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Periodic solutions of Boussinesq equations

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Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with the boundary $\partial \Omega$ such that

$$\partial \Omega = \Gamma_1 \cup \Gamma_2, \quad \Gamma_1 \cap \Gamma_2 = \phi.$$ 

We consider the following initial boundary value problem:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\frac{1}{\rho} \nabla p + \nu \Delta u + \beta g \theta$$

$$\text{div} u = 0,$$  \hspace{1cm} x \in \Omega, t > 0, \quad (1)

$$\frac{\partial \theta}{\partial t} + (u \cdot \nabla)\theta = \chi \Delta \theta,$$

$$\begin{cases}
    u(x, t) = 0, \quad \theta(x, t) = \xi(x, t), & x \in \Gamma_1, t > 0, \\
    u(x, t) = 0, \quad \frac{\partial}{\partial n} \theta(x, t) = \eta(x, t), & x \in \Gamma_2, t > 0, \\
    u(x, 0) = a_0(x), \quad \theta(x, 0) = \tau_0(x), & x \in \Omega, 
\end{cases} \quad (2)$$

where $u = (u_1, u_2)$ is the fluid velocity, $p$ is the pressure, $\theta$ is the temperature, $u \cdot \nabla = \sum_{j=1}^{2} u_j \frac{\partial}{\partial x_j}$ denotes the outer normal derivative of $\theta$ at $x$ to $\partial \Omega$, $g(x, t)$ is the gravitational vector function, and $\rho$ (density), $\nu$ (kinematic viscosity), $\beta$ (coefficient of volume expansion), $\chi$ (thermal diffusivity) are positive constants. $\xi(x, t)$ (resp. $\eta(x, t)$) is a function defined on $\Gamma_1 \times (0, T)$ (resp. $\Gamma_2 \times (0, T)$) and $a_0(x)$ (resp. $\tau_0(x)$) is a vector (resp. scalar) function defined on $\Omega$.

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In order to state our results, we introduce some

**Function spaces**([1],[2],[3]).

$L^p(\Omega)$ and the Sobolev space $W^l_p(\Omega)$ are defined as usual. We also denote $L^p(\Omega) = L^p(\Omega) \times L^p(\Omega), \; H^l(\Omega) = W^l_2(\Omega)$. Whether the elements of the space are scalar or vector functions is understood from the contexts unless stated explicitly.

$$D_\sigma = \{\text{vector function } \varphi \in C^\infty(\Omega) \mid \text{supp} \varphi \subset \Omega, \text{div} \varphi = 0 \text{ in } \Omega\},$$

$H$ = completion of $D_\sigma$ under the $L^2(\Omega)$-norm,

$V$ = completion of $D_\sigma$ under the $H^1(\Omega)$-norm,

$D_0 = \{\text{scalar function } \varphi \in C^\infty(\Omega) \mid \varphi \equiv 0 \text{ in a neighborhood of } \Gamma_1\},$

$W$ = completion of $D_0$ under the $H^1(\Omega)$-norm,

$V', W'$ are dual space of $V, W$.

**Definition 1**

$\{u, \theta\}$ is called a weak solution of evolitional problem (1),(2) if, for some function $\theta_0$ such that

$$\theta_0 \in L^2(0,T : H^1(\Omega)), \quad \theta_0 = \xi \text{ on } \Gamma_1,$$

$\{u, \theta\}$ satisfies following conditions:

$$u \in L^2(0,T : V), \quad \theta - \theta_0 \in L^2(0,T : W),$$

$$\begin{aligned}
\frac{d}{dt}(u,v) + \nu(\nabla u, \nabla v) + ((u \cdot \nabla) u, v) - (\beta g \theta, v) &= 0, \quad \forall v \in V, \\
\frac{d}{dt}(\theta, \tau) + \chi(\nabla \theta, \nabla \tau) + ((u \cdot \nabla) \theta, \tau) - \chi(\eta, \tau)_{\Gamma_2} &= 0, \quad \forall \tau \in W,
\end{aligned}$$

(4)

where

$$(\eta, \tau)_{\Gamma_2} = \int_{\Gamma_2} \eta(x') \tau(x') d\sigma.$$
As for the smoothness of $\partial \Omega$, we suppose

**Condition (H)**

$\partial \Omega$ is of class $C^1$ and divided as follows:

$$\partial \Omega = \Gamma_1 \cup \Gamma_2, \quad \Gamma_1 \cap \Gamma_2 = \phi,$$

and the intersection $\overline{\Gamma}_1 \cap \overline{\Gamma}_2$ consists of finite points.

In [3], we showed the existence and the uniqueness of weak solution of evolutorial problem for $2 \leq n \leq 4$. For $n = 2$, we have the following result:

**Theorem A**

Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with $C^1$ boundary satisfying Condition (H). If the function $g$ is in $L^\infty(\Omega \times (0, T))$, $\xi \in C^1(\overline{\Gamma}_1 \times [0, T])$, $\eta \in L^2(\Gamma_2 \times (0, T))$, $a_0 \in H$, $\tau_0 \in L^2(\Omega)$, then there exists one and only one weak solution $\{u, \theta\}$ of (1), (2) satisfying the initial condition (3). Furthermore

$$u \in C([0, T] : H), \quad \theta \in C([0, T] : L^2(\Omega)).$$

**Definition 2**

$\{u, \theta\}$ is called a periodic weak solution of (1), (2) with period $T_0$, if $\{u, \theta\}$ is a weak solution of (1), (2) for $T = T_0$ satisfying

$$u(x, T_0) = u(x, 0), \quad \theta(x, T_0) = \theta(x, 0). \quad (5)$$

We also obtained the existence of periodic weak solutions([3]).

**Theorem B**

Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with $C^1$ boundary satisfying Condition (H). Let $g(x, t), \xi(x, t), \eta(x, t)$ be periodic with respect to $t$ with period $T_0$, satisfying $g \in L^\infty(\Omega \times (0, T_0))$, $\xi \in C^1(\overline{\Gamma}_1 \times [0, T_0])$ and $\eta \in L^2(\Gamma_2 \times (0, T_0))$.

Set $g_\infty = \|g\|_{L^\infty(\Omega \times (0, T_0))}$. If $\frac{\beta g_\infty}{\sqrt{\nu \chi}}$ is sufficiently small, then there exists a periodic weak solution of (1), (2) with period $T_0$. Furthermore

$$u \in C([0, \infty) : H), \quad \theta \in C([0, \infty) : L^2(\Omega)).$$

Now we can state our results. As for the uniqueness of periodic weak solutions, we obtained:
Theorem 1
Let \( \{u_\pi, \theta_\pi\} \) be a weak periodic solution of (1), (2) with period \( T_0 \) such that for some \( p > 2 \),
\[
\text{ess.sup}_t \{c\|u_\pi(t)\|_p + \frac{1}{4\chi}(c\|\theta_\pi(t)\|_p + c'\beta g_{\infty})^2\} < \nu,
\]
where \( c \) and \( c' \) are constants depending on \( \Omega \). If \( \{u_\pi + u, \theta_\pi + \theta\} \) is a weak periodic solution of (1), (2) with period \( T_0 \), then \( u = 0, \theta = 0 \).

Let \( g \in L^\infty(\Omega \times (0, \infty)) \), \( \xi \in C^1(\overline{\Gamma}_1 \times [0, \infty)) \) \( \eta \in L^2(\Gamma_2 \times (0, \infty)) \), \( a_0 \in H \), \( \tau_0 \in L^2(\Omega) \). Let \( T \) be any positive number. Then there exists one and only one weak solution \( \{u_T, \theta_T\} \) of (1), (2) satisfying (3). Therefore, for \( T < T' \),
\[
 u_T(t) = u_{T'}(t), \quad \theta_T(t) = \theta_{T'}(t) \quad \text{for } \forall t \in (0, T)
\]
hold, and we can omit \( T \). This solution is called a global weak solution. We obtained the asymptotic property of solutions of Boussinesq equations as follows.

Theorem 2
Let \( g, \xi, \eta \) satisfy the condition of Theorem B, \( a_0 \in H \), \( \tau_0 \in L^2(\Omega) \). Let \( \{u, \theta\} \) be a global weak solution of (1), (2) satisfying (3), \( \{u_\pi, \theta_\pi\} \) a periodic weak solution satisfying (6). Then
\[
 \lim_{t \to \infty} \{\|u(t) - u_\pi(t)\|^2 + \|\theta(t) - \theta_\pi(t)\|^2\} = 0.
\]

Remark
(i) Since \( u_\pi \in L^2(0, T : V) \cap C([0, T] : H) \), \( u_\pi \) belongs to the space \( L^{2p/(p-2)}(0, T : L^p(\Omega)) \) for \( \forall p > 2 \). Similarly \( \theta_\pi \) is in \( L^{2p/(p-2)}(0, T : L^p(\Omega)) \). The condition (6) is stronger than this one.
(ii) When (6) holds, such periodic solution is unique (Theorem 1).

References
