the following claims is satisfied:

(1) $x$ and $y$ are elements of some gap subgroup of $G$.
(2) $|x|$ is even and the involutions of $\langle x \rangle$ is conjugate to the involutions of $\langle y \rangle$ in $G$.

We denote by $\pi(G)$ the set of all primes dividing the order of $G$. Note that $\langle x \rangle G^{nil} = \langle y \rangle G^{nil}$ as $f(x) = f(y)$. Recall that if $|x|$ is even, then for the involution $c$ of $\langle x \rangle$, $c \in O^2(G)$ or $|\pi(O^2(C_G(c)))| \geq 2$, then $\langle x \rangle O^2(G)$ is a gap group.

Theorem 5.3 ([PSu07]). If an Oliver group has a basic pair, it holds $LO(G) \cap CSm(G) \neq 0$. 

Recall that \( LO(G/G^{nil}) \subseteq LO(G) \). Furthermore we have

**Proposition 5.4.** \( 2LO(G/G^{nil}) \subseteq WLO(G) \) and in particular \( LO(G/G^{nil}) \neq 0 \) implies \( CSm(G) \neq 0 \).

Then \( LO(G) \cap CSm(G) = 0 \) implies \( LO(G/G^{nil}) = 0 \). Thus the following proposition is important.

**Proposition 5.5** ([PSu07]). Let \( H \) be a nilpotent group with \( LO(H) = 0 \). Then \( H \) is isomorphic to one of the following groups:

1. a \( p \)-group for a prime \( p \),
2. \( C_2 \times P \) for an odd prime \( p \) and a \( p \)-group \( P \), or
3. \( P \times C_3 \) for a 2-group \( P \) such that any element is self-conjugate.

**Lemma 5.6.** If \( a_G \geq 2 \) and \( LO(G) = 0 \) it holds \( |\pi(G/G^{nil})| = 1, 2 \).

**Proof.** If \( |\pi(G/G^{nil})| \geq 3 \), then \( G/G^{nil} \) is a gap group with \( LO(G/G^{nil}) \neq 0 \), a contrary. If \( |\pi(G/G^{nil})| = 0 \), then \( G \) is perfect and thus \( rank LO(G) = a_G - 1 > 0 \), a contrary. \( \square \)

**Theorem 5.7.** If \( LO(G) \cap CSm(G) = 0 \), then \( G \) has no element \( x \) with \( |\pi(\langle x \rangle)| \geq 3 \).

**Proof.** We assume that \( x \) is an element of \( G \) of order \( pqr \) such that \( p, q, r \) are distinct primes. It is clear that \( a_G \geq 4 \). We may assume that \( x^{pq} \in G^{nil} \) by Lemma 5.6. Then \( (x^{pq}x^{pr}, x^{qr}x^{pr}) \) is a basic pair, a contrary. \( \square \)

Thus \( |\pi(\langle c \rangle)| \leq 2 \) for each non-trivial element \( c \in Z(G) \).

### 6. Induced Modules and \( PO(G) \)

Let \( G \) be a finite group and \( NPP(G) \) be the set of all elements of \( G \) not of prime power order. Note that \( NPP(G) \) does not contain the identity element. For the real representation ring \( RO(G) \), the real vector space \( RO(G) \otimes \mathbb{R} \) is identified with the vector space consisting of all maps from the set of real conjugacy classes of \( G \) to the real number field \( \mathbb{R} \). We denote by \( 1^G_{\langle g \rangle_{\pm}^G} \) the map defined by \( 1^G_{\langle g \rangle_{\pm}^G}(\langle g \rangle_{\pm}^G) = 1 \) and \( 1^G_{\langle g \rangle_{\pm}^G}(\langle a \rangle_{\pm}^G) = 0 \) if \( a \) is not real conjugate to \( g \). Then

\[
RO(G) \otimes \mathbb{R} \cong \langle 1^G_{\langle g \rangle_{\pm}^G} | (g)^G_{\pm} \subseteq G \rangle
\]

and

\[
RO(G)_{PP(G)} \otimes \mathbb{R} \cong \langle 1^G_{\langle g \rangle_{k}^G} | g \in NPP(G) \rangle.
\]

Let \( K \) be a subgroup of \( G \). The induced map \( Ind^G_K 1^G_{\langle k \rangle_{\pm}^G} \) has a non-zero value at \( (g)^G_{\pm} \) only if \( g \) is real conjugate to \( k \) in \( G \), i.e. \( (g)^G_{\pm} = (k)^G_{\pm} \), since

\[
Ind^G_K 1^G_{\langle k \rangle_{\pm}^G}((a)^G_{\pm}) = \sum_{b \in G/K, b^{-1}ab \in K} 1^K_{\langle k \rangle_{\pm}^G}((b^{-1}ab)^K_{\pm}).
\]
We denote by $RO(G)_{P(G)}$ the subset of $RO(G)$ consisting the differences $U - V$ of real representations $U$ and $V$ such that $\text{Res}^{G}_{P}(U) \cong \text{Res}^{G}_{P}(V)$ for $P \in P(G)$. It is clear that $PO(G) = \text{Ker}((\text{Fix}^{G} : RO(G)_{P(G)} \rightarrow \mathbb{R})$.

We have the following commutative diagram.

$$
\begin{array}{ccc}
RO(K)_{P(K)} \otimes \mathbb{R} & \longrightarrow & (\text{Ind}^{G}_{K} RO(K)_{P(K)}) \otimes \mathbb{R} \\
\downarrow^{\cong} & & \downarrow^{\cong} \\
\langle 1^{K}_{(k)_{\pm}^{K}} | k \in \text{NPP}(K) \rangle & \longrightarrow & \langle 1^{G}_{(g)_{\pm}^{G}} | g \in \text{NPP}(G) \rangle
\end{array}
$$

It holds that

$$(\text{Ind}^{G}_{K} RO(K)_{P(K)}) \otimes \mathbb{R} = (\text{Ind}^{G}_{K} RO(K))_{P(G)} \otimes \mathbb{R}$$

and then that

$$(\text{Ind}^{G}_{K} RO(K)_{P(K)}) \otimes \mathbb{Q} = (\text{Ind}^{G}_{K} RO(K))_{P(G)} \otimes \mathbb{Q}.$$  

Since an element of $RO(G)$ is a linear combination with rational coefficients of induced modules of $RO(C)$ for cyclic subgroups $C$ of $G$, we obtain that

$$\sum_{C \in \mathcal{S}G} (\text{Ind}^{G}_{C} RO(C)_{P(C)}) \otimes \mathbb{Q} = RO(G)_{P(G)} \otimes \mathbb{Q}.$$  

Furthermore, noting $\text{Ind}^{G}_{C} RO(C)_{P(C)} = 0$ for $C \in P(G)$, it holds that

$$\sum_{g \in \text{NPP}(G)} (\text{Ind}^{G}_{(g)_{\pm}^{G}} RO((g))_{P(G)}) \otimes \mathbb{Q} = RO(G)_{P(G)} \otimes \mathbb{Q}.$$  

If $g$ has order $2p$ for an odd prime $p$, then $RO((g))_{P((g))} \otimes \mathbb{Q}$ is spanned by

$$(2\mathbb{R} - \mathbb{R}[(x^{p})]) \otimes (2\mathbb{R} - \eta)$$

for all real irreducible modules $\eta$ over $(g^{2})$ and $PO((g)) \otimes \mathbb{Q}$ is spanned by

$$(2\mathbb{R} - \mathbb{R}[(x^{p})]) \otimes (\eta - \eta')$$

for all non-trivial real irreducible modules $\eta, \eta'$ over $(g^{2})$. Hence we can investigate $LO(G)$ for a finite non-gap group $G$ with $G/O^{2}(G)$ an elementary abelian 2-group. Letting $C^{2}_{2}$ be an elementary abelian 2-group of order $2^{n}$, we obtain the following results.

**Theorem 6.1.** Let $G := K \times C^{n}_{2}$, $n \geq 2$ be an Oliver group such that $K/O^{2}(K)$ is an elementary abelian 2-group. Then it holds $MLO(G) \subseteq CSm(G) \subseteq LO(G)$. Furthermore if $G$ is a gap group, it holds the equality $CSm(G) = LO(G)$.

We will discuss in the case when $G$ is a non-gap group in Theorem 6.1.

**Proposition 6.2.** Let $G$ be an Oliver non-gap group such that $[G : O^{2}(G)] = 2$.

The following two claims are equivalent.

1. $MLO(G) = LO(G)$.  
2. $G$ is a group with a normal Sylow 2-subgroup.

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1. $MLO(G) = LO(G)$.
2. $G$ is a group with a normal Sylow 2-subgroup.
(2) If two involutions \( x \) and \( y \) of \( G \) outside of \( O^2(G) \) are not conjugate then \( C_G(x) \) or \( C_G(y) \) is a 2-group.

The author does not know a group \( G \) with \( MLO(G) \neq LO(G) \).

7. NON-TRIVIAL CENTRAL

In this section we consider whether \( CSm(G) = 0 \) or not for an Oliver group \( G \) with \( a_G \geq 2 \). In the section 5 we know completely it for a non-solvable group \( G \). From now on we assume that \( G \) is an Oliver solvable group with \( LO(G) \cap CSm(G) = 0 \) and \( a_G \geq 2 \). Recall that \( PO(G, G^{nil}) \neq 0 \) implies \( a_G \geq 2 \).

**Lemma 7.1.** If \( Z(G) \neq \{1\} \) then \( |\pi(G^{nil})| = 2 \).

**Proof.** Since \( LO(G/G^{nil}) = 0 \), \( G/G^{nil} \) is isomorphic to \( P, C_2 \times P \), or \( C_3 \times P_2 \), where \( P \) is a \( p \)-group and \( P_2 \) is a 2-group. Then for some subgroup \( K \) of \( G \), the sequence \( G^{nil} \leq K \leq G \) such that \( |\pi(G/K)| = 1 \) and \( K/G^{nil} \) is cyclic. Thus \( |\pi(G^{nil})| \geq 2 \). We assume that \( |\pi(G^{nil})| \geq 3 \). Take distinct primes \( p, q, r \) in \( \pi(G^{nil}) \). Let \( c \in Z(G) \) be an element of prime order. We may assume that \( |c| \neq q, r \). Take elements \( c_q \) and \( c_r \) of \( G^{nil} \) of order \( q \) and \( r \) respectively. Then \( xc_q \) and \( xc_r \) are NPP elements of distinct order. Therefore \( (cx_q, cx_r) \) is a basic pair. \( \square \)

**Lemma 7.2.** \( Z(G) \) has no NPP element.

**Proof.** We suppose that \( Z(G) \) has an NPP element \( c \) of order \( pq \) where \( p \) and \( q \) are primes. Then \( |\pi(G)| = 2 \) and \( \pi(G) = \pi(\langle c \rangle) = \{p, q\} \) by Theorem 5.7. First we show that \( G^{nil} \) is not a subgroup of \( \langle c \rangle \). Suppose \( G^{nil} \leq \langle c \rangle \). Let \( f: G \rightarrow G/\langle c \rangle \) be a canonical epimorphism. Note that \( \pi(G/\langle c \rangle) = \{p, q\} \). Since \( f(G) \) is nilpotent, \( O^f(f(G)) \) is a Sylow \( p \)-subgroup of \( f(G) \) and a Sylow \( p \)-subgroup \( O^f(G)_p \) of \( O^f(G) \) is normal and its quotient \( O^f(G)/O^f(G)_p \) is cyclic. This is a contrary against \( G \) is Oliver.

\[
\begin{array}{ccc}
\langle c \rangle & \longrightarrow & G \\
\uparrow & & \uparrow \\
\langle c \rangle \cap O^f(G) & \longrightarrow & O^f(G) \\
\uparrow & & \uparrow \\
\langle c \rangle \cap O^f(G)_p & \longrightarrow & O^f(G)_p \\
\end{array}
\]

Thus we can take an element \( x \) of \( G^{nil} \) which is not in \( \langle c \rangle \). Since \( f \) sends two NPP elements \( xc \) and \( c \) to elements of distinct order, \( xc \) and \( c \) are not real conjugate. It is clear that they are sent to the same element by \( G \rightarrow G/G^{nil} \). Then \( (xc, c) \) is a basic pair, which is a contrary. Thus \( Z(G) \) has no NPP element. \( \square \)

The following can be straightforward checked.
Lemma 7.3. Let \( c \in Z(G) \) be an element of order a prime \( p \). If \( G^{nil} \) has an element \( x \) of order \( q^2 \) for some prime \( q \neq p \), then \( G \) has a basic pair \((cx, cx^q)\).

We define the DressLength\((G)\) as the minimal length \( n \) of sequences
\[
G = G_0 > G_1 > G_2 > \cdots > G_n = \{1\}
\]
such that \( O^{p_i}(G_{j-1}) = G_j \) with some prime \( p_j \) for each \( j \). In convenient, we assume DressLength\((G)\) = \( \infty \) if there is no sequence as above. For example, DressLength\((G)\) = \( \infty \) for a non-solvable group. It is easy to see that DressLength\((G)\) \( \geq 3 \) if \( G \) is an Oliver group and that DressLength\((G)\) \( \geq 3 \) if \( G \) is a gap group.

Now we recall classical results. A finite group is called a CP group if it has no NPP elements.

Lemma 7.4 (Higman, cf. [PSO, Lemma 2.5]). Let \( H \) be a finite solvable CP group. Then one of the following conclusions holds:

1. \( H \) is a \( p \)-group for some prime \( p \); or
2. \( H = K \rtimes C \) is a Frobenius group with kernel \( K \) and complement \( C \), where \( K \) is a \( p \)-group and \( C \) is a \( q \)-group of \( q \)-rank \( 1 \) for two distinct primes \( p \) and \( q \); or
3. \( H = K \rtimes C \rtimes A \) is a 3-step group, in the sense that \( K \rtimes C \) is a Frobenius group as in the conclusion (2) with \( C \) cyclic, and \( C \rtimes A \) is a Frobenius group with kernel \( C \) and complement \( A \), a cyclic \( p \)-group.

Proposition 7.5 ([Hu, Proposition 22.3 and Remark on p.193]). \( \text{Aut}(C_{2^p}) = C_2 \times C_{2^{p-2}} \) where \( x \leftrightarrow x^5 \) is a generator of \( C_{2^{p-2}} \) and \( x \leftrightarrow x^{-1} \) is a generator of \( C_2 \). \( \text{Aut}(C_{p^r}) = C_{p^{r-1}(p-1)} \) for an odd prime \( p \).

With these results we use a Frattini subgroup and a Fitting subgroup and then we obtain the following results.

Theorem 7.6. Let \( G \) be an Oliver solvable group with \( a_G \geq 2 \) and \( Z(G) \neq \{1\} \). If \( CS m(G) = 0 \), then it holds the following.

1. \( Z(G) \) has no NPP element.
2. If \( Z(G) \) is a \( p \)-group, an element of \( G^{nil} \) not of \( p \) power order has prime order.
3. \( |\pi(G)| = 2 \).

References


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