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Graded algebras associated with indecomposable vector bundles over an elliptic curve

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§1. Introduction

Let $X$ be an elliptic curve over an algebraically closed field $k$ with $\text{char}(k) \neq 2$. Our object is to compute the graded algebra

$$
\bigoplus_{i \geq 0} \text{Hom}(\mathcal{E}, \mathcal{E} \otimes \mathcal{L}^{\phi i})
$$

for a line bundle $\mathcal{L}$ and a vector bundle $\mathcal{E}$ over $X$ defined as follows. Choose a point $P \in X$ and let $\mathcal{L} = \mathcal{L}(P)$ be the line bundle associated to the divisor $P$. Vector bundles over $X$ were classified by Atiyah [1]. Among them we choose the following ones. For each positive integer $n$ there exists uniquely an indecomposable vector bundle $\mathcal{E}_n$ of rank $n$ which is a successive extension of the trivial bundle. That is,

$$
\mathcal{O}_X = \mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots
$$

$$
0 \rightarrow \mathcal{E}_{n-1} \hookrightarrow \mathcal{E}_n \rightarrow \mathcal{O}_X \rightarrow 0
$$

exact, non split.

Now put

$$
\Lambda(n) = \bigoplus_{i \geq 0} \Gamma(X, \mathcal{E}nd(\mathcal{E}_n) \otimes \mathcal{L}^{\phi i}) = \bigoplus_{i \geq 0} \text{Hom}(\mathcal{E}_n, \mathcal{E}_n \otimes \mathcal{L}^{\phi i}).
$$

We aim to give an explicit description of the algebra $\Lambda(n)$.

§2. Homogeneous coordinate ring

First of all, we look at the algebra

$$
S = \bigoplus_{i \geq 0} \Gamma(X, \mathcal{L}^{\phi i}).
$$

We know the following presentation of $S$ [2, p. 336].

 generators: $t \in S_1$, $z \in S_2$, $y \in S_3$

 relation: $y^2 = z(z - t^2)(z - \lambda t^2)$ with $\lambda \in k - \{0,1\}$. 
Also we have $S_0 = k$, $\dim S_i = i$ for $i > 0$ and a $k$-basis of $S$ is given by $t^i z^j$, $t^i z^j y$ for $i, j \geq 0$. In addition, $X$ is determined by $\lambda$ as

$$X \cong \{z_1^2 z_2 = z_0 (z_0 - z_2)(z_0 - \lambda z_1)\} \subset \mathbb{P}^2$$

$$P \leftrightarrow (0:1:0)$$

We fix $t$, $x$, $y$, $\lambda$ throughout.

§3. First properties of $\Lambda(n)$

We collect here some properties of $\Lambda(n)$ which are easily proved.

- The functor

$$\Gamma_* : \text{quasi-coherent } \mathcal{O}_X - \text{mod} \rightarrow \text{graded } S - \text{mod}$$

$$\mathcal{F} \mapsto \bigoplus_{i \in \mathbb{Z}} \Gamma(X, \mathcal{F} \otimes \mathcal{L}^\otimes i)$$

is fully faithful, because $\mathcal{L}$ is ample. Hence we have an $S$-algebra isomorphism

$$\Lambda(n) \cong \text{End}_S(\Gamma_*(\mathcal{E}_n)).$$

We shall describe the $S$-module $\Gamma_*(\mathcal{E}_n)$ in §6.

- $\Lambda(n)$ is a maximal order in $\Lambda(n) \otimes_S \text{Frac}(S) \cong M_n(\text{Frac}(S)).$

- The degree 0 part $\Lambda(n)_0 = \text{End}(\mathcal{E}_n)$ is generated by a single endomorphism $f$ defined by

$$f : \mathcal{E}_n \rightarrow \mathcal{E}_n / \mathcal{E}_1 \cong \mathcal{E}_{n-1} \hookrightarrow \mathcal{E}_n.$$  

We have $f^n = 0$ and $\dim \Lambda(n)_0 = n$. We shall construct $f$ explicitly in §7.

- The degree $i$ part $\Lambda(n)_i$ has dimension $n^2 i$ for $i > 0$.

§4. $\Lambda$ as an $R$-algebra

Write $\Lambda = \Lambda(n)$. Put $R = k[t,x]$, a polynomial subalgebra of $S$. Then $S = R \oplus R y$. $\Lambda$ is an $R$-free module of rank $2n^2$. We shall give an $R$-basis of $\Lambda$.

There exist $g \in \Lambda_1$, $h \in \Lambda_2$, $l \in \Lambda_3$ such that the following diagrams commute.

$$\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\theta} & \mathcal{E} \otimes \mathcal{L} \\
\uparrow & & \downarrow \\
\mathcal{O} & \xrightarrow{t} & \mathcal{L}
\end{array}$$
$h$ \[ \mathcal{E} \otimes \mathcal{L}^{\Phi 2} \]

$\uparrow$

$\downarrow$

$L^{\Phi 2}$

$\mathcal{O}$

$\mathcal{L}^{\Phi 2}$

Here the left vertical arrows are the inclusion map and the right ones are induced by the surjection $\mathcal{E} \rightarrow \mathcal{O}$. An explicit form of $g$ will be given in §7. Then the following monomials form an $R$-basis of $A$.

$$f^{i}, f^{i}g^{j}, f^{i}h^{j}, f^{i}l \quad 0 \leq i \leq n-1, \quad 0 \leq j \leq n-2.$$  

The quotient $\tilde{A} = A/R_{+}A = A/(t, x)A$ is a symmetric graded $k$-algebra of dimension $2n^2$. We have the following isomorphisms of bimodules over $\tilde{A}_{0} = A_{0}$.

$$\tilde{A}_{1} \cong \tilde{A}_{2} \cong \text{Ker}(\tilde{A}_{0} \otimes \tilde{A}_{0} \xrightarrow{\text{mult}} \tilde{A}_{0})$$

$$\tilde{A}_{3} \cong \tilde{A}_{0}$$

$$\tilde{A}_{i} = 0 \quad i > 3.$$  

§5. $A$ as a $k$-algebra

Let $n > 2$. Regard $A$ as a left $A_{0} \otimes A_{0}$-module by $(a \otimes b) \cdot c = abc$.

**Proposition.** $A_{+} = A_{1} \oplus A_{2} \oplus \cdots$ is a free $A_{0} \otimes A_{0}$-module with basis

$$(gf^{n-1})^{i}g, \quad (gf^{n-1})^{i}(gf^{n-2})^{j}gf^{n-3}g \quad \text{for } i, j \geq 0.$$  

**Theorem.** The $k$-algebra $A$ is generated by $f$ and $g$. The relations between them are generated by the following ones.

**Case** $n$ is even: $f^{n} = 0$ and $n - 2$ quadratic relations of the form

$$gf^{k}g = A_{k} \cdot gf^{n-2}g + B_{k} \cdot gf^{n-1}g$$

with $A_{k}, B_{k} \in A_{0} \otimes A_{0}$ for $0 \leq k \leq n - 2, k \neq n - 3$.

**Case** $n$ is odd: $f^{n} = 0$ and $n - 2$ quadratic relations as above and one cubic relation of the form

$$gf^{n-3}gf^{n-3}g = C \cdot gf^{n-2}gf^{n-3}g + D \cdot gf^{n-1}gf^{n-3}g + E \cdot gf^{n-1}gf^{n-1}g$$

with $C, D, E \in A_{0} \otimes A_{0}$. 

§6. S-module $\Gamma_*(E_n)$

Put $v = x - (\lambda + 1)t^2$, $u = (x - t^2)(x - \lambda t^2)$. Define a graded $S$-module $M$ as follows. $M$ is $R$-free with basis $\alpha, \beta_i, \gamma_i$ for $i > 0$ with $\deg \alpha = 0$, $\deg \beta_i = 1$, $\deg \gamma_i = 2$. The action of $y$ on $M$ is given by

$$y\alpha = x\beta_1 + t\gamma_1$$
$$y\beta_i = -\lambda t^2\beta_{i-1} - tx\beta_{i+1} + v\gamma_{i-1} - t^2\gamma_{i+1}$$
$$y\gamma_i = x^2\beta_{i+1} + \lambda t^2\gamma_{i-1} + tx\gamma_{i+1}$$

where $\beta_0 = -t\alpha$, $\gamma_0 = x\alpha$ and $O_i = 1$ for an odd $i$, $O_i = 0$ for an even $i$, $E_i = 1 - O_i$.

For $n \geq 1$ define a graded $S$-submodule $M(n)$ of $M$ to be the free $R$-submodule generated by $\alpha, \beta_i, \gamma_i$ for $1 \leq i \leq n - 1$ and $x\beta_n + t\gamma_n$.

**Proposition.** $\Gamma_*(E_n) \cong M(n)$ as graded $S$-modules.

So we may identify $\Lambda(n) = \text{End}_S(M(n))$.

Though the $S$-module $M$ is not free, the $S[\frac{1}{y}]$-module $M[\frac{1}{y}] = S[\frac{1}{y}] \otimes_S M$ is free with basis $\alpha_i$, $i \geq 0$, given by

$$\alpha_i = \begin{cases} \frac{1}{x}\gamma_i & \text{if } i \text{ is odd} \\ -\frac{1}{u}((\lambda+1)v + \lambda t^2)\alpha_i & \text{if } i \text{ is even} \end{cases}$$

§7. Generators

Let us construct $f, g \in \Lambda$ as endomorphisms of the $S$-module $M(n)$. Define an $S[\frac{1}{y}]$-linear map $f: M[\frac{1}{y}] \rightarrow M[\frac{1}{y}]$ by

$$f(\alpha_i) = \alpha_{i-1} - \frac{\lambda t^2 y}{ux}\alpha_{i-2} + \frac{((\lambda + 1)v + \lambda t^2)x}{u}\alpha_{i-3}$$
$$- \frac{\lambda ty}{u}\alpha_{i-4} + \frac{\lambda vx}{u}\alpha_{i-5}$$

if $i$ is even

$$f(\alpha_i) = \alpha_{i-1} + \frac{\lambda t^2 y}{ux}\alpha_{i-2}$$
$$+ \frac{(\lambda + 1)x - \lambda t^2}{z}\alpha_{i-3} + \frac{\lambda ty}{u}\alpha_{i-4}$$

if $i$ is odd
where we understand $\alpha_i = 0$ for $i < 0$. Then

\[
\begin{align*}
f(\alpha) &= 0 \\
f(\beta_i) &= \beta_{i-1} + (\lambda + 1)\beta_{i-2} \quad i: \text{even} \\
&= \beta_{i-1} + (\lambda + 1)\beta_{i-2} + \lambda \beta_{i-5} \quad i: \text{odd} \\
f(\gamma_i) &= \gamma_{i-1} + (\lambda + 1)\gamma_{i-2} + \lambda \gamma_{i-5} - \lambda t \beta_{i-3} \quad i: \text{even} \\
&= \gamma_{i-1} + (\lambda + 1)\gamma_{i-2} + \lambda t \beta_{i-3} \quad i: \text{odd}
\end{align*}
\]

So $M$ and $M(n)$ are stable under $f$. We denote also by $f$ the restrictions of $f$ to $M$ and $M(n)$. Thus $f \in \Lambda(n)_0$ for all $n$.

Secondly, define an $S[\frac{1}{y}]$-linear map $g: M[\frac{1}{y}] \to M(n)[\frac{1}{y}]$ as follows. When $n$ is even,

\[
\begin{align*}
g(\alpha_0) &= t \alpha_{n-1} - \frac{y}{x} \alpha_{n-2} \\
g(\alpha_1) &= \frac{y}{x} \alpha_{n-1} + \frac{t((\lambda + 1)x - \lambda t^2)}{x} \alpha_{n-2} + \frac{\lambda t^2 y}{u} \alpha_{n-3} \\
g(\alpha_2) &= -\frac{\lambda t^2 y}{u} \alpha_{n-2} + \frac{\lambda t^2 u}{u} \alpha_{n-3} \\
g(\alpha_i) &= 0 \quad \text{for } i > 2,
\end{align*}
\]

and when $n$ is odd,

\[
\begin{align*}
g(\alpha_0) &= t \alpha_{n-1} - \frac{vy}{u} \alpha_{n-2} \\
g(\alpha_1) &= \frac{y}{x} \alpha_{n-1} + (\lambda + 1) t \alpha_{n-2} \\
g(\alpha_2) &= -\frac{\lambda t^2 y}{u} \alpha_{n-2} + \sum_{i \geq 3, \text{odd}} \lambda (-\lambda - 1)^{(i-3)/2} (t \alpha_{n-i} - \frac{vy}{u} \alpha_{n-i-1}) \\
g(\alpha_i) &= 0 \quad \text{for } i > 2.
\end{align*}
\]

Then it turns out that $g$ maps $M$ into $M(n)$. Its restriction $M(n) \to M(n)$ is denoted by $g$ again. $g$ increases degree by 1, so $g \in \Lambda_1$. These $f$, $g$ are the desired generators.
§8. Explicit equations in case $n$ even

When $n$ is even, we can give explicit defining equations for $A$, using additional generators. We define $e \in A_0$ and $g_+ \in A_1$ by

\[
e(\alpha_i) = \alpha_{i-2} \quad \text{for all } i
\]
\[
g_+(\alpha_0) = t\alpha_{n-1} - \frac{vy}{u}\alpha_{n-3}
\]
\[
g_+(\alpha_1) = t\alpha_{n-1} + (\lambda + 1)\alpha_{n-3}
\]
\[
g_+(\alpha_2) = \frac{vy}{u}\alpha_{n-1} + (\lambda + 1)\alpha_{n-2}
\]
\[
g_+(\alpha_i) = 0 \quad \text{for } i > 2.
\]

**Theorem.** If $n$ is even and $n > 2$, the $k$-algebra $A$ has the following presentation. The generators are $f$, $e$, $g$, $g_+$. The relations are

\[
e^3 = 0
\]
\[
f^2 = (1 + (\lambda + 1)e)(1 + \lambda e)(1 + e)e
\]
\[
f g(1 + (\lambda + 1)e) + (1 + (\lambda + 1)e)g f
\]
\[
= g_+ + (\lambda + 1)eg_+ + (\lambda + 1)g e + \lambda e^2 g + ((\lambda + 1)^2 + \lambda)e g_+ e + \lambda g_+ e^2
\]
\[
+ \lambda(\lambda + 1)e^2 g_+ e + \lambda(\lambda + 1)eg_+ e^2
\]
\[
ge^{\frac{n-4}{2}} g = \lambda g_+ e^{\frac{n-2}{2}} g_+
\]
\[
g_+ e^{\frac{n-4}{2}} g_+ = (\lambda + 1)g_+ e^{\frac{n-2}{2}} g_+
\]
\[
ge^j g = ge^j g_+ = 0 \quad \text{for } 0 \leq j \leq \frac{n-8}{2}.
\]

Finally we give another presentation of $A$ in line with the theorem of §5. Put

\[
c = e \otimes 1, d = 1 \otimes e, p = f \otimes 1, q = 1 \otimes f \in A_0 \otimes A_0
\]

and

\[
\alpha = (1 + (\lambda + 1)c)(1 + (\lambda + 1)d) - \lambda^2 c^2 d^2
\]
\[
\gamma = (\lambda + 1)(1 + \lambda c)(1 + c)(1 + \lambda d)(1 + d)
\]
\[
+ \lambda d(1 + \lambda c)(1 + c) + \lambda c(1 + \lambda d)(1 + d)
\]
\[
\beta = (1 + \lambda cd)\alpha - (\lambda + 1)cd\gamma
\]
\[
= 1 + (\lambda + 1)(c + d) + \lambda cd - (\lambda + 1)^2(c^2 d + cd^2)
\]
\[
- ((\lambda + 1)^4 + (\lambda + 1)^2 + \lambda^2)c^2 d^2 - \lambda(\lambda + 1)^2(c^3 d + cd^3)
\]
\[
- \lambda(\lambda + 1)((\lambda + 1)^2 + \lambda)(c^3 d^2 + c^3 d^2) - \lambda^2((\lambda + 1)^3 + \lambda)c^3 d^3.
\]

Then $\alpha, \beta, \gamma \in A_0 \otimes A_0$ and $\beta$ is invertible.
THEOREM. If $n$ is even and $n > 2$, the $k$-algebra $\Lambda$ has the following presentation. The generators are $f$, $e$, $g$. The relations are

\[ e^\frac{n}{2} = 0 \]

\[ f^2 = (1 + (\lambda + 1)e)(1 + \lambda e)(1 + e)e \]

\[ ge^{\frac{n}{2}} = (\square_1 p + \square_2 q)ge^{\frac{n-2}{2}} fg + (\square_3 p + \square_4 q)ge^{\frac{n-2}{2}} fg \]

\[ \square_1 = -\frac{1}{\beta}(1 + \lambda d)(1 + d)(1 + \lambda + 1)d + \lambda cd) \]

\[ \square_3 = \frac{1}{\beta}(1 + \lambda d)(1 + d)[(\lambda + 1)(1 + \lambda + 1)d) \]
\[ + (\lambda + 1 + \frac{\lambda c}{(1 + \lambda c)(1 + c)}(1 + \lambda + 1)d + \lambda cd)] \]

\[ \square_1 \leftrightarrow \square_2, \quad \square_3 \leftrightarrow \square_4 \quad \text{by interchange } c \leftrightarrow d \]

\[ ge^{\frac{n-4}{2}} = (\square_3 p + \square_4 q)ge^{\frac{n-2}{2}} fg + (\square_3 + \square_4 pq)ge^{\frac{n-2}{2}} fg \]

\[ \square_1 = -\frac{1}{\beta}d(1 + (\lambda + 1)d)(1 + \lambda + 1)c + \lambda cd) \]

\[ \square_3 = \frac{1}{\beta}(1 + \lambda + 1)d)(1 + \lambda + 1)c + \lambda cd) \]
\[ + \frac{1 + \lambda c}{(1 + \lambda c)(1 + c)}(1 + \lambda + 1)d + \lambda cd)] \]

\[ \square_1 \leftrightarrow \square_2, \quad \square_3 \leftrightarrow \square_4 \quad \text{by interchange } c \leftrightarrow d \]

\[ ge^{\frac{n-k}{2}} = 0 \quad \text{for } k > 4, \text{ even} \]

\[ ge^{\frac{n+k}{2}} fg = (\square_1 + \square_2 pq)ge^{\frac{n-4}{2}} fg + (\square_3 + \square_4 pq)ge^{\frac{n-2}{2}} fg \]

\[ \square_1 = \frac{1}{\beta}((\lambda + 1)\beta - \lambda c) \]

\[ \square_2 = -\frac{1}{\beta}\lambda(1 + \lambda cd) \]

\[ \square_3 = \frac{1}{\beta}[\lambda(1 + \lambda cd)(1 + (\lambda + 1)c)(1 + (\lambda + 1)d) \]
\[ - (\lambda + 1)^2\beta + \lambda(\lambda + 1)\gamma cd] \]

\[ \square_4 = \frac{1}{\beta}(\frac{\lambda \gamma}{(1 + \lambda cd)(1 + c)(1 + \lambda d)(1 + d)} + \lambda(\lambda + 1)(1 + \lambda cd)) \]

\[ ge^{\frac{n-k}{2}} fg = (\square_1 + \square_2 pq)ge^{\frac{n-4}{2}} fg + (\square_3 + \square_4 pq)ge^{\frac{n-2}{2}} fg \]
\[ \square_1 = \frac{1}{\beta}(1 + (\lambda + 1)c)(1 + (\lambda + 1)d) \times (1 - (\lambda + 1)^2cd - \lambda(\lambda + 1)(c^2d + cd^2) - \lambda^2c^2d^2) \]
\[ \square_2 = -\frac{1}{\beta}\lambda(\lambda + 1)cd \]
\[ \square_3 = -\frac{1}{\beta}(\lambda + 1)(1 + (\lambda + 1)c)(1 + (\lambda + 1)d) \times (1 - ((\lambda + 1)^2 + \lambda)cd - \lambda(\lambda + 1)(c^2d + cd^2) - \lambda^2c^2d^2)) \]
\[ \square_4 = \frac{1}{\beta}\frac{\lambda \alpha}{(1 + \lambda c)(1 + c)(1 + \lambda d)(1 + d)} + \lambda(\lambda + 1)^2cd \]

\[ ge^{\frac{b}{2}}fg = 0 \quad \text{for } k > 8, \text{ even.} \]

References