On a Local Energy Decay of Solutions of a Dissipative Wave Equation

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§1. Introduction.

This study is concerned with a local energy decay property of solutions to the following initial boundary value problem of the dissipative wave equation:

$$
(D) \begin{cases}
  u_{tt} + u_t - \Delta u = 0 & \text{in } \Omega \text{ and } t > 0, \\
  u = 0 & \text{on } \Gamma \text{ and } t > 0, \\
  u(0, x) = u_0(x), \ u_t(0, x) = u_1(x) & \text{in } \Omega,
\end{cases}
$$

where $\Omega$ is an exterior domain in an $n$-dimensional Euclidean space $\mathbb{R}^n$, whose boundary $\Gamma$ is a $C^\infty$ and compact hypersurface. Below, $r_0 > 0$ is a fixed constant such that $\Omega^c \subset B_{r_0} = \{x \in \mathbb{R}^n \mid |x| < R\}$. ($\Omega^c$ is the complement of $\Omega$.)

In the wave equation case, the local energy decays exponentially fast if $n$ is odd and polynomially fast if $n$ is even, when $\Omega$ is at least non-trapping (cf. [9], [10], [11], [16]). In fact, from a physical point of view the energy propagates along the wave fronts, so that the motion stops after time passes unless the wave front is trapped in a bounded set.

In the dissipative wave case, the energy also propagates along the wave front. Moreover, the trapped energy also decreases in virtue of the dissipative term $u_t$, so that we can expect to get the local energy decay result for any domains. In fact, in 1983 Shibata [14] proved the following theorem.

**Theorem 1.1.** Assume that $n \geq 3$. Let $R > r_0$ and let $u(t, x)$ be a smooth solution of (D) such that suppu(0, x), suppu(0, x) $\subset \Omega_R = \{x \in \Omega \mid |x| < R\}$. Then, there exists
a constant $C > 0$ depending on $n$ and $R$ such that

$$
\int_{\Omega_R} \left\{ |u_t(t, x)|^2 + \sum_{|\alpha| \leq 1} |\partial_x u(t, x)|^2 \right\} dx \\
\leq C(1 + t)^{-n} \left\{ \sum_{|\alpha| \leq 3} \int_{\Omega} |\partial^\alpha_x u(0, x)|^2 dx + \sum_{|\alpha| \leq 4} \int_{\Omega} |\partial^\alpha_x u_0(x)|^2 dx \right\},
$$

where $\partial^\alpha_x v = \partial^{\alpha_1} v / \partial x_{\alpha_1} \cdots \partial^{\alpha_n} x_{\alpha_n}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

The purpose of this study is to show the decay rate of the local energy of even the weak solutions of $(D)$ is also $n/2$ when $n \geq 2$, that is, we shall prove the following theorem.

**Theorem 1.2.** Assume that $n \geq 2$. Let $R > r_0$ and $u_0 \in H^1_{0,R}(\Omega)$ and $u_1 \in L^2_R(\Omega)$, where

$$
L^2_R(\Omega) = \{ f \in L^2(\Omega) \mid \text{suppf} \subset \Omega_R \}, \\
H^1_{0,R}(\Omega) = \{ f \in H^1(\Omega) \mid \text{suppf} \subset \Omega_R, \text{f = 0 on } \Gamma \}.
$$

Let $u(t, x)$ be a weak solution of $(D)$. Then, there exists a constant $C$ depending on $n$ and $R$ such that

$$
\int_{\Omega_R} \left\{ |u_t(t, x)|^2 + \sum_{|\alpha| \leq 1} |\partial^\alpha_x u(t, x)|^2 \right\} dx \\
\leq C(1 + t)^{-n} \left\{ \int_{\Omega} |u_1(x)|^2 dx + \sum_{|\alpha| \leq 1} \int_{\Omega} |\partial^\alpha_x u_0(x)|^2 dx \right\}.
$$

Compared with Theorem 1.1, in Theorem 1.2 we remove the smoothness assumption on solutions of $(D)$ and we consider the case that $n = 2$ as well as the case that $n \geq 3$.

For the Cauchy problem of the dissipative wave equation (i.e. $\Omega = \mathbb{R}^n$), A. Matsumura [8] studied the decay rate of solutions in 1976. His argument was based on the concrete representation of solutions by using the Fourier transform. When $\Omega$ is bounded, it is well-known that the energy of solutions decays exponentially fast. In fact, this fact is easily proved by the multiplications of the equation with $u_t$ and $u$.
and by use of Poincaré's inequality. Since $\Omega$ is unbounded in our case, we cannot use Poincaré's inequality. And also, because of the boundary, we can not use the Fourier transform. Our method is based on a spectral analysis to the corresponding stationary problem.

§2. A construction of $C_0$ semigroup solving $(D)$.

Putting $u_t = v$, let us rewrite the problem (D) in the following form:

$$\frac{d}{dt} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \Delta & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = A \begin{bmatrix} u \\ v \end{bmatrix}.$$ 

To consider $A$ to be dissipative, we introduce a space $H_D(\Omega)$. For any open set $\mathcal{O} \subset \mathbb{R}^n$, $C_0^\infty(\Omega)$ denotes the space of all $C^\infty$ functions on $\mathbb{R}^n$ whose support is compact and lies in $\mathcal{O}$ (in particular, such functions vanish near the boundary of $\mathcal{O}$), $L^2(\mathcal{O})$ a usual $L^2$ space on $\mathcal{O}$ with norm $\| \cdot \|_\mathcal{O}$ inner product $(\ , \ )_\mathcal{O}$ and $H^s(\mathcal{O})$ a usual Sobolev space of order $s$ on $\mathcal{O}$ with norm $\| \cdot \|_{s,\mathcal{O}}$. $\| \cdot \|_{k,\Omega}$ will be denoted simply by $\| \cdot \|_k$. Likewise for $\| \cdot \|_\mathcal{O}$ and $(\ , \ )_\mathcal{O}$. Then, we put

$$H_D(\Omega) = \{ u \in H^1_{loc}(\Omega) \mid \nabla u = (\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}) \in L^2(\Omega), \ u = 0 \text{ on } \Gamma, \ \exists \{ u_n \} \subset C_0^\infty(\Omega) \text{ s.t. } \| \nabla(u_n - u) \| \to 0 \text{ as } n \to \infty \},$$

where $H^1_{loc}(\Omega) = \{ u \in \mathcal{D}'(\Omega) \mid u \in H^1(\Omega_R) \ \forall R > r_0 \}$. $H_D(\Omega)$ has the following properties.

**Theorem 2.1.** If $u \in H_D(\Omega)$, then $u$ satisfies the following inequalities:

$$\| u \|_{0,\Omega_R} \leq C(R) \| \nabla u \|_{0,\Omega_R},$$

$$\int_{\Omega} \frac{|u(x)|^2}{d(x)^2} \, dx \leq C \| \nabla u \|^2.$$ 

Moreover, $H_D(\Omega)$ is a Hilbert space equipped with an inner product $(u, v)_D = (\nabla u, \nabla v)$. 

Then, an underlying space for $A$ is
\[
\mathcal{H} = \left\{ \begin{bmatrix} u \\ v \end{bmatrix} \mid u \in H_D(\Omega), v \in L^2(\Omega) \right\}.
\]

From Theorem 2.1 we know that $\mathcal{H}$ is a Hilbert space equipped with the inner product
\[
\left( \begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} w \\ z \end{bmatrix} \right)_{\mathcal{H}} = (u, w)_D + (v, z).
\]

The domain of $A$ is
\[
D(A) = \left\{ \begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{H} \mid A \begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{H} \right\}
\]
\[
= \left\{ \begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{H} \mid v \in H_D(\Omega), \Delta u \in L^2(\Omega) \right\}.
\]

Then, $A$ has the following properties.

**Proposition 2.2.** (1) $A$ is a closed operator. (2) $A$ is a dissipative operator.

(3) $\mathcal{R}(I - A) = \mathcal{H}$. (4) $D(A)$ is dense in $\mathcal{H}$.

Lumer and Phillips theorem [13, Chapter 1, Theorem 4.3] implies that $A$ generates a $C^0$ semigroup $\{T(t)\}$ on $\mathcal{H}$.

§3. A proof of Theorem 1.2.

Our purpose in this section is to prove the following result, which implies our main theorem.

**Theorem 3.1.**
\[
\|\phi_R T(t)x\|_{\mathcal{H}} \leq C(1 + t)^{-n/2}\|x\|_{\mathcal{H}},
\]
for $x \in \mathcal{H}_{1,R}$, where $C = C(R)$.

**Sketch of proof.**
Since $A$ is dissipative, $T(t)$ is a $C_0$ semigroup of contractions, so that

\begin{equation}
\|T(t)\| \leq 1 \quad \forall t \geq 0.
\end{equation}

Let $\alpha$ be a positive number. In view of (3.1), we have the following expression:

\begin{equation}
T(t)x = \lim_{\omega \to \infty} \frac{1}{2\pi i} \int_{\alpha-i\omega}^{\alpha+i\omega} e^{\lambda t}(\lambda I - A)^{-1}x d\lambda \quad \text{for } x \in D(A^2).
\end{equation}

(cf. [12, p.295] or [13, Chapter 1, Corollary 7.5]). By a lemma due to F. Huang in [4, §1, Lemma 1] (also see [7]), we have the following lemma.

**Lemma 3.2.** For any $\alpha > 0$ and $x \in \mathcal{H}$, put

$$g(\omega) = \|(\alpha + i\omega)I - A)^{-1}x\|_{\mathcal{H}}.$$  

Then $g(\omega) \in L^2(\mathbb{R})$ and

$$\lim_{|\omega| \to \infty} g(\omega) = 0,$$

$$\int_{-\infty}^{\infty} g(\omega)^2 d\omega \leq \frac{\pi}{\alpha} \|x\|^2_{\mathcal{H}}.$$  

In view of Lemma 3.2, the high frequency part decays sufficiently fast, so that we have to investigate the low frequency part. Now we shall introduce some functional spaces. Let $E$ be a Banach space with norm $|\cdot|_E$, $N \geq 0$ an integer and $k = N + \sigma$ with $0 < \sigma \leq 1$. Put

$$C^k(\mathbb{R}^1; E) = \{u \in C^{N-1}(\mathbb{R}^1; E) \cap C^\infty(\mathbb{R}^1 - \{0\}; E); \ll u \gg k,E < \infty\},$$

where

$$\ll u \gg k,E = \sum_{j=0}^{N} \int_{\mathbb{R}} |\left(\frac{d}{d\tau}\right)^j u(\tau)|_E d\tau + \sup_{h \neq 0} |h|^{-\sigma} \int_{\mathbb{R}} |\left(\frac{d}{d\tau}\right)^N u(\tau + h) - \left(\frac{d}{d\tau}\right)^N u(\tau)|_E d\tau$$

if $0 < \sigma < 1$,
\[ + \sup_{h \neq 0} |h|^{-1} \int_{\mathbb{R}} \left| (\frac{d}{d\tau})^N u(\tau + 2h) - 2(\frac{d}{d\tau})^N u(\tau + h) + (\frac{d}{d\tau})^N u(\tau) \right|_E d\tau, \]

if \( \sigma = 1 \). Here, \( (\frac{d}{d\tau})^0 = 1 \). The following lemma is concerned with the properties of the Fourier transformation of functions belonging to \( C^k(\mathbb{R}^1, E) \), which was proved in [14, Part 1, Theorem 3.7].

**Lemma 3.3.** Let \( E \) be a Banach space with norm \( | \cdot |_E \). Let \( N \geq 0 \) be an integer and \( \sigma \) a positive number \( \leq 1 \). Assume that \( f \in C^{N+\sigma}(\mathbb{R}^1; E) \). Put

\[ F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\tau) \exp(\sqrt{-1}\tau t) d\tau. \]

Then,

\[ |F(t)|_E \leq C(1 + |t|)^{-(N+\sigma)} \ll f \gg N+\sigma, E. \]

Here and hereafter, we put \( \mathcal{H}_R = \{ \left[ \begin{array}{l} u \\ v \end{array} \right] \in \mathcal{H} \mid \text{supp } u, \text{supp } v \subset \Omega_R \} \). \( \varphi_R(x) \) always refers to a function in \( C_0^\infty(\mathbb{R}^n) \) such that \( \varphi_R(x) = 1 \) if \( |x| \leq R \) and \( = 0 \) if \( |x| \geq R+1 \). Moreover, we put

\[ \mathcal{H}_{loc} = \{ \left[ \begin{array}{l} u \\ v \end{array} \right] \mid u \in H^1(\Omega_R), v \in L^2(\Omega_R), \forall R \geq r_0 \}, \]

\[ \mathcal{H}_{comp} = \bigcup_{R \geq r_0} \mathcal{H}_R, \]

and \( \mathcal{L}(B_1, B_2) \) denotes the set of all bounded linear operators from \( B_1 \) into \( B_2 \) and \( \text{Anal}(I, B) \) the set of all \( B \)-valued analytic functions in \( I \). In view of Lemma 3.3, if we prove the following fact, the proof of Theorem 3.1 is complete.

(F) Put \( Q_d = \{ \lambda \in \mathbb{C} \mid 0 < \Re \lambda < d, |\Im \lambda| < d \} \). Then, there exists a \( d > 0 \) and \( R(\lambda) \in \text{Anal}(Q_d; \mathcal{L}(\mathcal{H}_{comp}, \mathcal{H}_{loc})) \) such that:

(a) \( R(\lambda)x = (\lambda I - A)^{-1}x \) for \( x \in \mathcal{H}_{comp} \) and \( \lambda \in Q_d \);

(b) For any \( R \geq r_0 \) and \( \rho(s) \in C_0^\infty(\mathbb{R}) \) such that \( \rho(s) = 1 \) if \( |s| < d/2 \) and \( = 0 \) if \( |s| > d \), there exist \( M_1 > 0 \) depending on \( R, \rho \) and \( d \) such that

\[ \ll \rho(\cdot)(\varphi_R R(\alpha + i\cdot)x, y)_{\mathcal{H}} \gg n/2, \Re \leq M_1 \|x\|_{\mathcal{H}}\|y\|_{\mathcal{H}}, \]
for any \(x \in \mathcal{H}_R, y \in \mathcal{H}\) and \(0 < \alpha < d\).

We shall conclude this report by giving a brief proof of (F).

**Proof of (F).**

When \(n \geq 3\), (F) was proved by Shibata [14, Part 1], so that we shall consider the case that \(n = 2\). Corresponding stationary problem is

\[(\lambda^2 + \lambda - \Delta)u = f \quad \text{and} \quad u = 0 \quad \text{on} \quad \Gamma.\]

If \(|\lambda|\) is small, then in stead of (3.3), it is sufficient to consider the following problem:

\[(A_{\lambda}) \quad (\lambda - \Delta)u = f \quad \text{in} \quad \Omega \subset \mathbb{R}^2 \quad \text{and} \quad u = 0 \quad \text{on} \quad \Gamma,\]

where \(\lambda \in S_{r,\epsilon} = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\lambda| < r, \ |\arg\lambda| < \pi - \epsilon\}, 0 < r < 1\) and \(0 < \epsilon < \pi/2\), because \(\lambda^2 + \lambda\) is equivalent to \(\lambda\) for small \(|\lambda|\). In view of Lemma 3.4 of [14], in order to prove (F), it is sufficient to prove the following propositions.

**Proposition 3.4.** For \(\lambda \in S_{r,\epsilon}\) and \(r_0 \leq R < \infty\), there exists \(A(\lambda) : L^2_R(\Omega) \rightarrow H^1_{loc}(\Omega)\) satisfying that

\[(\lambda - \Delta)A(\lambda)f = f \quad \text{in} \quad \Omega \quad \text{and} \quad A(\lambda)f = 0 \quad \text{on} \quad \Gamma,\]

for \(f \in L^2_R(\Omega)\). Moreover, it satisfies that

\[\|\varphi_R A(\lambda)f\|_1 \leq C\|f\| \quad \text{as} \quad \lambda \in S_{r,\epsilon}.\]

**Proposition 3.5.** For \(\lambda\) and \(R\) as mentioned above, following estimates hold;

\[
\|\varphi_R \frac{d}{d\lambda} A(\lambda)f\|_1 \leq \frac{C(R)}{|\lambda||\log\lambda|^2}\|f\| \leq \frac{C(R)}{|3\lambda|}\|f\|,
\]

\[
\|\varphi_R \frac{d^2}{d\lambda^2} A(\lambda)f\|_1 \leq \frac{C(R)}{|\lambda|^2|\log\lambda|^2}\|f\| \leq \frac{C(R)}{|3\lambda|^2}\|f\|,
\]
for $f \in L_{R}^{2}(\Omega)$.

Our main idea to prove Propositions 3.4 and 3.5 is to use the single layer potential and the double layer potential, and to reduce $(A_{\lambda})$ to a boundary integral equation. Put $v = (\lambda - \Delta)^{-1}f$. Then $v$ is represented by the modified Bessel function:

$$ (\lambda - \Delta)^{-1}f = \int_{\mathbb{R}^{2}} E_{\lambda}(x - y)f(y)dy, $$

where $E_{\lambda}(x) = (2\pi)^{-1}K_{0}(|x|\sqrt{\lambda})$, $K_{m}$ ($m \in \mathbb{N} \cup \{0\}$) denotes the modified Bessel function of order $m$. So we want to solve the equation

$$(A'_{\lambda}) \quad (\lambda - \Delta)w = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad w = f_{\lambda} \quad \text{on} \quad \Gamma,$$

where $f_{\lambda} = (\lambda - \Delta)^{-1}f|_{\Gamma}$. To do this, let us introduce the integral operator $B_{\lambda}:

$$ B_{\lambda} \Phi = D_{\lambda} \Phi - \eta E_{\lambda} M \Phi + \frac{2\pi \alpha}{\log \sqrt{\lambda}} E_{\lambda} \Phi \quad \text{for} \quad \Phi \in C^{0}(\Gamma). $$

Here $\alpha, \eta > 0$, $E_{\lambda}$ is a single layer potential defined by

$$ E_{\lambda} \Psi(x) = \int_{\Gamma} E_{\lambda}(x - y)\Psi(y)d\sigma_{y} $$

and $D_{\lambda}$ is a double layer potential defined by

$$ D_{\lambda} \Psi(x) = \int_{\Gamma} D_{\lambda}(x, y)\Psi(y)d\sigma_{y}, $$

where

$$ D_{\lambda}(x, y) = \nabla_{x}E_{\lambda}(x - y) \cdot N(y) $$

$$ = -\frac{1}{2\pi}K_{1}(|x - y|\sqrt{\lambda})\frac{\sqrt{\lambda}}{|x - y|}(x - y) \cdot N(y). $$

The projection $M : C^{0}(\Gamma) \rightarrow C^{0}(\Gamma)$ is defined by

$$ \Phi \rightarrow M \Phi = \Phi - \Phi_{M} \quad \text{with} \quad \Phi_{M} = \frac{1}{|\Gamma|} \int_{\Gamma} \Phi d\sigma \quad \text{and} \quad |\Gamma| = \text{meas}(\Gamma). $$
Obviously $B_{\lambda}\Phi$ satisfies that $(\lambda - \Delta)B_{\lambda}\Phi = 0$ in $\Omega$, so that we obtain the following boundary integral equation:

\begin{equation}
B_{\lambda}\Phi|_{\Gamma} = K_{\lambda}\Phi = (-\frac{1}{2} + D_{\lambda} - \eta E_{\lambda}M + \frac{2\pi \alpha}{\log \sqrt{\lambda}} E_{\lambda})\Phi = f_{\lambda}.
\end{equation}

If $\Phi$ is a solution of (3.4), $B_{\lambda}\Phi$ satisfies $(A_{\lambda}')$, and $A(\lambda)f$ is expressed by

\begin{equation}
A(\lambda)f = (\lambda - \Delta)^{-1}f - B_{\lambda}\Phi.
\end{equation}

Therefore, $(A_{\lambda})$ was reduced to a boundary integral equation (3.4). $K_{\lambda}$ is a Fredholm operator, so that by using the Fredholm alternative theorem, we can solve the boundary equation (3.4). If we consider that $A(\lambda)$ is an operator from $L_{R}^{2}(\Omega)$ to $L_{loc}^{2}(\Omega)$, by the properties of Bessel function, we know that the expansion of $A(\lambda)$ at $\lambda \to 0$ is

\[ A(\lambda) = C_{0} + C_{1}\frac{1}{\log \lambda} + \cdots. \]

Therefore, we have Propositions 3.4 and 3.5.

**References**