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Large indiscernible sets of a structure

Akito Tsuboi

1 Introduction

An indiscernible set of a given structure is by definition a set $I$ such that every finite subset of the same cardinality has the same type. A singleton $I = \{a\}$ is trivially an indiscernible set, so it is called a trivial one. A transcendental basis of an algebraically closed field $K$ is a good example of a non-trivial indiscernible set. In this example, if $K$ is an uncountable, then it has a large indiscernible set $I$, i.e. an indiscernible set $I$ with $|I| = |K|$. Generally speaking, if a theory $T$ is $\omega$-stable then every uncountable model of $T$ has such a large indiscernible set. However, in the structure $\mathbb{R} = (\mathbb{R}, 0, 1, +, \cdot)$, there is no non-trivial indiscernible set, i.e. $tp(a) = tp(b)$ implies $a = b$.

In this note we show that every $L$-structure $M$ can be embedded into a structure $M^*$ of an expanded language $L^*$ such that any $L^*$-structure $N \equiv M^*$ has a large indiscernible set. We also show that if $T$ is stable and non-$\omega$-stable then there is a model of power $\aleph_1$ which has no large indiscernible sets.

2 Preliminaries

In what follows, $T$ is a complete theory formulated in a countable language $L$. We give some necessary definitions and review some basic results.

Definition 1. (1) Let $I$ be a subset of a structure $M$. $I$ is said to be an indiscernible set if whenever $F \subset I$ and $G \subset I$ are finite sequences of the same length then $tp(F) = tp(G)$. 

We will say that an indiscernible set \( I \) in a structure \( M \) is large if \( I \) has the same cardinality as \( M \).

**Fact 1** (Theorem 2.8 of [S, CH.I, §2]). If \( T \) is \( \omega \)-stable, then every uncountable model of \( T \) includes a large indiscernible set.

If \( T \) is not \( \omega \)-stable, then any \((a,\omega)\)-model is uncountable. And any \((a,\kappa(T))\)-prime model does not have indiscernible set of power greater than \( \kappa(T) \). So we have:

**Fact 2.** If \( T \) is a non-\( \omega \)-stable, superstable theory, then there is a model of power \( \aleph_1 \) without a large indiscernible set.

Let \( T \) be the theory of refining equivalence relations. i.e., \( T \) is the theory of the structure \((2^\omega, E_1, E_2, \ldots)\), where \( E_i = \{(\eta_1, \eta_2) \in (2^\omega)^2 : \eta_1|i = \eta_2|i\} \). Then \( T \) is a superstable theory with \( |S(T)| = 2^{\aleph_0} \). Let \( M \) be any uncountable elementary submodel of \((2^\omega, E_1, E_2, \ldots)\). \( M \) has no large indiscernible sets.

**Definition 2.** A model \( M \supset A \) is said to be \( \ell \)-atomic over \( A \) if for every \( \bar{a} \in M \), and every finite set \( \Delta \) of formulas, \( \text{tp}_\Delta(\bar{a}/A) \) is a principal type.

**Fact 3.** Let \( T \) be stable.

(1) For every set \( A \), there is an \( \ell \)-atomic model over \( A \).

(2) Let \( a_1 \) and \( a_2 \) be independent over \( M \). Let \( M_i \) be an \( \ell \)-atomic model over \( M \cup \{a_i\} \). Then \( M_1 \) and \( M_2 \) are independent over \( M \).

## 3 Main Result

We want to extend fact 2 to a non-\( \omega \)-stable, stable theory \( T \). The following lemma will play a crucial role.

**Lemma.** Let \( T \) be a non-\( \omega \)-stable, stable theory and \( \kappa \leq 2^{\aleph_0} \) an uncountable cardinal. Then there is a set \( R \) of types over a set \( A \), \( |A| < \kappa \) such that whenever \( B \supset A \) is a set with \( |B| < \kappa \) and \( S \) is a set of stationary types over \( B \) with \( |S| < \kappa \) then there is a non-algebraic type \( r \in R \) which is almost orthogonal to any type in \( S \).

**Proof.** This lemma remains true for a superstable theory, but we concentrate on an unsuperstable theory. (Superstable case is easier.) Since \( T \) is not superstable, there are infinitely long continuous sequence \( \{p_i : i \leq \alpha\} \) of types such that
(1) dom\(p\) is a countable set;
(2) \(p_i\) is a forking extension of \(p_j\), if \(i > j\);
(3) \(\alpha < \omega_1\) is a countable limit ordinal;
(4) \(U(p_\alpha) < \infty\).

By choosing a subsequence of \(\{p_i : i \leq \alpha\}\), we can assume that \(\alpha = \omega\). Now by the definition of forking, we can easily find a countable set \(A_0\), and continuously many types \(\{q_i : i < 2^{\aleph_0}\}\) over \(A_0\) such that each \(q_i\) is \(U\)-ranked \((U(q_i) < \infty)\). We can assume that each type \(q_i\) is stationary.

Suppose that our lemma does not hold. By induction on \(j < \omega\), we define a set \(A_j\) of cardinality < \(\kappa\) and types \(q_{i,j} \in S(A_j)\) \((i < 2^{\aleph_0}\) such that for any \(i < 2^{\aleph_0}, k < j\),

\[q_{i,k}\text{ is algebraic or } q_{i,j}\text{ is a forking extension of } q_{i,k}.$

For each \(i < 2^{\aleph_0}\), let \(q_{i,0} = q_i\). Suppose we have defined \(q_{i,k} \in S(A_k)\) for \(i < 2^{\aleph_0}\) and \(k < j\). Let \(\Lambda = \{i < 2^{\aleph_0} : q_{i,j-1}\text{ is non-algebraic}\}\). Since we are assuming the negation of the statement in our lemma, there are a set \(B \supset A_{j-1}, |B| < \kappa\) and a set \(S \subset S(B), |S| < \kappa\) such that every \(q_{i,j-1}\) \((i \in \Lambda)\) is not almost orthogonal to some \(s_i \in S\). For \(i \in \Lambda\), choose \(a_i \models q_{i,j-1}|B\) and \(b_i \models s_i\) such that \(a_i\) and \(b_i\) are dependent over \(B\). We can assume that if \(s_i = s_j\) then \(b_i = b_j\). Now let

\[A_j = acl(A_{j-1} \cup \{b_i : i \in \Lambda\});
\]

\[q_{i,j} = \begin{cases} \text{tp}(a_i/A_j) & i \in \Lambda \\ \text{arbitrary extension of } q_{i,j-1} & i \notin \Lambda \end{cases} \]

Finally let \(A_\omega = \bigcup_{j < \omega} A_j\). Note that \(|A_\omega| < 2^{\aleph_0}\). (If \(\kappa = 2^{\aleph_0}\), then \(cf(\kappa) > \omega\), so \(|A_\omega| = \kappa = 2^\omega\). If \(\kappa < 2^{\aleph_0}\), then \(|A_\omega| \leq \kappa < 2^{\aleph_0}\).) Since \(q_i\) is \(U\)-ranked by (4), \(q_i^* = \bigcup_{j < \omega} q_{i,j} \in S(A_\omega)\) must be an algebraic type. (Otherwise there is an infinitely long forking sequence starting from \(q_i\).) So we have constructed continuously many distinct algebraic types over a fixed set \(A_\omega, |A_\omega| < 2^\omega\). However this is a contradiction, since we are assuming that \(L\) is countable.
**Theorem A.** Let $T$ be a non-$\omega$-stable, stable theory. Then for any uncountable cardinal $\kappa \leq 2^{\aleph_0}$, there is a model of power $\kappa$ without a large indiscernible set.

**Proof.** Choose a set $A$ and types $R \subseteq S(A)$ which satisfy the condition in the above lemma. Let $\lambda = |A|$. Clearly $\lambda < \kappa$. We construct an elementary chain of models $\{M_i : i \leq \kappa\}$ such that each model $M_i$ has cardinality $\leq |i| + \lambda$. Without loss of generality, $A$ is a model. Let $M_0 = A$, and $M_1$ an arbitrary proper extension of $M_0$ with the same cardinality. Suppose that we have constructed $\{M_i : i < \alpha\}$. If $\alpha$ is a limit ordinal, then let $M_\alpha = \bigcup_{i<\alpha} M_i$. So we assume that $\alpha = \beta + 1$, and let

$$S_\beta = \bigcup_{i<\beta} \{q(x) \in S(M_\beta) : q \text{ is based on } M_i, q|M_i \text{ is realized in } M_\beta\}$$

Clearly $|S_\beta| \leq |\beta| + \lambda < \kappa$. By the property of $R$, there is a type $r \in R$ which is almost orthogonal to each type in $S_\beta$. Let $M_{\beta+1}$ be an $\ell$-atomic model over $M_\beta \cup \{e_\beta\}$, where $e_\beta$ is a realization of $r|M_\beta$. Of course we can assume $|M_{\beta+1}| < |\beta + 1| + \lambda$.

**Claim.** There is no large indiscernible set in $M_\kappa$.

Suppose that there was a large indiscernible set $I \subseteq M_\kappa$. By stability, there is a countable set $I_0 \subseteq I$ such that $J = I - I_0$ is a Morley sequence over $I_0$. Choose $M_i$ $(i < \kappa)$ which includes $I_0$. Since $M_i < \kappa$, we may assume that $J$ is a Morley sequence over $M_i$, by choosing a subset of $J$ if necessary. Choose $M_j$ $(j < \kappa)$ which intersects with $J$. Let $a \in J \cap M_j$. Since $|J| = \kappa$, there is $b \in J$ which is independent from $M_j$ over $M_i$. Choose the least $k$ such that $b$ and $M_k$ are dependent over $M_i$. Then $k$ is a successor ordinal greater than $j$, and

1. $b$ and $M_k$ are dependent over $M_{k-1}$;  
2. $b$ and $M_{k-1}$ are independent over $M_i$.

Remember that $M_k$ is $\ell$-atomic over $M_{k-1} \cup \{e_{k-1}\}$. From (1), using fact 3, we know that $b$ and $e_{k-1}$ are dependent over $M_{k-1}$. By our choice of $e_{k-1}$, $tp(e_{k-1}/M_{k-1})$ is almost orthogonal to every type in $S_{k-1}$, hence $tp(b/M_{k-1})$ does not belong to $S_{k-1}$. Note that $tp(b/M_i)$ is realized by $a \in M_{k-1}$. Then we must have
(3) $tp(b/M_{k-1})$ is a forking extension of $tp(a/M_i)$.

(2) and (3) yield a contradiction.

Next theorem shows that theorem A cannot be extended to an unstable theory.

Theorem B. Let $M$ be an infinite $L$-structure. Then there is a structure $M^*$ for an expanded language $L^* \supset L$ with the following properties:

(i) $M$ is $\emptyset$-definable in $M^*$;

(ii) In any $L^*$-structure $N \equiv M^*$, there is a large indiscernible set in $N$.

Proof. For $i < \omega$, let $L_i = L \cup \{F_j(*) : j = 0, \ldots, i\} \cup \{U(*)\} \cup \{R_j(*,*,*) : j = 1, \ldots ,i\}$, where $F_i$'s and $U$ are unary predicate symbols, and $R_j$'s are 3-ary predicate symbols. Let $L^* = \bigcup_{i<\omega} L_i$. We construct inductively countable $L_j$-structures $M_j$ and countable subgroups $S_j$ of $\text{Aut}(M_j)$ ($j < \omega$) with the following properties:

(1) $M_0 = F_0^{M_0} \cup U^{M_0}$, where $F_0^{M_0} = M$, and $U^{M_0}$ is an infinite set disjoint from $F_0^{M_0}$.

(2) $S_0$ is a countable subgroup of $\text{Aut}(M_0)$ such that for given finite sequences $\bar{a} \in U^{M_0}$ and $\bar{b} \in U^{M_0}$ of the same length, there is a $\sigma \in S_0$ with $\sigma(\bar{a}) = \bar{b}$. Any two automorphisms $f \in S_0$ and $g \in S_0'$ differ at finitely many points.

(3) $M_{j+1} = M_j \cup F_{j+1}^{M_{j+1}}$,

(4) $S_j = \{\sigma[M_j : \sigma \in S_{j+1}]\}$.

Assume that we have already constructed $M_j$ and $S_j$ for $j < i$. Choose a bijective function $f_0 : F_{i-1}^{M_{i-1}} \to U^{M_{i-1}}$ arbitrarily and let

$$F_i^{M_i} = \{\sigma \circ f_0 \circ \sigma^{-1} : \sigma \in S_{i-1}\}.$$ 

$F_i^{M_i}$ is a countable set of functions from $F_{i-1}^{M_{i-1}}$ to $U^{M_{i-1}}$. Define $R_i^{M_i} \subset F_i^{M_i} \times F_{i-1}^{M_{i-1}} \times U^{M_{i-1}}$ by

$$(f, a, b) \in R_i^{M_i} \iff f(a) = b.$$
Now let $M_i = M_{i-1} \cup F^M_i$. We can extend each $\tau \in S_{i-1}$ to an automorphism $\tau^*$ of $M_i$. Let $f = \sigma \circ f_0 \circ \sigma^{-1} \in S_{i-1}$. Then define

$$\tau^*(f) = \tau \circ f \circ \tau^{-1} = (\tau \sigma) \circ f_0 \circ (\tau \sigma)^{-1} \in S_{i-1}.$$ 

The following equivalence shows that $\tau^*$ is really an automorphism:

$$M_i \models R(f, a, b) \iff f(a) = b \iff \tau^*(f)(\tau^*(a)) = \tau^*(b) \iff M_i \models R(\tau^*(f), \tau^*(a), \tau^*(b)).$$

Finally we set $M^* = \bigcup_{1<\omega} M_i, \tau^* = \text{Th}_{L^*}(M)$. Now it is sufficient to prove the following two claims.

**Claim 1.** In any model $N$ of $T^*$, $U^N$ is an indiscernible set.

It is sufficient to prove the statement for the case $N = M^*$. Let $\overline{a}, \overline{b} \in U^{M^*}$ be given. By the assumption on $S_0$, there is a $\sigma \in S_0$ such that $\sigma(\overline{a}) = \overline{b}$. $\sigma$ can be extended to an automorphism of $M^*$. So $\overline{a} \equiv \overline{b}$.

**Claim 2.** If $N \models T^*$, then there is a large indiscernible set.

Clearly $U^N \cup \bigcup_i F^N_i$ has the same cardinality as $N$, or the complement $N - (U^N \cup \bigcup_i F^N_i)$ has the same cardinality as $N$. The second case clearly implies that $N - (U^N \cup \bigcup_i F^N_i)$ is a large indiscernible set. Let the second case hold. Note that an element in $F_{i+1}$ gives a bijection between $F^N_i$ and $U^N$. Then we see that $U^N$ has the same cardinality as $N$. By claim 1, $U^N$ is a large indiscernible set in this case.

**Remark.** (i) Any model of $T = \text{Th}(\mathbb{Z}, <)$ has a large indiscernible sequence. (ii) The construction of $M^*$ was inspired by [F], in which Fuhrken showed the existence of an uncountable complete theory without the omitting types property. Note that our $T^*$ is not stable: By our choice of $S_0$ and $F_1$, there is a sequence $\{(f_i, g_i) : i < \omega\} \subset F^M_1 \times F^M_1$ such that the formulas $\forall y \in F_0(R(f_i, x, y) \leftrightarrow R(g_i, x, y)) (i < \omega)$ define a strictly decreasing subsets of $F_0$.

**Question.** Does theorem A remain true, if we we replace 'large indiscernible set' by 'uncountable indiscernible set'?
4 References