Bifurcation from Classical to Quantum distinguishability

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1 Extended abstract

Physical observable is not unique target on which we can speak of "dynamics". In fact, as to be seen here, it is also possible to discuss dynamics of "informatic ability" defined through some "informatic tasks" associated with physical (evoluting) states. Such dynamics can be highly nonlinear even in cases where the state evolution is linear itself.

In the present work, we consider, as an example of such "informatic ability", distinguishability of two thermalization processes which can be defined through the following two-party game2:

(1) Alice (with heatbath) and Bob (with ancilla) agree to use two (inverse) temperature ($\beta_0, \beta_1$) as parameters of the game. (2) Choosing one of the two temperature $\beta_j$ with probability $\pi_j$ ($j = 0$ or $1$), Alice prepares heatbath whose state is equilibrium of the chosen temperature. (3) Getting information from the ancilla, which is interacting with the heatbath prepared by Alice, Bob guesses the temperature chosen by Alice. (4) If his guess is correct, Bob wins. Otherwise Alice wins. In this game, the distinguishability of two thermalization processes can be defined as the probability of Bob’s winning.

Here let us discuss what Bob can do when he has one qubit as ancilla. Let us suppose that the evolution of qubit affected by heatbath whose temperature $\beta_j$ can be described as

$$\rho_t^{(\beta_j)} = e^{-\mathcal{L}^{(\beta_j)}_t} \rho_0,$$

where

$$\mathcal{L}^{(\beta)} = -\left( \sigma_- \cdot \sigma_+ - \frac{1}{2} \{ \sigma_+ \sigma_- \}, \cdot \right) - e^{-\beta \omega} \left( \sigma_+ \cdot \sigma_- - \frac{1}{2} \{ \sigma_- \sigma_+ \}, \cdot \right).$$

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2 Our purpose here is not to propose game being fair against malicious players, but just to explain situation we focus on. Therefore we are assuming that Alice and Bob always follow the rule of the game.
3 As known well, this equation can be derived by simple model with some reasonable assumptions. Pay attention that we chose a representation so that the free Hamiltonian of the qubit is represented as $H_S = \frac{\gamma}{2} \sigma_z$. 

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An optimal success probability of the distinction between $\rho_t^{(\beta_0)}$ (with probability $\pi_0$) and $\rho_t^{(\beta_1)}$ (with probability $\pi_1$) can be obtained as

$$p(\rho_t^{(\beta_0)}, \rho_t^{(\beta_1)}) = \max_{E_0, E_1} \left[ \pi_0 \text{tr} \left( E_0 \rho_t^{(\beta_0)} \right) + \pi_1 \text{tr} \left( E_1 \rho_t^{(\beta_1)} \right) \right]$$

where $E_0$ and $E_1$ ($E_0, E_1 \geq 0$ and $E_0 + E_1 = I$) are a POVM to distinguish the two states. Therefore, time evolution of the distinguishability of the two thermal processes are given as

$$P_{\text{qubit}}(t) = \max_{\rho_0} \left[ p(\rho_t^{(\beta_0)}, \rho_t^{(\beta_1)}) \right].$$

Let us consider also a classical counterpart of the above. Since we are discussing thermalization process, the classical counterpart of the state evolution (1) would be a balance equation of a classical two level system. In fact, it is known that a physically relevant balance equation can be obtained by (1) with a restricted choice of initial states within diagonal state. Thus we can introduce the classical counterpart of (3) as

$$P_{\text{bit}}(t) = \max_{\rho_0 \in \Omega_D} \left[ p(\rho_t^{(\beta_0)}, \rho_t^{(\beta_1)}) \right].$$

where $\Omega_D$ is a set of all diagonal states.

For simplicity, let us put $\pi_0 = \pi_1 = 1/2$ in the following. By simple observation, we can see the following facts: (a) $P_{\text{bit}}(0) = P_{\text{qubit}}(0) = 1/2$, which simply means that what Bob can do at $t = 0$ is pure guess without additional hint. (b) $P_{\text{bit}}(t) \leq P_{\text{qubit}}(t)$ and $P_{\text{bit}}(t \to \infty) \to P_{\text{qubit}}(t \to \infty)$ hold as naturally expected. (c) The initial state which gives optimal distinguishability is always pure state. (d) Advantage of qubit ancilla with respect to classical two level ancilla appears at $t = t^* < \infty$ in a bifurcational way. (For given two temperature $\beta_0$ and $\beta_1$, $P_{\text{bit}}(t) = P_{\text{qubit}}(t)$ for $0 \leq t \leq t^*$ and $P_{\text{bit}}(t) < P_{\text{qubit}}(t)$ for $t^* < t$.) It might be possible to construct a new cryptographic primitive based on the above facts.

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4 Diagonal state in a representation which diagonalizes $\sigma_z$. Notice that the time evolution of diagonal part of (1) has a closed form.
5 Note that this definition automatically restricts POVM to diagonal projective measurement.