Commutativity of localized self-homotopy groups of symplectic groups

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Abstract

The self-homotopy group of a topological group $G$ is the set of homotopy classes of self-maps of $G$ equipped with the group structure inherited from $G$. We determine the set of primes $p$ such that the $p$-localization of the self-homotopy group of $\text{Sp}(n)$ is commutative. As a consequence, we see that this group detects the homotopy commutativity of $p$-localized $\text{Sp}(n)$ by its commutativity almost all cases.

1 Introduction

For a group-like space $G$, the pointed homotopy set $[X, G]$ has a natural group structure inherited from $G$. We will always assume $[X, G]$ as a group with this group structure. This group has been studied for a long time, and there are many applications especially to the $H$-structure of $G$. See [1] and [9], for example. Put $X = G$. Then the group $[G, G]$ is called the self-homotopy group of $G$ and denoted by $\mathcal{H}(G)$. The self homotopy group $\mathcal{H}(G)$ has also been studied extensively, especially, in connection with the $H$-structure of $G$, see [2], [12] and [11]. In particular, it is shown in [12] the following.

**Theorem 1.1** (Kono and Ōshima [12]). Let $G$ be a compact, connected Lie group. Then $\mathcal{H}(G)$ is commutative if and only if $G$ is isomorphic with $T^n$ $(n \geq 0)$, $T^n \times \text{Sp}(1)$ $(0 \leq n \leq 2)$ or $\text{SO}(3)$, where $T^n$ denotes the $n$-dimensional torus.

Then we can say that for a connected Lie group $G$, $\mathcal{H}(G)$ reflects the homotopy commutativity of $G$ to its commutativity effectively, since we have Hubbuck’s torus theorem [8].

Localize at the prime $p$ in the sense of [7]. Then it is an interesting problem to consider for a fixed $G$, how the $H$-structure of $G_{(p)}$ changes when we vary $p$. Kaji and the first named author obtained a result for a Lie group $G$ when $p$ is relatively large [9], [10]. Let us turn to the self homotopy group $\mathcal{H}(G)$. Let $X$ be a finite complex, and let $G$ be a path-connected group-like space. Then the group $[X, G]$ is known to be nilpotent, and then we can consider its localization $[X, G]_{(p)}$ at the prime $p$ in the sense of [7]. Moreover, there is a natural isomorphism of groups:

$$[X, G]_{(p)} \cong [X_{(p)}, G_{(p)}]$$
See [7]. Then if $G$ is a connected Lie group, it is also an interesting problem to consider how the group structure of $\mathcal{H}(G)_{(p)}$ changes if we vary $p$ as is considered for $G_{(p)}$. Recently, Hamanaka and the second named author obtained:

**Theorem 1.2** (Hamanaka and Kono [5]). $\mathcal{H}({\text{SU}}(n))_{(p)}$ is commutative if and only if $p > 2n$ except for $n = 2$ and $(p, n) = (5, 3), (7, 4), (11, 6), (13, 7)$.

As is shown in [13], $\text{SU}(n)_{(p)}$ is homotopy commutative if and only if $p > 2n$. Then we can say that $\mathcal{H}(\text{SU}(n)_{(p)})$ detects the homotopy commutativity of $\text{SU}(n)_{(p)}$ very well.

The aim of this paper is to consider the above problem for $G = \text{Sp}(n)$, and we will prove:

**Theorem 1.3.** The group $\mathcal{H}({\text{Sp}}(n))_{(p)}$ is commutative if and only if $p > 4n$ except for $n = 1$ and $(p, n) = (3, 2), (5, 3), (7, 2), (11, 3), (19, 5), (23, 6)$.

Since $\text{Sp}(n)_{(p)}$ is homotopy commutative if and only if $p > 4n$ except for $(p, n) = (3, 2)$ by [13], we get:

**Corollary 1.1.** $\text{Sp}(n)_{(p)}$ is homotopy commutative if and only if $\mathcal{H}(\text{Sp}(n))_{(p)}$ is commutative except for $n = 1$ and $(p, n) = (5, 3), (7, 2), (11, 3), (19, 5), (23, 6)$.

**Remark 1.1.** Let $p$ be an odd prime. As is well known [4], there is a homotopy equivalence $B\text{Sp}(n)_{(p)} \simeq B\text{Spin}(2n+1)_{(p)}$, and then, in particular, we have $\mathcal{H}(\text{Sp}(n))_{(p)} \cong \mathcal{H}(\text{Spin}(2n+1))_{(p)}$. Thus the above results for $\mathcal{H}(\text{Sp}(n))_{(p)}$ implies those for $\mathcal{H}(\text{Spin}(2n+1))_{(p)}$. We also have a similar result for $\mathcal{H}(\text{Spin}(2n))_{(p)}$ when $p$ is an odd prime [6].

## 2 Calculating commutators in the group $[X, \text{Sp}(n)]$

Throughout this section, all spaces will be localized at the prime $p$.

Put $G_n = \text{Sp}(n)$ and $X_n = G_\infty/G_n$. Let $q_k \in H^{4k}(BG_n; \mathbb{Z}_{(p)})$ be the $k$-th universal symplectic Pontrjagin class. Then the cohomology of $G_n$ is given by

$$H^\ast(G_n; \mathbb{Z}_{(p)}) = \Lambda(x_3, x_7, \ldots, x_{4n-1}), \quad x_{4k-1} = \sigma(q_k),$$

where $\sigma$ is the cohomology suspension. We also have

$$H^\ast(X_n; \mathbb{Z}_{(p)}) = \Lambda(y_{4n+3}, y_{4n+7}, \ldots), \quad \pi^\ast(y_i) = y_i$$

for the projection $\pi : G_\infty \to X_n$. Put $b_{4k+2} = \sigma(y_{4k+3}) \in H^\ast(\Omega X_n; \mathbb{Z}_{(p)})$ for $k \geq n$. We write a map $X \to K(\mathbb{Z}_{(p)}, k)$ corresponding to the cohomology class $x \in H^k(X; \mathbb{Z}_{(p)})$ by $x$, ambiguously. Then, in particular, since $b_{4k+2}$ is a loop map, the map $b_{4k+2} : [X, \Omega X_n] \to H^{4k+2}(X; \mathbb{Z}_{(p)})$ is a homomorphism.

Now we recall from [15] how to determine the (non)triviality of commutators in the group $[X, G_n]$. Apply the functor $[X, -]$ to the fibre sequence

$$\Omega G_\infty \xrightarrow{\Omega \pi} \Omega X_n \xrightarrow{\delta} G_n \to G_\infty$$
in which all arrows are loop maps. Then we get an exact sequence of groups:

\[
\overline{KSp}^{-2}(X)_{(p)} \xrightarrow{(\Omega \pi)_*} [X, \Omega X_n] \xrightarrow{\delta_*} [X, G_n] \rightarrow \overline{KSp}^{-1}(X)_{(p)}
\]  

(2.1)

Since \(\overline{KSp}^{-1}(X)_{(p)}\) is abelian, commutators in \([X, G_n]\) are in the image of \(\delta_*\). We determine the (non)triviality of commutators in \([X, G_n]\) by the following proposition which is easily deduced by (2.1).

**Proposition 2.1.** Let \(\alpha, \beta \in [X, G_n]\), and put \(\Phi = \bigoplus_{i=1}^k (b_{4n_i+2})_* : [X, \Omega X_n] \rightarrow \bigoplus_{i=1}^k H^{4n_i+2}(X; \mathbf{Z}_{(p)})\).

1. If there exists \(\lambda \in [X, \Omega X_n]\) such that \(\delta_*(\lambda) = [\alpha, \beta]\) and \(\Phi(\lambda)\) is not in the image of \(\Phi \circ (\Omega \pi)_*\), then \([\alpha, \beta]\) is not trivial.

2. Suppose that \(\Phi\) is injective. Then \([\alpha, \beta]\) is not trivial if and only if there exists the above \(\lambda\).

In order to use Proposition 2.1, we need to describe \(\lambda^*(b_{4m+2})\) explicitly, where \(\lambda\) is as in Proposition 2.1. In [15], it is shown that we can choose \(\lambda\) as:

**Lemma 2.1.** For \(\alpha, \beta \in [X, G_n]\), there exists \(\lambda \in [X, \Omega X_n]\) such that \(\delta_*(\lambda) = [\alpha, \beta]\) and for \(k \geq n\),

\[
\lambda^*(b_{4k+2}) = \sum_{i+j=k+1}^{\alpha^*(x_{4i-1})\beta^*(x_{4j-1})}.
\]

We next describe \((\Omega \pi)_*(\xi)\) through the map \(b_{4k+2} : [X, \Omega X_n] \rightarrow H^{4k+2}(X; \mathbf{Z}_{(p)})\) for \(\xi \in \overline{KSp}^{-2}(X)_{(p)}\) to use Proposition 2.1. Let \(c' : G_n \rightarrow U(2n)\) denote the complexification map. We also denote the complexification \(\overline{KSp}^*(X)_{(p)} \rightarrow \overline{K}^*(X)_{(p)}\) by \(c'\). Let \(\text{ch}_k\) denote the 2\(k\)-dimensional part of the Chern character.

**Lemma 2.2.** For \(\xi \in \overline{KSp}^{-2}(X)_{(p)}\), we have

\[
(b_{4k+2} \circ \Omega \pi)_*(\xi) = (-1)^{k+1}(2k+1)!\text{ch}_{2k+1}(c'(\xi)).
\]

**Proof.** Let \(c_k\) be the \(k\)-th universal Chern class. Then we have \(c'(c_{2k}) = (-1)^k q_k\), and thus

\[
(b_{4k+2} \circ \Omega \pi)_*(\xi) = \alpha^2(q_{k+1})(\xi) = (-1)^{k+1}(2k+1)!\text{ch}_{2k+1}(c'(\xi)).
\]

\(\square\)

### 3 Proof of Theorem 1.3 for \(p\) odd

Throughout this section, we localize all spaces at the odd prime \(p\) unless otherwise is specified.

For a given positive integer \(n\), let \(m\) be an arbitrary integer satisfying \(m < n \leq 2m\). Let \(\epsilon_{4k-1}\) be a generator of \(\pi_{4k-1}(G_n) \cong \mathbf{Z}_{(p)}\) for \(k \leq n\). Then we have

\[
(\epsilon_{4k-1})^*(x_{4k-1}) = \begin{cases} (2k - 1)!u_{4k-1} & k \text{ is odd} \\ 2(2k - 1)!u_{4k-1} & k \text{ is even} \end{cases}
\]

(3.1)
where $u_l$ is a generator of $H^1(S^l; \mathbf{Z}(p))$. Define a map $\theta : S^{4m-1} \times S^{4m+3} \to G_n$ by the composition

$$S^{4m-1} \times S^{4m+3} \xrightarrow{c_{4m-1} \times c_{4m+3}} G_n \times G_n \xrightarrow{\mu} G_n,$$

where $\mu$ is the multiplication of $G_n$. Then by (3.1), we have for $k < l$:

$$\theta^*(x_{4k-1} x_{4l-1}) = \begin{cases} 2(2m-1)! (2m+1)! u_{4m-1} \otimes u_{4m+1} & (k, l) = (m, m+1) \\ 0 & \text{otherwise} \end{cases} \quad (3.2)$$

Let $j : G_n \to G_{2m}$ be the inclusion, and let $\psi^2 : BG_n \to BG_n$ be the unstable Adams operation of degree 2 [18]. We can consider the commutator $[j \circ \Omega \psi^2, j]$ in $[G_n, G_{2m}]$ by pulling back to $S^{4m-1} \times S^{4m+3}$ through $\theta$. By Lemma 2.1, there exists $\lambda \in [G_n, \Omega X_{2m}]$ such that $\delta(\lambda) = [j \circ \Omega \psi^2, j]$ and

$$\lambda^*(b_{8m+2}) = \sum_{i+j=n+1} (\Omega \psi^2)^*(x_{4i-1}) x_{4j-1}.$$

By definition of $\psi^2$, we have $$(\Omega \psi^2)^*(x_{4k-1}) = 2^{2k} x_{4k-1}.$$ Then we get

$$\lambda^*(b_{8m+2}) = \sum_{i+j=n+1} 2^{2i} x_{4i-1} x_{4j-1}$$

and thus by (3.2),

$$\theta^* \circ \lambda^*(b_{8m+2}) = 2^{2m} (-3)(2m-1)! (2m+1)! u_{4m-1} \otimes u_{4m+1}.$$ 

On the other hand, we have $KSp^{-2}(S^{4m-1} \times S^{4m+3})_{(p)} \cong \mathbf{Z}(p)$ and its generator $\xi$ can be chosen to satisfy

$$\text{ch}_{4m+1}(\xi) = (4m+1)! u_{4m-1} \otimes u_{4m+3}.$$ 

If we see that $\theta^* \circ \lambda^*(b_{8m+2})$ is not in the $\mathbf{Z}(p)$-module generated by $(4m+1)! u_{4m-1} \otimes u_{4m+3}$, by Proposition 2.1, we can conclude that $\theta^*([j \circ \Omega \psi^2, j]) = j \circ [\Omega \psi^2, 1_{G_n}] \circ \theta$ is non-trivial which implies $\mathcal{H}(G_n)$ is not commutative. Put $m$ as in the following table. Then we can easily see that $m$ satisfies $m < n \leq 2m$ and $\frac{(4m+1)!}{(2m-1)!(2m+1)!} = (4m+1)\left(\begin{smallmatrix} 3m \\ 2m-1 \end{smallmatrix}\right) \equiv 0 \pmod{p}$ by Lucas’ formula, and thus $\mathcal{H}(G_n)$ is not commutative in these cases.

<table>
<thead>
<tr>
<th>$p &lt; n$</th>
<th>$p = n$</th>
<th>$n &lt; p &lt; n + 3 \ (p \geq 13)$</th>
<th>$n + 3 \leq p &lt; 2n$</th>
<th>$2n &lt; p &lt; 4n - 1 \ (p \equiv -1 \pmod{4})$</th>
<th>$2n &lt; p &lt; 4n - 1 \ (p \equiv 1 \pmod{4}, p &gt; 5)$</th>
<th>$(p, n) = (5, 2)$</th>
<th>$(p, n) = (7, 6)$</th>
<th>$(p, n) = (11, 9), (11, 10)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m \equiv 0 \pmod{p}$, $0 &lt; n - m \leq p$</td>
<td>$m = p - 1$</td>
<td>$m = p - 3$</td>
<td>$m = \frac{p+3}{2}$</td>
<td>$m = \frac{p+1}{4}$</td>
<td>$m = \frac{p+3}{4}$</td>
<td>$m = 1$</td>
<td>$m = 5$</td>
<td>$m = 8$</td>
</tr>
</tbody>
</table>
Recall from [13] that $G_n$ is homotopy commutative if $p > 4n$ or $(p, n) = (3, 2)$ which implies $\mathcal{H}(G_n)$ is commutative for $p > 4n$ or $(p, n) = (3, 2)$ . Then the remaining cases are:

1. $p = 4n - 1$
2. $(p, n) = (7, 5)$
3. $(p, n) = (5, 4)$
4. $(p, n) = (5, 3)$

### 3.1 Case 1

In this case we have a homotopy equivalence [14] $G_n \cong \prod_{k=1}^{n} S^{4k-1}$. Assume $n \geq 14$. We define $\alpha \in \mathcal{H}(G_n)$ by the composite

$$G_n \xrightarrow{\rho} S^3 \times S^7 \times S^{11} \times S^{15} \times S^{4n-37} \xrightarrow{q} S^{4n-1} \xrightarrow{\epsilon_{4n-1}} G_n,$$

where $\rho$ is the projection and $q$ is the pinch map onto the top cell. We also define $\beta \in \mathcal{H}(G_n)$ by

$$G_n \xrightarrow{\rho'} S^{4n-1} \xrightarrow{\epsilon_{4n-1}} G_n,$$

where $\rho'$ is the projection. Then we have

$$[\alpha, \beta] = \gamma \circ (\epsilon_{4n-1} \times \epsilon_{4n-1}) \circ ((q \circ \rho) \times \rho') \circ \Delta,$$

where $\gamma \colon G_n \times G_n \to G_n$ and $\Delta \colon G_n \to G_n \times G_n$ denote the commutator map of $G_n$ and the diagonal map, respectively. Now one can easily see $((q \circ \rho) \times \rho') \circ \Delta$ induces an injection $[S^{4n-1} \times S^{4n-1}, G_n] \to \mathcal{H}(G_n)$. On the other hand, we have $\gamma \circ (\epsilon_{4n-1} \times \epsilon_{4n-1}) = (\epsilon_{4n-1}, \epsilon_{4n-1}) \circ q'$, where $q' : S^{4n-1} \times S^{4n-1} \to S^{8n-2}$ is the pinch map onto the top cell and $(-, -)$ means a Samelson product. Then since $q'$ induces an injection $\pi_{8n-2}(G_n) \to [S^{4n-1} \times S^{4n-1}, G_n]$ and the Samelson product $\langle \epsilon_{4n-1}, \epsilon_{4n-1} \rangle \in \pi_{8n-2}(G_n)$ is non-trivial by [3], we obtain that the commutator $[\alpha, \beta]$ is non-trivial. Thus $\mathcal{H}(G_n)$ is not commutative.

We next assume $8 \leq n \leq 13$. By looking at the homotopy groups of spheres [16], the above Samelson product $\langle \epsilon_{4n-1}, \epsilon_{4n-1} \rangle$ factors as $\langle \epsilon_{4n-1}, \epsilon_{4n-1} \rangle = i \circ \alpha_1(3)$, where $i : S^3 \to G_n$ is the inclusion and $\alpha_1(2k-1)$ is a generator of $\pi_{2k+2p-4}(S^{2k-1}) \cong \mathbb{Z}/p$. Put $X = S^3 \times S^7 \times S^{11} \times S^{4n-13} \times S^{4n-9} \times S^{4n-5}$. We define $\alpha, \beta \in \mathcal{H}(G_n)$ by

$$G_n \xrightarrow{\rho} X \xrightarrow{q} S^{3p-3} \xrightarrow{\alpha_1(p)} S^p \xrightarrow{\epsilon_p} G_n$$

and

$$G_n \xrightarrow{\rho'} S^p \xrightarrow{\epsilon_p} G_n,$$

respectively, where $\rho$ and $\rho'$ are the projections and $q$ is the pinch map onto the top cell. Then we get

$$[\alpha, \beta] = i \circ \alpha_1(3) \circ \alpha_1(2p) \circ q' \circ ((q \circ \rho) \times \rho') \circ \Delta,$$
where \( q' : S^{9p-3} \times S^p \rightarrow S^{4p-3} \) is the pinch map. As is seen above, the maps \( i \) and \( q' \circ ((q \circ \rho) \times \rho') \circ \Delta \) induce injections \( \pi_{4p-3}(S^3) \rightarrow \pi_{4p-3}(G_n) \) and \( \pi_{4p-3}(G_n) \rightarrow \mathcal{H}(G_n) \), respectively. Since \( \alpha_1(3) \circ \alpha_1(2p) \neq 0 \) as in [16], we obtain that the commutator \([\alpha, \beta]\) is non-trivial. Thus \( \mathcal{H}(G_n) \) is not commutative.

For \( n \leq 7 \), the case 1 occurs only when \( n = 1, 2, 3, 5, 6 \). We only prove the case \( n = 6 \) since the remaining cases are quite similarly proved. Note that for \( n = 6 \) in the case 1, we have \( p = 23 \). One can easily see that the dimension of cells of \( G_6/\bigvee_{k=1}^6 S^{4k-1} \) is in the set \( I = \{0\} \cup \bigcup_{k=2}^6 \{4(n_1 + \cdots + n_k) - k \mid 1 \leq n_1 < \cdots < n_k \leq 6\} \). On the other hand, Since \( G_6 \cong \prod_{k=1}^6 S^{4k-1} \), we see that the homotopy groups of \( G_6 \) in dimension \( k \in I \) for all \( k \in I \) are trivial by looking at the homotopy groups of spheres [16]. Then the inclusion \( \bigvee_{k=1}^6 S^{4k-1} \rightarrow G_6 \) induces an injection \( \mathcal{H}(G_6) \rightarrow \bigoplus_{k=1}^6 \pi_{4k-1}(G_6) \), and so \( \mathcal{H}(G_6) \) is commutative.

### 3.2 Case 2

In this case, we have \( G_5 \cong B_1 \times B_2 \times S^{11} \), where \( B_k \) is an \( S^{4k-1} \)-bundle over \( S^{4k+1} \) for \( k = 1, 2 \), see [14]. We first calculate \( K^*(G_5)_{(7)} \). Note that \( K^*(B_k) \) for \( k = 1, 2 \) and \( K^*(S^{11})_{(7)} \) are free \( \mathbb{Z}_{(7)} \)-modules, we have

\[
K^*(G_5)_{(7)} \cong K^*(B_1)_{(7)} \otimes K^*(B_2)_{(7)} \otimes K^*(S^{11})_{(7)}.
\]

Let \( A_k \) be the \((4k + 11)\)-skeleton of \( B_k \) for \( k = 1, 2 \). Then we have \( A_2 \cong \Sigma^4 A_1 \).

Let \( u' \) be the composite of the inclusions \( \Sigma A_1 \rightarrow \Sigma G_5 \rightarrow BG_5 \rightarrow BU(\infty) \). Since \( A_1 \) is a retract of \( \Sigma CP^7 \), we get \( ch(u') = \Sigma t_3 + \frac{1}{7} \Sigma t_{15} \) where \( t_3, t_{15} \) are generators of \( H^*(A_1; \mathbb{Z}_{(7)}) \) with \(|t_k| = k\) and \( \Sigma \) stands for the suspension isomorphism. Let \( v' \) be the composite of the pinch map \( \Sigma A_1 \rightarrow S^{16} \) and a generator of \( \pi_{16}(BU(\infty)) \cong \mathbb{Z}_{(7)} \). Then we see \( ch(v') = \Sigma t_{15} \) by choosing a suitable generator of \( \pi_{16}(BU(\infty)) \). Consider the exact sequence

\[
0 \rightarrow \tilde{K}^{-1}(S^{15})_{(7)} \rightarrow \tilde{K}^{-1}(A_1)_{(7)} \rightarrow \tilde{K}^{-1}(S^3)_{(7)} \rightarrow 0
\]

induced from the cofibre sequence \( S^3 \rightarrow A_1 \rightarrow S^{15} \). Then we get \( \tilde{K}^{-1}(A_1)_{(7)} \) is generated by \( u' \) and \( v' \). Since the inclusion \( A_k \rightarrow B_k \) induces an isomorphism \( \tilde{K}^{-1}(B_k)_{(7)} \rightarrow \tilde{K}^{-1}(A_k)_{(7)} \), we get

\[
K^*(G_5)_{(7)} = \Lambda(u_1, u_2, v_1, v_2, w), \quad |u_k| = |v_k| = |w| = -1
\]

such that for \( k = 1, 2 \),

\[
ch(u_k) = \Sigma x_{4k-1} + \frac{1}{7} \Sigma x_{4k+11}, \quad ch(v_k) = \Sigma x_{4k+11}, \quad ch(w) = \Sigma x_{11}.
\]

Since \( q \circ c' = 2 \) for the quaternionization \( q : K^*(G_5)_{(7)} \rightarrow KSp^*(G_5)_{(7)} \), we obtain:

**Lemma 3.1.** \( KSp^{-2} G_5_{(7)} \) is a free \( \mathbb{Z}_{(7)} \)-module with a basis \( \{a_1, \ldots, a_{10}\} \) such that

\[
ch_{15}(c'(a_k)) = \begin{cases} 
\frac{1}{7} x_{11} x_{19} & k = 1 \\
x_{11} x_{19} & k = 2 \\
0 & k \neq 1, 2.
\end{cases}
\]
Let $\alpha$ be the composite of the projection $G_5 \to B_2$ and the inclusion $B_2 \to G_5$. We consider the commutator $[1_{G_5}, \alpha]$. By Lemma 2.1, there exists $\lambda \in [G_5, \Omega X_3]$ such that $\delta_*(\lambda) = [1_{G_5}, \alpha]$ and $\lambda^*(b_{30}) = x_{11}x_{19}$. On the other hand, it follows from Lemma 2.2 and Lemma 3.1 that the image of the map $b_{30} \circ (\Omega \pi)_* : \widetilde{KSp}^2(G_5) \to H^{30}(G_5; \mathbb{Z}_{(7)})$ is generated by $7x_{11}x_{19}$. Then by Proposition 2.1, we conclude that $[1_{G_5}, \alpha]$ is non-trivial which implies $\mathcal{H}(G_5)$ is not commutative.

### 3.3 Case 3

In this case, we have a homotopy equivalence $G_4 \cong B_1 \times B_2$ where $B_k$ is an $S^{4k-1}$-bundle over $S^{4k+7}$ for $k = 1, 2$ [14]. As in the previous case, we have

$$K^*(G_4) = \Lambda(u_1, u_2, v_1, v_2), \quad |u_k| = |v_k| = -1$$

such that for $k = 1, 2$,

$$\text{ch}(u_k) = \Sigma x_{4k-1} + \frac{1}{5!} \Sigma x_{4k+7}, \quad \text{ch}(v_k) = \Sigma x_{4k+7},$$

and thus we obtain:

**Lemma 3.2.** $\widetilde{KSp}^2(G_4)$ is a free $\mathbb{Z}$-module with a basis $\{a_1, \ldots, a_6\}$ such that

$$\text{ch}_{11}(c'(a_k)) = \begin{cases} 
\frac{1}{5} x_7 x_{15} & k = 1 \\
x_7 x_{15} & k = 2 \\
0 & k \neq 1, 2.
\end{cases}$$

Let $\psi^2 : BG_4 \to BG_4$ be the unstable Adams operation of degree 2 as above. We consider $[\Omega\psi^2, 1_{G_1}]$. By Lemma 2.1, there exists $\lambda \in [G_4, \Omega X_4]$ such that $\delta_*(\lambda) = [\Omega\psi^2, 1_{G_4}]$ and

$$\lambda^*(b_{22}) = 2^4 x_7 x_{15} + 2^8 x_{15} x_7 = 2^4 \cdot 3 \cdot 5 x_7 x_{15}.$$

Then by Lemma 2.2 and Lemma 3.2, we see that $\lambda^*(b_{22})$ is not in the image of $b_{22} \circ (\Omega \pi)_*$. Then by Proposition 2.1, we obtain $[\Omega\psi^2, 1_{G_4}]$ is not trivial, and thus $\mathcal{H}(G_4)$ is not commutative.

### 3.4 Case 4

This case is very special. We first show:

**Lemma 3.3.** The map $(b_{14} \times b_{18})_* : [G_3, \Omega X_3] \to H^{14}(G_3; \mathbb{Z}_{(5)}) \oplus H^{18}(G_3; \mathbb{Z}_{(5)})$ is injective.

**Proof.** Note that the 23-skeleton of $X_3$ is $A = S^{15} \cup e^{19} \cup e^{23}$. Then since $G_3$ is of dimension 21, the inclusion $A \to X_3$ induces an isomorphism of groups $[G_4, \Omega A] \cong [G_4, \Omega X_3]$. Since for $k \leq 23$, $\pi_k(A)$ is in the stable range. Then one can easily see that

$$\pi_k(A) \cong \begin{cases} 
\mathbb{Z} & k = 15, 19, 23 \\
0 & k \neq 15, 19, 23 \text{ and } k \leq 23.
\end{cases}$$

Thus we can easily deduce that $[G_3, \Omega X_3]$ is a free $\mathbb{Z}_{(5)}$-module. On the other hand, the rationalization of the map $(b_{14} \times b_{18})_*$ is injective. Then the proof is completed. \qed
As in the case 2, we have

\[ K^*(G_3)_{(5)} = \Lambda(u, v, w), \quad |u| = |v| = |w| = -1 \]

such that

\[ \text{ch}(u) = \Sigma x_3 + \frac{1}{5!}\Sigma x_{11}, \quad \text{ch}(v) = \Sigma x_{11}, \quad \text{ch}(w) = \Sigma x_7. \]

Then we get \( \widetilde{KSp}^{-2}(G_3)_{(5)} \) is a free \( \mathbb{Z}_{(5)} \)-module with a basis \( \{a_1, a_2, a_3\} \) such that

\[ \text{ch}(c'(a_1)) = x_3x_{11}, \quad \text{ch}(c'(a_2)) = \frac{1}{5}x_7x_{11}, \quad \text{ch}(c'(a_3)) = x_7x_{11}. \]

Thus we obtain:

**Lemma 3.4.** The image of \((b_{14} \times b_{18})_* \circ (\Omega \pi)_* : \widetilde{KSp}^{-2}(G_3)_{(5)} \to H^{14}(G_3; \mathbb{Z}_{(5)}) \oplus H^{18}(G_3; \mathbb{Z}_{(5)}) \) is generated by \( 5x_3x_{11} \) and \( x_7x_{11} \).

Let \( \alpha, \beta \in \mathcal{H}(G_1) \). Then for a degree reason, we have \( \alpha^*(x_{4k-1}) = \alpha_{4k-1}x_{4k-1} \) and \( \beta^*(x_{4k-1}) = \beta_{4k-1}x_{4k-1} \), where \( \alpha_i, \beta_i \in \mathbb{Z}_{(5)} \). Moreover, since \( P^1x_3 = x_{11} \), we have \( \alpha_3 \equiv \alpha_{11}, \beta_3 \equiv \beta_{11} \) (5).

Let us consider the commutator \([\alpha, \beta]\). By Lemma 2.1, there exists \( \lambda \in [G_3, \Omega X_3] \) such that \( \delta_*^\lambda(\lambda) = [\alpha, \beta] \) and

\[ \lambda^*(b_{14}) = (\alpha_3\beta_{11} - \alpha_{11}\beta_3)x_3x_{11}, \quad \lambda^*(b_{18}) = (\alpha_7\beta_{11} - \alpha_{11}\beta_7)x_7x_{11}. \]

Since \( \alpha_3\beta_{11} - \alpha_{11}\beta_3 \equiv 0 \) (5), we obtain that \((b_{14} \times b_{18})_*^\lambda(\lambda)\) is in the image of \((b_{14} \times b_{18})_* \circ (\Omega \pi)_* \) by Lemma 3.4. Thus by Proposition 2.1, \( \mathcal{H}(G_3) \) is commutative.

## 4 Proof of Theorem 1.3 for \( p = 2 \)

Throughout this section, spaces will be localized at the prime 2. We only consider \( \mathcal{H}(G_n) \) for \( n \geq 2 \) since \( \mathcal{H}(G_1) \) is obviously commutative.

For \( m \geq 2 \), put \( N = 2^{m-2} \). Let \( A = S^3 \cup e^7 \) be the 7-skeleton of \( G_\infty \), and let \( i : \Sigma A \to BG_\infty \) be the composite of inclusions \( \Sigma A \to \Sigma G_\infty \to BG_\infty \). We write generators of \( \widetilde{H}^*(A; \mathbb{Z}(2)) \) by \( t_3, t_7 \) where \( |t_k| = k \). Then by [17], we can deduce

\[ \text{ch}(c'(i)) = \Sigma u_3 - \frac{1}{6}\Sigma u_7. \tag{4.1} \]

For a generator \( \beta_R \) of \( \widetilde{KO}(S^8)_2 \), let \( \bar{\alpha} : \Sigma^{8N-8}A \to G_\infty \) be the adjoint of \( i \wedge \beta_R^{N-1} : \Sigma^{8N-7}A \to BG_\infty \). Then by (4.1), we get

\[ \bar{\alpha}^*(x_{8N-1}) = (4N - 1)!\Sigma^{8N-7} \text{ch}(c'(i)) = -(4N - 1)!\frac{1}{6}\Sigma^{8N-8}t_7. \]

Since the inclusion \( G_{4N} \to G_\infty \) is an \((16N + 2)\)-equivalence and \( \Sigma^{8N-8}A \) is of dimension \( 8N - 1 \), the map \( \bar{\alpha} : \Sigma^{8N-8}A \to G_\infty \) factors as the composite of the map \( \alpha : \Sigma^{8N-8}A \to G_{4N} \) and the inclusion \( G_{4N} \to G_\infty \). In particular, we have

\[ \alpha^*(x_{8N-1}) = -(4N - 1)!\frac{1}{6}\Sigma^{8N-8}t_7. \]
Let \( \epsilon \) be a generator of \( \pi_{8N+3}(G_{4N}) \). Then we get
\[
\epsilon^*(x_{8N+3}) = (4N + 1)!w,
\]
where \( w \) denotes a generator of \( H^{8N+3}(S^{8N+3}; \mathbb{Z}(2)) \). Define a map \( \theta : \Sigma^{8N-8}A \times S^{8N+3} \to G_{4N} \) by the composite
\[
\Sigma^{8N-8}A \times S^{8N+3} \xrightarrow{\alpha \times \epsilon} G_{4N} \times G_{4N} \xrightarrow{\mu} G_{4N},
\]
where \( \mu \) is the multiplication of \( G_{4N} \). Then by definition, we have:
\[
\theta^*(x_{4k-1}) = \begin{cases} 
-(4N - 1)!\frac{1}{6}\Sigma^{8N-8}t_7 \otimes 1 & k = 2N \\
(4N + 1)!\otimes w & k = 2N + 1 \\
0 & k \neq 2N, 2N + 1 \end{cases} \tag{4.2}
\]
Consider the commutator \([\Omega \psi^3, 1_{G_{4N}}] \) in \( H(G_{4N}) \) for the unstable Adams operation \( \psi^3 : BG_{4N} \to BG_{4N} \) of degree 3. Then by Lemma 2.1, there exists \( \lambda \in [G_{4N}, \Omega X_{4N}] \) such that
\[
\lambda^*(b_{16N+2}) = \sum_{i+j=4N+1 \atop 1 \leq i, j \leq 4N} (\Omega \psi^3)^*(x_{4i-1})x_{4j-1} = \sum_{i+j=4N+1 \atop 1 \leq i, j \leq 4N} 3^{2i}x_{4i-1}x_{4j-1}.
\]
Hence by (4.2), we get
\[
\theta^* \circ \lambda^*(b_{16N+2}) = 3^{4N-1} \cdot 4(4N - 1)!\frac{1}{6}\Sigma^{8N-8}t_7 \otimes w, \tag{4.3}
\]
here \( \delta_*(\lambda \circ \theta) \) equals to the commutator \([\Omega \psi^3, \theta, \theta] \) in \([\Sigma^{8N-8}A \times S^{8N+3}, G_{4N}]\).

In order to apply Proposition 2.1, we next calculate the free part of \( \widetilde{KSp}^{-2}(\Sigma_{4N-8}A \times S^{8N+3}) \). We know that the pinch map \( \varphi : \Sigma^{8N-8}A \times S^{8N+3} \to \Sigma^{16N-8}A \) induces an isomorphism between the free parts in \( \widetilde{KSp}^{-2}(\Sigma_{4N-8}A) \). Consider the following commutative diagram of exact sequences induced from the cofibre sequence
\[
S^{16N-2} \to \Sigma^{16N-5}A \to S^{16N+2}.
\]
Put \( u' = \beta_{c}^{8N-2} \wedge c'(i) \) and \( v' \) to be the complexification of the composite of the pinch map \( \Sigma^{16N-3}A \to S^{16N+4} \) and a generator of \( \pi_{16N+4}(BSp(\infty)) \), where \( \beta_{c} \) is a generator of \( K^{0}(S^2)_{(2)} \). Then by (4.1), one sees that \( \widetilde{K}^{-2}(\Sigma_{16N-5}A)_{(2)} \) is generated by \( u' \) and \( v' \) such that
\[
\text{ch}(u') = \Sigma^{16N-5}t_3 - \frac{1}{6}\Sigma^{16N-5}t_7, \quad \text{ch}(v') = \Sigma^{16N-5}t_7.
\]
Put \( u = \lambda \wedge i \) and \( v \) to be the composite of the pinch map \( \Sigma^{16N-3}A \to S^{16N+4} \) and a generator of \( \pi_{16N+4}(BSp(\infty)) \), where \( \lambda \) is a generator of \( K\overline{O}(S^{16N-4})_{(2)} \). Then by the above diagram, we obtain that \( \widetilde{KSp}^{-2}(\Sigma_{16N-5}A)_{(2)} \) is a free \( \mathbb{Z}(2) \)-module generated by \( u, v \) such that
\[
\text{ch}(c'(u)) = 2\Sigma^{16N-5}t_3 - \frac{1}{3}\Sigma^{16N-5}t_7, \quad \text{ch}(c'(v)) = \Sigma^{16N-5}t_7.
\]
Summarizing, we get:
Lemma 4.1. The free part of $\overline{KS^p}^{-2}(\Sigma^{8N-8} A \times S^{8N+3})_{(2)}$ is generated by $\bar{u}$ and $\bar{v}$ such that

$$\text{ch}(c'(\bar{u})) = 2\Sigma^{8N-8} t_3 \otimes w - \frac{1}{3} \Sigma^{8N-8} t_7 \otimes w, \quad \text{ch}(c'(\bar{v})) = \Sigma^{8N-8} t_7 \otimes w.$$ 

For an integer $k$, we put $\nu_2(k) = m$ if $k = 2^m(2l - 1)$. Then in general, we have

$$\nu_2(k!) = \left\lfloor \frac{k}{2} \right\rfloor + \left\lfloor \frac{k}{2^2} \right\rfloor + \left\lfloor \frac{k}{2^3} \right\rfloor + \cdots , \quad (4.4)$$

where $\lfloor x \rfloor = \max\{n \in \mathbb{Z} \mid n \leq x\}$.

Note that $H^{16N+2}(\Sigma^{8N-8} A \times S^{8N+3})$ is a free $\mathbb{Z}_{(2)}$-module. Then it follows from the above lemma, we obtain that the image of $b_{16N+2} \circ (\Omega\pi)_* : [\Sigma^{8N-8} A \times S^{8N+3}, \Omega X_{4N}] \rightarrow H^{16N+2}(\Sigma^{8N-8} A \times S^{8N+3}; \mathbb{Z}_{(2)})$ is generated by $(8N + 1)!\Sigma^{8N-8} t_7 \otimes w$. It follows from (4.4) that $\nu_2((8N + 1)!) = 2^{m+2} - 1$ and $\nu_2(4(4N - 1)!(4N + 1)!) = 2^{m+2} - m$. Then by Lemma 2.1, (4.3) and Lemma 4.1, we get that the commutator $[\Omega\phi^3] \circ \theta, \theta$ is non-trivial. If $N < n \leq 2N$, the map $\alpha$ and $\epsilon$ factors through the inclusion $j : G_n \rightarrow G_{4N}$, and so there exists a map $\hat{\theta} : \Sigma^{8N-8} A \times S^{8N+3} \rightarrow G_n$ such that $\theta = j \circ \hat{\theta}$. Then we obtain that $[(\Omega\psi^3) \circ \theta, \theta = j \circ [\Omega\psi^3, 1_{G_n}] \circ \hat{\theta}$ is non-trivial which implies that $\mathcal{H}(G_n)$ is not commutative.

References


