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The Gradient Theory of the Phase Transitions in Cahn-Hilliard Fluids with the Dirichlet boundary conditions

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1. Introduction

In this note we will investigate the asymptotic behavior of minimizer $\{u_\epsilon\}_{\epsilon>0}$ (as $\epsilon \to 0$) of the following variational problem :

$$(\boldsymbol{P_\epsilon}) \qquad \inf\left\{ \int_\Omega [\,\epsilon|\nabla u|^2 + \frac{1}{\epsilon}\, W(x,u)\,]dx \;\middle|\; u \in W^{1,2}(\Omega : \mathbb{R}^n),\ u = g \text{ on } \partial\Omega \right\},$$

where Ω is a bounded domain in $\mathbb{R}^N$ with C^2 smooth boundary $\partial\Omega$ and g is a Lipschitz continuous function from $\partial\Omega$ into $\mathbb{R}^n$. Here $W(x,\cdot)$ is a nonnegative continuous function which has two potential wells with equal depth. This type of problem is related to the study of the phase transitions of the Cahn-Hilliard fluids. See [**8**] and [**9**].

In [**7**] R.V.Kohn & P.Sternberg conjectured that minimizer of the variational problem, which is special case of $(\mathcal{P}_\epsilon)$,

$$(SP_\epsilon) \qquad \inf\left\{ \int_\Omega [\epsilon|\nabla u|^2 + \frac{1}{\epsilon}(u^2-1)^2]dx \;\middle|\; u \in W^{1,2}(\Omega),\ u|_{\partial\Omega} = g \right\}$$

converges to a solution of

$$\inf\left\{ \frac{8}{3} P_\Omega\{u=1\} + 2\int_{\partial\Omega} |d(u)-d(g)|d\mathcal{H}_{N-1} \;\middle|\; u \in BV(\Omega),\ |u| = 1 \text{ a.e.} \right\},$$

where $d(t) = \int_{-1}^t |s^2-1|ds$. Here $\mathcal{H}_{N-1}$ is the $N-1$ dimensional Hausdorff measure.

In this note, we will study the asymptotic behavior of minimizer of (P_ϵ), and as a byproduct, we will state the affirmative results to the conjecture in [**7**].

Recently using the theory of Gamma-convergence, several authors studied the asymptotic behavior of the minimizer of the problem:

$$(E_\epsilon) \qquad \inf\left\{ \int_\Omega [\,\epsilon|\nabla u|^2 + \frac{1}{\epsilon}\, W(u)\,]dx \;\middle|\; u \in W^{1,2}(\Omega : \mathbb{R}^n),\ \int_\Omega u(x)dx = m \right\},$$

where m is a constant vector in $\mathbb{R}^n$. For the scalar case (i.e. $n = 1$), see [**8**] and [**9**]. For the vector case (i.e. $n \geq 2$), see [**1**] and [**4**]. Our results on the problem (P_ϵ) depend mainly on the study of asymptotic behavior of minimizer of (E_ϵ). However there are several different aspects between the asymptotic behavior of minimizer of (P_ϵ) and that of (E_ϵ). In fact, minimizer of (E_ϵ) generates the only interior layer, but minimizer of (P_ϵ) generates both the interior and the boundary layers as $\epsilon \to 0$.

On the other hand, we can easily see that minimizer of (SP_ϵ) satisfies the equation:

$$(CP_\epsilon) \qquad \begin{cases} \epsilon^2 \Delta u - u(u-1)(u+1) = 0 & \text{in} \quad \Omega, \\ u(x) = g(x) & \text{on} \quad \partial\Omega. \end{cases}$$

Then there exist several results for the solutions of (CP_ϵ) obtained by using the method of matched expansion. Our results also seem to be closely related to [**2**] and [**3**].

We will give the precise conditions of the functions $W(x,u)$ and $g(x)$. Let $W(x,u) : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}$ be a continuous nonnegative function, and for any $x \in \overline{\Omega}$ $W(x,u) = 0$ if and only if $u = \alpha$ or β. Here we note α and β are constant vectors independent of x. We assume that there exist two constants K_1 and K_2 such that

$$(1.1) \qquad \sup_{u \in \partial[K_1,K_2]^n} W(x,u) \leq W(x,v) \qquad \text{for all } x \in \overline{\Omega},\ v \notin [K_1,K_2]^n$$

and

$$(1.2) \qquad g(x) \in [K_1,K_2]^n \qquad \text{for all } x \in \partial\Omega.$$

Moreover we set $W_\infty(\cdot) = \inf_{x \in \overline{\Omega}} W(x,\cdot)$ and assume that for any $\epsilon > 0$ there exists a positive constant δ such that

$$(1.3) \qquad |W^{1/2}(x,u) - W^{1/2}(y,u)| \leq \epsilon W_\infty^{1/2}(u)$$

for all $x, y \in \overline{\Omega}$ with $|x - y| \leq \delta$ and all $u \in \mathbb{R}^n$. Here from the definition of $W_\infty(u)$ and (1.3) we have the following relation

$$(1.3') \qquad |W^{1/2}(x,u) - W^{1/2}(y,u)| \leq \epsilon W^{1/2}(x,u)$$

for all $x, y \in \overline{\Omega}$ with $|x - y| \leq \delta$ and for all $u \in \mathbb{R}^n$.

We think that the conditions (1.1) and (1.3) are not restrictive. In fact, consider continuous functions $W(u)$, $h(x)$, where $W(u)$ satisfies the condition (1.1) and where $h(x)$ is positive function in $\overline{\Omega}$. If the function $W(x,u)$ has a form of $h(x)W(u)$, then we can see that $W(x,u)$ satisfies the conditions (1.1) and (1.3).

In order to state the main theorem, we will introduce a Riemannian metric on $\mathbb{R}^n$, $d(x,a,b)$ which depends on $x \in \overline{\Omega}$. For $x \in \overline{\Omega}$ and $a, b \in \mathbb{R}^n$, let $d(x,a,b)$ be the metric defined by

$$d(x,a,b) = \inf\left\{ \int_0^1 W^{1/2}(x,\gamma(t))|\,\dot{\gamma}(t)\,|\,dt \,\middle|\, \gamma \in C^1([0,1]:\mathbb{R}^n),\ \gamma(0)=a,\ \gamma(1)=b \right\}. \tag{1.4}$$

For example, in the case of $W(x,u) = (u^2-1)^2$ and $n = 1$, we have

$$d(x,-1,b) = \int_{-1}^{b} |s^2-1|ds \qquad \text{for} \quad b \geq -1.$$

We now state our main theorem of this note.

Theorem 1. *(See [6].) Suppose that function W satisfies (1.1) and (1.3) and that g satisfies (1.2). For $\epsilon > 0$, let u_ϵ be a solution of the variational problem:*

$$\inf\{\int_\Omega [\,\epsilon|\nabla u|^2 + \frac{1}{\epsilon}\, W(x,u)\,]dx \quad | \quad u \in W^{1,2}(\Omega:\mathbb{R}^n),\ u|_{\partial\Omega}(x) = g(x)\}.$$

If there exist a positive sequence $\{\epsilon_i\}_{i=1}^{\infty}$ and a function $u_0(x) \in L^1(\Omega:\mathbb{R}^n)$ such that

$$\lim_{i\to\infty} \epsilon_i = 0 \quad \text{and} \quad \lim_{i\to\infty} u_{\epsilon_i} = u_0 \quad \text{in } L^1(\Omega:\mathbb{R}^n), \tag{1.5}$$

then the function u_0 is characterized by

$$W(x,u_0(x)) = 0 \text{ for almost all } x \in \Omega, \text{ that is, } u_0(x) = \alpha \text{ or } \beta \text{ for almost all } x \in \Omega.$$

Moreover the set $E_0 = \{x \in \Omega \,|\, u_0(x) = \alpha\}$ is a solution of the variational problem (P_0):

$$\textbf{(}P_0\textbf{)} \qquad \inf\{\int_{\Omega\cap\partial^* E} d(x,\alpha,\beta)d\mathcal{H}_{N-1} + \int_{\partial\Omega} d(x,v|_{\partial\Omega}(x),g(x))d\mathcal{H}_{N-1} \quad | \quad E \subset \Omega,\ P_\Omega(E) < \infty,\ v = \alpha\chi_E + \beta\chi_{\Omega\setminus E}\},$$

where $P_\Omega(E)$ is a perimeter of E in Ω and $v|_{\partial\Omega}$ is the trace of v to $\partial\Omega$. Furthermore we have

$$\lim_{i\to\infty} \int_\Omega [\,\epsilon_i|\nabla u_{\epsilon_i}|^2 + \frac{1}{\epsilon_i}\, W(x,u_{\epsilon_i})\,]\,dx = 2\int_{\Omega\cap\partial^* E_0} d(x,\alpha,\beta)d\mathcal{H}_{N-1} + 2\int_{\partial\Omega\cap\partial^* E_0} d(x,\alpha,g(x))d\mathcal{H}_{N-1} + 2\int_{\partial\Omega\setminus\partial^* E_0} d(x,\beta,g(x))d\mathcal{H}_{N-1}.$$

Here $\partial^* E_0$ is the reduced boundary of E_0.

Remark. *It is not restrictive to assume that there exists a subsequence $\{u_{\epsilon_i}\}_{i=1}^{\infty}$ satisfying (1.5). In fact, the following is proved in [4] and [5]: if there exist constants C and R such that*

$$(1.6) \qquad W_\infty(u) \geq C|u| \quad \text{for} \quad |u| \geq R,$$

then there exists a subsequence $\{u_{\epsilon_i}\}_{i=1}^{\infty}$ satisfying (1.5).

It is worth noting that the study of asymptotic behavior of minimizer of (P_ϵ) occurs a completely different difficulty from that of (SP_ϵ). One of the difficulties is that the selection of minimizing sequence $\{\gamma_k\}_{k=1}^{\infty}$ achieving the value of $d(x, \alpha, g(x))$ depends on the space variable x. In order to overcome this difficulty, we approximate $W(\cdot, u)$ and $g(\cdot)$ by suitable piecewise smooth functions near the transition layer and the boundary $\partial\Omega$.

2. The Main Propositions

At first, we will give functionals F_ϵ and F_0 from $L^1(\Omega : \mathbb{R}^N)$ into $[0, \infty]$. For $u \in L^1(\Omega : \mathbb{R}^n)$ and $\epsilon > 0$, we define $F_\epsilon(u)$, $F_0(u)$ by

$$F_\epsilon(u) = \begin{cases} \displaystyle\int_\Omega [\epsilon|\nabla u|^2 + \frac{1}{\epsilon} W(x, u)]\, dx, & \text{if } u \in W^{1,2}(\Omega : \mathbb{R}^n) \text{ and } u = g \text{ on } \partial\Omega, \\ +\infty, & \text{otherwise}, \end{cases}$$

$$F_0(u) = \begin{cases} \displaystyle 2\int_\Omega d(x, \alpha, \beta)|\nabla \chi_{\{u(x)=\alpha\}}| + 2\int_{\partial\Omega} d(x, u|_{\partial\Omega}(x), g(x)) d\mathcal{H}_{N-1}, \\ \qquad \text{if } u \in BV(\Omega : \mathbb{R}^n) \text{ and } W(x, u(x)) = 0 \text{ for almost all } x \in \Omega, \\ +\infty, \qquad \text{otherwise}. \end{cases}$$

In order to prove our main theorem, we need the following two propositions which are crucial in our analysis.

Proposition A. *Suppose that $\{v_\epsilon\}_{\epsilon>0}$ is a sequence in $L^1(\Omega : \mathbb{R}^n)$ which converges in $L^1(\Omega : \mathbb{R}^n)$ as $\epsilon \to 0_+$ to a function v_0. If*

$$\liminf_{\epsilon \to 0_+} F_\epsilon(v_\epsilon) < +\infty,$$

then v_0 is a function in $BV(\Omega : \mathbb{R}^n)$ such that

$$F_0(v_0) \leq \liminf_{\epsilon \to 0_+} F_\epsilon(v_\epsilon).$$

Proposition B. *Suppose that $w_0 \in L^1(\Omega : \mathbb{R}^n)$ is a function with $w_0 = \alpha\chi_E + \beta\chi_{\Omega\setminus E}$ where E is a measurable subset in Ω with finite perimeter. Then there exists a sequence $\{w_\epsilon\}_{\epsilon>0}$ in $W^{1,2}(\Omega : \mathbb{R}^n)$ which converges in $L^1(\Omega : \mathbb{R}^n)$ as $\epsilon \to 0_+$ to w_0 such that*

$$\limsup_{\epsilon\to 0_+} F_\epsilon(w_\epsilon) \le F_0(w_0). \tag{2.1}$$

Using Propositions A and B, we can prove Theorem 1 as in the same matter with in [**8**]. Therefore we have only to prove Proposition A and B. In this note, we will only prove Proposition B for the special case.

On the other hand, in Theorem 1, the minimizers $\{u_\epsilon\}_{\epsilon>0}$ do not always generate interior layers. For example, if we consider the problem (SP_ϵ) with $g \equiv 0$, we have $E_0 = \Omega$ or $\emptyset$. In contrast, considering the family of *local* minimizers, from Theorem 1 and the results of [**7**], we obtain the following theorem.

Theorem 2. *Let $u_0 \in L^1(\Omega : \mathbb{R}^n)$ be a isolated L^1-local minimizer of F_0, that is,*

there exists a positive constant δ such that $F_0(u_0) < F_0(v)$

whenever $u \neq v$ and $\|u_0 - v\|_{L^1(\Omega : \mathbb{R}^n)} \le \delta$.

Then there exist a constant $\epsilon_0 > 0$ and a sequence $\{u_\epsilon\}_{\epsilon<\epsilon_0}$ such that u_ϵ is a local minimizer of F_ϵ and $u_\epsilon \to u_0$ in $L^1(\Omega : \mathbb{R}^n)$ as $\epsilon \to 0$.

3. Proof of Proposition B

In this section, we will only prove Proposition B for the special case that $w_0 \equiv \alpha$ in Ω. In order to prove Proposition B for the case of $w_0 \equiv \alpha$, we need the following two lemmas. The first lemma is obtained easily by the inverse mapping theorem.

Lemma 3–1. *Let Ω be a bounded domain with C^2-smooth boundary $\partial\Omega$. For $x \in \partial\Omega$ let $\nu(x)$ be a inner normal vector to $\partial\Omega$ at x. Define a mapping $\pi : \partial\Omega \times [0,\infty) \to \mathbb{R}^N$ by*

$$\pi(x,t) = \pi_t(x) = x + t\nu(x). \tag{3.1}$$

Then there exists a constant s_0 such that the image of π in $\partial\Omega \times (0, s_0]$ is contained in Ω and the C^1-smooth inverse mapping π^{-1} of π exists in $\pi(\partial\Omega \times [0, s_0])$.

Lemma 3–2. *(See [8] and [9].) Let Ω be an open bounded subset of $\mathbb{R}^N$ with Lipschitz-continuous boundary. Let A be an open subset of $\mathbb{R}^N$ with C^2, compact, nonempty* *boundary such that* $\mathcal{H}_{N-1}(\partial A \cap \partial\Omega) = 0$. *Define a distance function to* ∂A, $d_{\partial A} : \Omega \to \mathbb{R}$, *by* $d_{\partial A}(x) = dist\,(x, A)$. *Then, for some* $s_1 > 0$, $d_{\partial A}$ *is a* C^2*-function in* $\{0 < d_{\partial A}(x) < s_1\}$ *with*

$$|\nabla d_{\partial A}| = 1. \tag{3.2}$$

Furthermore, $\lim_{s\to 0} \mathcal{H}_{N-1}(\{d_{\partial A}(x) = s\}) = \mathcal{H}_{N-1}(\partial A \cap \Omega)$ *and*

$$|\{x \,|\, |d_{\partial A}(x)| < s\}| = O(s). \tag{3.3}$$

By $d_{\partial\Omega}(x)$ we denote a function $\text{dist}(x, \partial\Omega)$. From Lemma 3–2, we can see that $d_{\partial\Omega}$ is a C^2-function. We set $s^* = \min\{s_0, s_1\}$. For any $\nu \in \mathbb{S}^{N-1}$ we denote by Q_ν the open unit cube centered at the origin with two of its surfaces normal to ν. Furthermore for $x \in \partial\Omega$, $\eta > 0$, and sufficiently small δ with $0 < \delta < s^*$, we set $\partial\Omega_\eta(x) = \partial\Omega \cap (x + \eta Q_{\nu(x)})$ and $\Omega_\eta^\delta(x) = \bigcup_{\delta < t < s^*} \pi_t(\partial\Omega_\eta(x))$.

We will start to prove Proposition B for the special case $w_0 \equiv \alpha$. The proof of Proposition B for the case of $w_0 = \alpha$ requires three steps.

The First Step: Let x_0 be any point in $\partial\Omega$. In this step, for any sufficiently small $\eta > 0$ we will construct a family $\{w_\epsilon^\delta\}_{\epsilon,\delta>0} \subset W^{1,2}(\Omega_\eta^\delta(x_0) : \mathbb{R}^n)$ such that

$$\limsup_{\epsilon,\delta\to 0} \int_{\Omega_\eta^\delta} [\,\epsilon|\nabla w_\epsilon^\delta|^2 + \frac{1}{\epsilon} W(x_0, w_\epsilon^\delta)\,]dx \le 2d(x_0, \alpha, g(x_0))\mathcal{H}_{N-1}(\partial\Omega_\eta(x_0)). \tag{3.4}$$

In this step, for simplicity, we set $\Omega_\eta^\delta = \Omega_\eta^\delta(x_0)$.

In order to construct $\{w_\epsilon^\delta\}_{\epsilon,\delta>0}$, we fix $\epsilon, \delta > 0$, and consider the following ordinary differential equation:

$$\begin{cases} \dfrac{d}{dt}\, y_\epsilon(t) = \dfrac{[\epsilon^{1/2} + W(x_0, \gamma(y_\epsilon(t)))]^{1/2}}{\epsilon|\,\dot\gamma(y_\epsilon(t))\,|}, \\ y_\epsilon(\delta) = 0. \end{cases} \tag{3.5}$$

Here by $\dot\gamma$ we denote $d\gamma(t)/dt$, and assume that $\gamma \in C^1([0,1] : [K_1, K_2]^n)$, $\gamma(0) = \alpha$, $\gamma(1) = g(x_0)$. We set

$$\psi_\epsilon(t) = \int_0^t \frac{\epsilon|\dot\gamma(t)|}{[\epsilon^{1/2} + W(x_0, \gamma(t))]^{1/2}}dt$$

for $t \in (0, 1)$. Then $\psi_\epsilon(t)$ is a monotone increasing function and

$$\tau_\epsilon \equiv \psi_\epsilon(1) \le \epsilon^{3/4} \cdot \text{ length of } \gamma. \tag{3.6}$$

Here we set $\tilde{y}_\epsilon(t) = \psi_\epsilon^{-1}(t-\delta)$, and we can see that $\tilde{y}_\epsilon(t)$ satisfies (3.5) in $[\delta, \delta+\tau_\epsilon]$ and we define $y_\epsilon(t)$ by

$$y_\epsilon(t) \equiv \max\{0, \min\{1, \tilde{y}_\epsilon(t)\}\}. \tag{3.7}$$

We separate Ω_η^δ to three domains $\Omega_{\eta,i}^\delta$, $i = 1,2,3$ as follows:

$$\begin{aligned} \Omega_{\eta,1}^\delta &\equiv \{x \in \Omega_\eta^\delta : d_{\partial\Omega}(x) < \delta + \tau_\epsilon,\ d_S(x) \le \eta\tau_\epsilon\}; \\ \Omega_{\eta,2}^\delta &\equiv \{x \in \Omega_\eta^\delta : d_{\partial\Omega}(x) < \delta + \tau_\epsilon,\ d_S(x) \ge \eta\tau_\epsilon\}; \\ \Omega_{\eta,3}^\delta &\equiv \{x \in \Omega_\eta^\delta : d_{\partial\Omega}(x) \ge \delta + \tau_\epsilon\}, \end{aligned} \tag{3.8}$$

where $d_S(x)$ is a distance function to $\bigcup_{\delta<t<s^*} \pi_t[\partial\Omega \cap (x_0 + \eta\partial Q_{\nu(x_0)})]$. Here we define $w_\epsilon(x)$ on $\bigcup_{i=2,3} \Omega_{\eta,i}^\delta$ as follows:

$$w_\epsilon(x) = \begin{cases} \gamma(y_\epsilon(d_{\partial\Omega}(x))), & \text{if } x \in \Omega_{\eta,2}^\delta, \\ \alpha, & \text{if } x \in \Omega_{\eta,3}^\delta. \end{cases} \tag{3.9}$$

and extend w_ϵ to $\Omega_{\eta,1}^\delta$ such that for any $x \in \Omega_\eta^\delta$ with $d_S(x) = 0$ or $d_{\partial\Omega}(x) = \delta + \tau_\epsilon$, $w_\epsilon(x) = \alpha$ and

$$|\nabla w_\epsilon| \le 2/(K_2 - K_1)\eta\tau_\epsilon + C/\epsilon \le C(\eta\tau_\epsilon)^{-1} + C\epsilon^{-1}.$$

For sufficiently small $\epsilon > 0$, we have the length of $\gamma < \epsilon^{-1/8}$ and $\tau_\epsilon \le \epsilon^{5/8}$. Therefore we obtain

$$\begin{aligned} \int_{\Omega_{\eta,1}^\delta} [\epsilon|\nabla w_\epsilon|^2 + \frac{1}{\epsilon} W(x_0, w_\epsilon)]dx &\le C[\epsilon/\eta^2\tau_\epsilon^2 + 1/\epsilon]\tau_\epsilon^N \mathcal{H}_{N-1}(\partial\Omega_\eta) \\ &\le C(\epsilon/\eta^2 + \epsilon^{1/4})\tau_\epsilon^{N-2}\mathcal{H}_{N-1}(\partial\Omega_\eta). \end{aligned} \tag{3.10}$$

Here we note that constants C are independent of ϵ and η. On the other hand, for sufficiently small $\delta > 0$ and $\epsilon > 0$ we have $\delta + \tau_\epsilon < s^* \equiv \min\{s_0, s_1\}$ and obtain from Lemma 3–2 and (3.9)

$$\int_{\bigcup_{i=2,3} \Omega_{\eta,i}^\delta} [\epsilon|\nabla w_\epsilon|^2 + \frac{1}{\epsilon} W(x_0, w_\epsilon)]dx \le \int_{\Omega_{\eta,2}^\delta} \frac{2}{\epsilon}[\epsilon^{1/2} + W(x_0, \gamma(y_\epsilon(d_{\partial\Omega}(x))))]|\nabla d_{\partial\Omega}(x)|dx,$$

and from the co-area formula in BV functions, we get

$$\leq 2\int_{\delta}^{\tau_\epsilon+\delta} dt \int_{\Omega_\eta^\delta\cap\{d_{\partial\Omega}(x)=t\}} \epsilon^{-1}[\epsilon^{1/2}+W(x_0,\gamma(y_\epsilon(t)))]d\mathcal{H}_{N-1}$$

$$\leq 2\kappa_\epsilon^\delta \int_{\delta}^{\tau_\epsilon+\delta} \epsilon^{-1}(\epsilon^{1/2}+W(x_0,\gamma(y_\epsilon(t))))dt,$$

where $\kappa_\epsilon^\delta = \sup_{\delta\leq d_S(x)\leq\delta+\epsilon} (\Omega_\eta^\delta \cap \pi_t(\partial\Omega))$. Then from (3.5) we obtain

$$\int_{\underset{i=1,2}{\cup}\Omega_{\eta,i}^\delta} [\epsilon|\nabla w_\epsilon|^2+\frac{1}{\epsilon}W(x_0,w_\epsilon)]dx \leq 2\kappa_\epsilon^\delta \int_0^1 [\epsilon^{1/2}+W(x_0,\gamma(t))]^{1/2}|\dot\gamma(t)|dt. \tag{3.11}$$

From the regularity of $\partial\Omega$ and the definition of $\Omega_\eta^0(x_0)$, there exist a constant η_0 independent of x_0 (dependent only on $\partial\Omega$) such that for any $0<\eta<\eta_0$, we have $\mathcal{H}_{N-1}(\partial\Omega_\eta^0(x_0)\cap\partial\Omega) = 0$. So from Lemma 3–2 we have $\lim_{\epsilon,\delta\to 0}\kappa_\epsilon^\delta = \mathcal{H}_{N-1}(\partial\Omega_\eta(x_0))$ for any $\eta\in(0,\eta_0)$. Here we set $w_\epsilon^{\delta,\gamma} = w_\epsilon$. Therefore from (3.10) and (3.11), for any $\eta\in(0,\eta_0)$ we obtain

$$\int_{\Omega_\eta^\delta(x_0)} [\epsilon|\nabla w_\epsilon^{\delta,\gamma}|^2+\frac{1}{\epsilon}W(x_0,w_\epsilon^{\delta,\gamma})]dx \tag{3.12}$$

$$\leq 2\mathcal{H}_{N-1}(\partial\Omega_\eta)\int_0^1 W^{1/2}(x_0,\gamma(t))|\,\dot\gamma(t)\,|dt$$

$$+\mathcal{H}_{N-1}(\partial\Omega_\eta)[0(\epsilon/\eta^2)+0(\epsilon^{1/4})+0_{\sqrt{\epsilon^2+\delta^2}}(1)].$$

Here by $0_\epsilon(1)$ we mean $\lim_{\epsilon\to 0}0_\epsilon(1)=0$. Since for any $\epsilon>0$ there exist a sequence of C^1-curves $\{\gamma_i\}_{i=1}^\infty$ such that the length of $\gamma_i\leq\epsilon^{-1/8}$ and

$$\lim_{i\to\infty}\int_0^1 W^{1/2}(x_0,\gamma_i(t))|\,\dot\gamma_i(t)\,|\,dt = d(x_0,a,b),$$

by the diagonal argument and (3.12), we can construct a sequence $\{w_\epsilon^\delta\}_{\epsilon,\delta>0}$ satisfying (3.4). Therefore the aim of the first step is completed. ∎

The Second Step: Let Ω_δ be a domain $\{x\in\Omega : \delta<d_{\partial\Omega}(x)<s^*\} = \underset{\delta<t<s^*}{\cup}\pi_t(\partial\Omega)$. At the second step, we construct a sequence $\{w_\epsilon^\delta\}_{\epsilon,\delta>0}$ in $W^{1,2}(\Omega_\delta,\mathbb{R}^n)$ such that

$$\limsup_{\delta,\epsilon\to\infty}\int_{\Omega_\delta}[\,\epsilon|\nabla w_\epsilon^\delta|^2+\frac{1}{\epsilon}W(x,w_\epsilon^\delta)\,]dx \leq 2\int_{\partial\Omega} d(x,a,g(x))d\mathcal{H}_{N-1}. \tag{3.13}$$

In order to construct a sequence $\{w_\epsilon^\delta\}_{\epsilon,\delta>0}$, we will separate $\partial\Omega$ into small pieces. From the regularity of $\partial\Omega$, for sufficiently small $\eta>0$, there exist p points $\{x_i\}_{i=1}^p\subset\partial\Omega$ and a subset ω_η of $\partial\Omega$ such that

$$\partial\Omega\setminus\underset{1\leq i\leq p}{\cup}\partial\Omega_\eta(x_i)\subset\omega_\eta,\qquad \partial\Omega_\eta(x_i)\cap\partial\Omega_\eta(x_j)=\emptyset,\ i\neq j,\ i,j=1,2,\cdots,p \tag{3.14}$$

and $\lim_{\eta\to 0}\mathcal{H}_{N-1}(\omega_\eta)=0$. Here we note that p depends on η and $\lim_{\eta\to 0}p(\eta)=\infty$.

For any $\eta,\delta,\epsilon>0$, fix η, δ, and ϵ. Then for any $i\in\{1,2,\cdots,p\}$, from (3.10) we can construct functions $w_\epsilon^{i,\delta,\eta}\in W^{1,2}(\Omega_\eta^\delta(x_i))$ such that

$$\begin{aligned}(3.15)\qquad \int_{\Omega_\eta^\delta(x_i)}&[\,\epsilon|\nabla w_\epsilon^i|^2+\frac{1}{\epsilon}W(x_i,w_\epsilon^i)\,]dx\\ &\le 2\mathcal{H}_{N-1}(\partial\Omega_\eta(x_i))d(x_i,\alpha,g(x_i))\\ &\quad+\mathcal{H}_{N-1}(\partial\Omega_\eta(x_i))[0(\epsilon/\eta^2)+0(\epsilon^{1/4})+0_{\sqrt{\epsilon^2+\delta^2}}(1)].\end{aligned}$$

Then we define $w_\epsilon^{\delta,\eta}\in W^{1,2}(\Omega_\delta:\mathbb{R}^n)$ as follows:

$$w_\epsilon^{\delta,\eta}=\begin{cases} w_\epsilon^{i,\delta,\eta}, & \text{if } x\in\Omega_\eta^\delta(x_i),\\ \alpha, & \text{otherwise }.\end{cases}$$

By the argument of Step 1, we can see $w_\epsilon^{\delta,\eta}\in W^{1,2}(\Omega_\delta:\mathbb{R}^n)$ easily. Then we have

$$(3.16)\quad \int_{\Omega_\delta}[\,\epsilon|\nabla w_\epsilon^{\delta,\eta}|^2+\frac{1}{\epsilon}W(x,w_\epsilon^{\delta,\eta})\,]dx=\sum_{i=1}^{p}\int_{\Omega_\eta^\delta(x_i)}[\,\epsilon|\nabla w_\epsilon^{i,\delta,\eta}|^2+\frac{1}{\epsilon}W(x,w_\epsilon^{i,\delta,\eta})\,]dx$$

On the other hand, we have (for simplicity we omit the index δ,η of $w_\epsilon^{i,\delta,\eta}$.)

$$\begin{aligned}&\int_{\Omega_\eta^\delta(x_i)}[\,\epsilon|\nabla w_\epsilon^i|^2+\frac{1}{\epsilon}W(x,w_\epsilon^i)\,]dx\\ &=\int_{\Omega_\eta^\delta(x_i)}[\,\epsilon|\nabla w_\epsilon^i|^2+\frac{1}{\epsilon}W(x_i,w_\epsilon^i)\,]dx+\int_{\Omega_\eta^\delta(x_i)}\frac{1}{\epsilon}[W(x,w_\epsilon^i)-W(x_i,w_\epsilon^i)]dx\\ &\equiv I_1^i+I_2^i.\end{aligned}$$

From (3.15) we obtain

$$(3.17)\quad \sum_{i=1}^{p(\eta)}I_1^i\le 2\sum_{i=1}^{p(\eta)}[d(x_i,\alpha,g(x_i))\mathcal{H}_{N-1}(\partial\Omega_\eta(x_i))]+0(\epsilon/\eta^2)+0(\epsilon^{1/4})+0_{\sqrt{\epsilon^2+\delta^2}}(1),$$

and from (1.2) and (3.15)

$$\sum_{i=1}^{p(\eta)}|I_2^i|\le\sum_{i=1}^{p(\eta)}\int_{\Omega_\eta^\delta(x_i)}0_{|x-x_i|}(1)\frac{1}{\epsilon}W(x_i,w_\epsilon^i)dx\le 0_\eta(1)\sum_{i=1}^{p(\eta)}I_1^i.$$

We set $\eta^2=\epsilon^{3/4}$. Then combinating (3.16) and (3.17), we obtain

$$\begin{aligned}(3.18)\qquad \limsup_{\delta,\epsilon\to 0}&\int_{\Omega_\delta}[\,\epsilon|\nabla w_\epsilon^{\delta,\eta(\epsilon)}|^2+\frac{1}{\epsilon}W(x,w_\epsilon^{\delta,\eta(\epsilon)})\,]dx\\ &\le\limsup_{\epsilon\to 0}2\sum_{i=1}^{p(\eta)}d(x_i,\alpha,g(x_i))\mathcal{H}_{N-1}(\partial\Omega_\eta(x_i)).\end{aligned}$$

From the continuity of the function $d(x, \alpha, g(x))$, we obtain

$$\sum_{i=1}^{p(\eta)} d(x_j, \alpha, g(x_j))\mathcal{H}_{N-1}(\partial\Omega_\eta(x_i)) \le \int_{\underset{1\le j\le p}{\cup} \partial\Omega_\eta(x_i)} d(x, \alpha, g(x))d\mathcal{H}_{N-1} + 0_\eta(1)$$
$$\le \int_{\partial\Omega} d(x, \alpha, g(x))d\mathcal{H}_{N-1} + 0_\eta(1).$$

Therefore combinating (3.18), we can see that the sequence $\{w_\epsilon^{\delta,\eta(\epsilon)}\}_{\epsilon,\delta>0}$ satisfies (3.13). Hence we set $w_\epsilon^\delta = w_\epsilon^{\delta,\eta(\epsilon)}$, and so the purpose of Step 2 is completed. ∎

The Third Step: In this step, we will complete th proof of Proposition B for the special case $w_0 \equiv \alpha$. For any $\delta, \epsilon > 0$ we define w_ϵ^δ as follows:

$$w_\epsilon^\delta = \begin{cases} \alpha, & \text{if } x \in \Omega \setminus \Omega_0, \\ {w^*}_\epsilon^\delta, & \text{if } x \in \Omega_\delta \end{cases}$$

where $\Omega_0 = \underset{0<t<s^*}{\cup} \pi_t(\partial\Omega)$ and where ${w^*}_\epsilon^\delta$ is a function constructed in Step 2. In $\Omega^\delta \equiv \Omega_0 \setminus \Omega_\delta$, we construct w_ϵ^δ by combinating between $g(x)$ and ${w^*}_\epsilon^\delta(\pi_\delta(x))$ *i.e.* for $x \in \Omega_0 \setminus \Omega_\delta$,

$$w_\epsilon^\delta(x) = \frac{d_{\partial\Omega}(x)}{\delta} {w^*}_\epsilon^\delta|_{(\partial\Omega)_\delta}(\pi_\delta \circ \pi_{d_{\partial\Omega}(x)}^{-1}(x)) + \left(1 - \frac{d_{\partial\Omega}(x)}{\delta}\right) g(\pi_{d_{\partial\Omega}(x)}^{-1}(x)). \tag{3.19}$$

Here $\pi_\delta(x)$ and $\pi_{d_{\partial\Omega}}(x)$ are functions appearing in Lemma 3–1. Then we can see easily $w_\epsilon^\delta \in W^{1,2}(\Omega)$ and $w_\epsilon^\delta(x) = g(x)$ for all $x \in \partial\Omega$.

In order to estimate the gradient of w_ϵ^δ, we fix ϵ, δ, and fix $\{\Omega_\eta^\delta(x_i)\}_{i=1}^p$ and ω_η. Then we set

$$\begin{aligned} \Omega_1^\delta &= \{x \in \Omega^\delta : \pi_\delta \circ \pi_{d_{\partial\Omega}(x)}^{-1}(x) \in \underset{1\le i\le p}{\cup} \partial(\Omega_{\eta,1}^\delta(x_i))\}, \\ \Omega_2^\delta &= \{x \in \Omega^\delta : \pi_\delta \circ \pi_{d_{\partial\Omega}(x)}^{-1}(x) \in \underset{1\le i\le p}{\cup} \partial(\Omega_{\eta,2}^\delta(x_i))\}, \\ \omega_\eta^\delta &= \underset{0<t<\delta}{\cup} \pi_t(\omega_\eta), \end{aligned} \tag{3.20}$$

and have $\Omega^\delta = \Omega_1^\delta \cup \Omega_2^\delta \cup \omega_\eta^\delta$. Here $\Omega_{\eta,i}^\delta(x)$, $i = 1, 2$ is a domain appearing in Step 1. Furthermore for simplicity, we set

$$\hat{g}(x) = g(\pi_{d_{\partial\Omega}(x)}^{-1}(x)) \quad \text{and} \quad \hat{w}_\epsilon^\delta(x) = {w^*}_\epsilon^\delta|_{(\partial\Omega)_\delta}(\pi_\delta \circ \pi_{d_{\partial\Omega}(x)}^{-1}(x))$$

for $x \in \Omega^\delta$. Then from Lemma 3–1 we can see that there exists a constant C such that $|\nabla \hat{g}(x)| \le C$ for almost all $x \in \Omega^\delta$.

Now in the domains $\Omega^\delta_{\eta,1}$, $\Omega^\delta_{\eta,2}$, and Ω^δ, we will estimate the gradient of w^δ_ϵ, and obtain the inequality (2.1). If $x \in \omega^\delta_\eta$, then from the construction of w_ϵ in Step 2 we see $v^{\delta,\eta}_\epsilon \equiv \alpha$ in a neighborhood of x, and so for almost all $x \in \omega^\delta_\eta$ we have

$$|\nabla w^{\delta,\eta}_\epsilon| \le C(1 + 1/\delta).$$

So we obtain

$$\int_{\omega^\delta} [\,\epsilon|\nabla w^{\delta,\eta}_\epsilon|^2 + \frac{1}{\epsilon}W(x, w^{\delta,\eta}_\epsilon)\,]dx \le C\left(\frac{\epsilon}{\delta^2} + \epsilon + \frac{1}{\epsilon}\right)\delta\mathcal{H}_{N-1}(\omega). \tag{3.21}$$

For almost all $x \in \Omega^\delta_{\eta,1}(x_i)$, then we have

$$\begin{aligned}|\nabla w^{\delta,\eta}_\epsilon| \le \frac{|\nabla d_{\partial\Omega}(x)|}{\delta}\hat{w}^{\delta,\eta}_\epsilon(x) + \frac{d_{\partial\Omega}(x)}{\delta}|\nabla \hat{w}^{\delta,\eta}_\epsilon(x)| \\ + \frac{|\nabla d_{\partial\Omega}(x)|}{\delta}\hat{g}(x) + \left(1 - \frac{d_{\partial\Omega}(x)}{\delta}\right)|\nabla\hat{g}(x)|.\end{aligned}$$

Here from the argument in Step 1, there exists a constant C_2 such that $|\nabla v^{\delta,\eta}_\epsilon(x)| \le C/(\epsilon^{5/8}\eta)$ for all $x \in \Omega^\delta_{\eta,1}$. Moreover we have $|\Omega^\delta_{\eta,1}| \le C\delta(\epsilon^{5/8}\eta^{N-1})(\mathcal{H}_{N-1}(\partial\Omega)/\eta^{N-1}) \le C\delta\epsilon^{5/8}$. So we obtain

$$\begin{aligned}\int_{\Omega^\delta_1} [\epsilon|\nabla w^{\delta,\eta}_\epsilon|^2 + \frac{1}{\epsilon}W(x, w^{\delta,\eta}_\epsilon)]dx &\le C\left[\epsilon\left(\frac{1}{\delta} + \frac{1}{\epsilon^{5/8}\eta} + 1\right)^2 + \frac{1}{\epsilon}\right]\delta\epsilon^{5/8} \\ &\le C\left(\frac{\epsilon}{\delta} + \frac{\delta}{\eta^2\epsilon^{1/4}} + \frac{\delta}{\epsilon}\right)\epsilon^{5/8}.\end{aligned} \tag{3.22}$$

For any $x \in \Omega^\delta_{\eta,2}(x_i)$, from Step 1 we see $w^*_\epsilon(x) \equiv g(x_i)$ in a neighborhood of x. Then from the Lipschitz continuity of $g(x)$ on $\partial\Omega$ and (3.19) we have

$$\begin{aligned}|\nabla w^{\delta,\eta}_\epsilon| &\le \frac{|\nabla d_{\partial\Omega}(x)|}{\delta}|g(x_i) - \hat{g}(x)| + \left(1 - \frac{d_{\partial\Omega}(x)}{\delta}\right)|\nabla\hat{g}| \\ &\le \frac{C}{\delta}|g(x_i) - \hat{g}(x)| + C \le C\frac{\eta}{\delta} + C.\end{aligned}$$

So we obtain

$$\int_{\Omega^\delta_2} [\,\epsilon|\nabla w^{\delta,\eta}_\epsilon|^2 + \frac{1}{\epsilon}W(x, w^{\delta,\eta}_\epsilon)\,]dx \le C\left(\epsilon\left(\frac{\eta}{\delta}\right)^2 + \epsilon + \frac{1}{\epsilon}\right)\delta\mathcal{H}_{N-1}(\partial\Omega). \tag{3.23}$$

Let $\sigma(\cdot)$ be a positive function with $\sigma(0) = 0$ such that $\lim_{\epsilon\to 0}\mathcal{H}_{N-1}(\omega_{\eta(\epsilon)})/\sigma(\epsilon) = 0$ and $\lim_{\epsilon\to 0}\epsilon^{5/8}/\sigma(\epsilon) = 0$. Here we set $\delta_\epsilon = \epsilon\sigma(\epsilon)$, and define $w_\epsilon = w^{\delta_\epsilon}_\epsilon$. Then from (3.21)–(3.23) we obtain

$$\lim_{\epsilon\to 0}\int_{\Omega^{\epsilon\sigma(\epsilon)}} [\,\epsilon|\nabla w_\epsilon|^2 + \frac{1}{\epsilon}W(x, w_\epsilon)\,]dx = 0. \tag{3.24}$$

Therefore from (3.13) and (3.24) we obtain

$$\limsup_{\epsilon\to 0}\int_\Omega [\epsilon|\nabla w_\epsilon|^2+\frac{1}{\epsilon}W(x,w)]dx \le 2\int_{\partial\Omega} d(x,\alpha,g(x))d\mathcal{H}_{N-1}.$$

Hence the proof of Proposition B for the special case $w_0=\alpha$ is completed. ∎

Finally we remark that the proof of this section is an essential part of complete proof of Proposition B.

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