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CONBINATORIAL PROPERTIES OF FINITE FULL TRANSFORMATION SEMIGROUPS

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Let $X$ be the finite set $\{1, 2, \ldots, n\}$ and let $T(X)$ be the semigroup (under composition of mappings from $X$ into $X$). The symmetric group $G(X)$, consisting of all permutations of $X$, is a subgroup of $T(X)$, while the set $S_n = T(X) \setminus G(X)$ of all singular mappings from $X$ into $X$ is a subsemigroup of $T(X)$. We denote the image of $\alpha$ of $S_n$ by $\text{ima}\alpha$, i.e., $\text{ima}\alpha = \{x\alpha | x \in X\}$, and define the rank of $\alpha$ to be $\text{rank}\alpha = |\text{ima}\alpha|$. Let $E$ be the set of idempotents of $S_n$. In [1], it has shown that $S_n$ is generated by the $n(n-1)$ idempotents of rank $n-1$. Then there arise the following two problems:

Problem 1. Find the least integer $k$ for which $E^k = S$.

Problem 2. For each $\alpha \in S_n$, find the least integer $k(\alpha)$ for which $\alpha \in E^{k(\alpha)}$.

Let $E_1$ be the set of idempotents of rank $n-1$ in $S_n$. Iwahori [3] and Howie [2] found the least integer $l(\alpha)$ for which $\alpha \in E_1^{l(\alpha)}$. By using this result, Howie [2] solved Problem 1, that is $k = [3(n-1)/2]$.

In this survey, we discuss on Problem 2. The proofs of the results here are not given. But to make the results understandable, we will give examples.

Let $\alpha \in S_n$. We define $\text{fix}\alpha = \{x \in X | x\alpha = x\}$, and an orbit of $\alpha$ to be an equivalence class under the equivalence $\omega = \{(x, y) \in X \times X | x\alpha^l = y\alpha^m \text{ for some } l, m \geq 0\}$. Then each orbit $\Omega$ of $\alpha$ has a kernel $K(\Omega)$ characterised by the property (for each $x$ in $\Omega$) $x \in K(\Omega)$ if and only if $x \in x\alpha^N$ where $x\alpha^N = \{y \in X | y\alpha^i = x \text{ for some } i \geq 1\}$. Then orbits classified into the following four types:

- standard orbit: $|\Omega| > |K(\Omega)| > 1$
- acyclic orbit: $|\Omega| = |K(\Omega)| = 1$
- cyclic orbit: $|\Omega| > |K(\Omega)| > 1$
- singleton orbit: $|\Omega| = |K(\Omega)| = 1$.

Example 1. Let $n = 14$ and let

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 2 & 3 & 4 & 5 & 3 & 5 & 8 & 9 & 9 & 12 & 13 & 11 & 14 \end{pmatrix}$$

The orbits of $\alpha$ can be depicted as follows:
Then $|\Omega_1| = 6 > |K(\Omega_1)| = 3 > 1$, $|\Omega_2| = 4 > |K(\Omega_2)| = 1$, $|\Omega_3| = |K(\Omega_3)| = 3 > 1$, $|\Omega_4| = 1$, so that $\Omega_1$ is standard, $\Omega_2$ is acyclic, $\Omega_3$ is cyclic and $\Omega_4$ is singleton.

It is easy to see that $\alpha \in S_n$ is an idempotent if and only if $\text{im} \alpha = \text{fix} \alpha$. Thus we have that, if $\epsilon$ is an idempotent of rank $n-1$, then there exist $a$ and $b$ in $X$ such that $a\epsilon = b$ and $x\epsilon = x$ if $a \neq b$. We write $\epsilon = \begin{pmatrix} a \\ b \end{pmatrix}$. For example, $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 4 \end{pmatrix}$.

Let $\alpha \in S_n$. Then the number of cyclic orbits of $\alpha$ is denoted by $c(\alpha)$. We define the gravity of $\alpha$ to be $g(\alpha) = n - |\text{fix} \alpha| + c(\alpha)$, and the defect of $\alpha$ to be $d(\alpha) = n - \text{rank} \alpha$.

**THEOREM 1.** (Nobuko Iwahori [3] and J. M. Howie [2])

Let $S_n$ be the semigroup of all singular mappings from $X$ into $X$ where $X$ is the finite set $\{1, 2, ..., n\}$ and let $E_1$ be the set of idempotents of defect 1 (rank $n-1$) in $S_n$. For each $\alpha \in S_n$ the least $l(\alpha)$ for which $\alpha \in E^{k(\alpha)}$ is $g(\alpha)$, where $g(\alpha)$ is the gravity of $\alpha$.

We state the outline of the proof of Theorem 1 by using the $\alpha$ in Example 1. In this case, $|\text{fix} \alpha| = 2$ and $c(\alpha) = 1$, so that $g(\alpha) = 14 - 2 + 1 = 13$. For $\Omega_1$, take $x \in \Omega_1$ such that $x \notin K(\Omega_1)$ and $x\alpha \in K(\Omega_1)$, say $x = 6$, and take $y \in K(\Omega_1)$ such that $x\alpha = y\alpha$, i.e., $x = 4$. Then

$$\Omega_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 3 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 5 & 6 \\ 3 & 5 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$  

For $\Omega_2$, $\Omega_2 = \begin{pmatrix} 7 & 8 & 9 & 10 \\ 8 & 9 & 9 & 9 \end{pmatrix} = \begin{pmatrix} 8 \\ 9 \end{pmatrix} \begin{pmatrix} 10 \\ 9 \end{pmatrix}$.

For $\Omega_3$, take $x \in X \setminus \text{im} \alpha$, say $x = 1$. Then

$$\Omega_3 = \begin{pmatrix} 11 & 12 & 13 \\ 12 & 13 & 11 \end{pmatrix} = \begin{pmatrix} 11 \\ 12 \\ 13 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$  

We obtain $\alpha = \begin{pmatrix} 4 & 6 \\ 6 & 4 \\ 3 & 5 \\ 5 & 3 \\ 6 & 2 \\ 2 & 8 \\ 8 & 7 \\ 7 & 10 \\ 10 & 11 \\ 11 & 13 \\ 13 & 12 \end{pmatrix}$. 

Let $a_1, ..., a_k$ be distinct elements in $X$, and let $b_1, ..., b_k$ be elements (not necessarily distinct) in $X$ such that $\{a_1, ..., a_k\} \cap \{b_1, ..., b_k\} = \emptyset$. Then the semigroup generated by the
idempotents \( \begin{array}{l} a_1 \\ b_1 \end{array}, \ldots, \begin{array}{l} a_k \\ b_k \end{array} \) is a semilattice of order \( 2^{k-1} \) in which the rank of each element is greater than \( n - k - 1 \). We write \( \begin{array}{l} a_1 \\ b_1 \end{array}, \ldots, \begin{array}{l} a_k \\ b_k \end{array} = \begin{array}{l} a_1 \ldots a_k \\ b_1 \ldots a_k \end{array} \).

Then \( \begin{array}{l} a_1 \ldots a_k \\ b_1 \ldots a_k \end{array} \) is an idempotent of defect \( k \) (rank \( n - k \)). Conversely, an idempotent of defect \( k \) can be written in the above form.

For \( \alpha, \beta \in S_n \), it is easy to see that \( \text{rank}(\alpha \beta) \leq \text{rank} \alpha \) and \( \text{rank}(\alpha \beta) \leq \text{rank} \beta \), so that \( \text{d}(\alpha) \leq \text{d}(\alpha \beta) \) and \( \text{d}(\beta) \leq \text{d}(\alpha \beta) \).

**Lemma 1.** Let \( \alpha \in S_n \). Then \( g(\alpha)/d(\alpha) \leq k(\alpha) \), where \( k(\alpha) \) means that of Problem 2.

**Proof.** Let \( \alpha = \epsilon_1 \epsilon_2 \ldots \epsilon_{k\alpha} \), where each \( \epsilon_i \) \((i = 1, 2, \ldots, k(\alpha))\) is an idempotent with \( d(\epsilon_i) \leq d(\alpha) \). Let \( d(\epsilon_i) = d_i \). Since an idempotent of defect \( d_i \) is a product of \( d_i \) idempotents of defect \( 1 \), \( \alpha \) is a product of \( d_1 + \ldots + d_{k\alpha} \) idempotents of defect \( 1 \). By Theorem 1, \( g(\alpha) \leq d_1 + \ldots + d_{k\alpha} \leq d(\alpha) k(\alpha) \). Thus \( g(\alpha)/d(\alpha) \leq k(\alpha) \).

**Lemma 2.** Let \( a, b, c \in X \). Then

1. \( \begin{array}{l} a \\ b \end{array} = \begin{array}{l} a \\ c \end{array} \), where \( a \neq b, a \neq c \).

2. \( \begin{array}{l} a \\ b \end{array} = \begin{array}{l} b \\ c \end{array} = \begin{array}{l} a \\ c \end{array} \), where \( a \neq b, b \neq c, a \neq c \).

We introduce a new notation to be more easily visible.

We write \( \begin{array}{l} a \\ b \end{array} = (b \leftarrow a) \), \( \begin{array}{l} b \\ c \end{array} = (a \leftarrow b)(b \leftarrow c) = (a \leftarrow b \leftarrow c) \)

and \( \begin{array}{l} a \\ c \\ b \\ d \end{array} = (b \leftarrow a) \) \( (d \leftarrow c) \).

**Lemma 3.** Let \( a_1, a_2, \ldots, b_1, \ldots, b_m \) be distinct elements in \( X \), and let \( c \in X \) with \( c \neq a_1, c \neq a_{k+1}, c \neq b_1 \). Then

\[
( c \leftarrow a_1 \leftarrow \ldots \leftarrow a_i \leftarrow \ldots \leftarrow a_j)(a_i \leftarrow b_m \leftarrow \ldots \leftarrow b_j)
\]

= \[
( c \leftarrow a_1 \leftarrow \ldots \leftarrow a_i \leftarrow \ldots \leftarrow a_j) \).
\]

We suggest a proof of Lemma 3 by using the following example.

**Example 2.** \((4 \leftarrow 3 \leftarrow 2 \leftarrow 1)(3 \leftarrow 5 \leftarrow 6) = \begin{array}{l} 3 \\ 4 \end{array} \begin{array}{l} 2 \\ 3 \end{array} \begin{array}{l} 1 \\ 2 \end{array} \begin{array}{l} 5 \\ 3 \end{array} \begin{array}{l} 6 \end{array} \)
Example 3. Let \( \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 3 & 4 & 5 & 3 & 5 & 8 & 9 & 9 & 9 \end{pmatrix} \).

By the previous result of \( \alpha \) in Example 1, we have
\[
\beta = \begin{pmatrix} 4 & 6 \\ 3 & 4 \\ 5 & 6 \\ 2 & 1 \\ 8 & 7 \\ 9 & 10 \end{pmatrix} = (6 \leftarrow 4 \leftarrow 3 \leftarrow 5 \leftarrow 6) (3 \leftarrow 2 \leftarrow 1) = (6 \leftarrow 4 \leftarrow 3 \leftarrow 5 \leftarrow 6) (5 \leftarrow 2 \leftarrow 1) (9 \leftarrow 8 \leftarrow 7) (9 \leftarrow 10).
\]

Then we have that in the above expression of \( \beta \) the last member of each series \((\ldots \leftarrow \ldots)\) belongs to \( X \setminus \text{im} \beta \) and they are mutually distinct.

The \( \alpha \) of Example 1 can be expressed as follows:
\[
\alpha = \begin{pmatrix} 6 & 4 & 7 & 5 & 6 \\ 5 & 2 & 1 & 11 & 13 & 12 & 1 \\ 9 & 8 & 7 \\ 9 & 10 \end{pmatrix}.
\]

Then the number of series in the above expression of \( \alpha \) coincides with \( d(\alpha) \) and the number of all arrows coincides with \( g(\alpha) \).

**Lemma 4.** Let \( a_p, \ldots, a_m (m \geq 3) \) be distinct elements in \( X \) and let \( \begin{pmatrix} a_m & b \\ c & d \end{pmatrix} \) be an idempotent of defect 2. Then
\[
\begin{pmatrix} c & a_m & \ldots & a_i & \ldots & a_p \\ d & b \end{pmatrix} = \begin{pmatrix} c & a_m & \ldots & a_{i+1} & \ldots & a_p \\ d & b & a_i & \ldots & a_i \end{pmatrix}.
\]

We also suggest a proof of Lemma 4 by using the following example.
Example 3. 

\[
(5 \leftarrow 4 \leftarrow 3 \leftarrow 2 \leftarrow 1) = \begin{pmatrix} 4 & 7 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ 2 \\ 3 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{pmatrix} \\
(6 \leftarrow 7)
\]

\[
= \begin{pmatrix} 4 & 7 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 7 \\ 2 \\ 3 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ 2 \\ 3 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} \quad \text{(by (1) of Lemma 2)}
\]

\[
= \begin{pmatrix} 4 & 7 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} \quad \text{(by (2) of Lemma 2)}
\]

\[
= \begin{pmatrix} 4 & 7 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 4 & 7 \end{pmatrix} \begin{pmatrix} 7 \\ 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix}
\]

\[
= (5 \leftarrow 4 \leftarrow 3 \leftarrow 7) \\
(6 \leftarrow 7 \leftarrow 2 \leftarrow 1) .
\]

The length of \((a_m \leftarrow \ldots \leftarrow a_1)\) is the number of arrows in it. Lemma 4 shows that the length of \((c \leftarrow a_m \leftarrow \ldots \leftarrow a_1)\) decreases by \(k\) and the length of \((d \leftarrow b)\) increases by \(k + 1\).

Let \(V_0 = \{v_1, v_2, \ldots, v_d\}\) be a multi-set of positive integers \((d \geq 2)\), where \(v_1, v_2, \ldots, v_d\) are not necessarily distinct. Let us subtract \(k\) from some \(v_i\) and add \(k + 1\) to some \(v_j\) where \(k\) is an integer. Let \(V_1 = \{v_1, \ldots, v_i - k, \ldots, v_j + k + 1, \ldots, v_d\}\). By repeating this procedure on \(V_0\), we obtain a new multi-set \(V_2\).

**Lemma 5.** Let \(V_0 = \{v_1, v_2, \ldots, v_d\}\) be a multi-set of positive integers \((d \geq 2)\) with \(v_1 + v_2 + \ldots + v_d = g\). By suitable repeating of the above procedure, there exists \(V_s\) such that \(\lceil g/d \rceil \leq \max V_s \leq \lceil g/d \rceil + 1\) and \(\max V_s = \lfloor g/d \rfloor + 1\) if \(g \equiv 1 \pmod{d}\), where \(\lfloor x \rfloor\) denotes the least integer \(m\) for which \(m \geq x\).

Example 5. Let \(V_1 = \{1, 8, 26, 32, 54\}\). Then \(V_1 = \{31, 8, 25, 32, 25\}, V_2 = \{31, 16, 26, 25, 25\}, V_3 = \{25, 23, 26, 25, 25\}\) and \(V_4 = \{25, 25, 25, 25, 25\}\).

Let \(\alpha\) be as in Example 1. Then \(\alpha = \begin{pmatrix} 6 \leftarrow 4 \leftarrow 3 \leftarrow 5 \leftarrow 6 \\ 5 \leftarrow 2 \leftarrow 1 \leftarrow 11 \leftarrow 13 \leftarrow 12 \leftarrow 1 \\ 9 \leftarrow 8 \leftarrow 7 \\ 9 \leftarrow 10 \end{pmatrix} .\)

Let \(V_0\) be the multi-set of the lengths of the series in the above expression of \(\alpha\), i.e., \(V_0 = \{4, 6, 2, 1\}\). By applying Lemma 5 to the expression of \(\alpha\), we have
\[ \alpha = \begin{pmatrix} 6 & 4 & 3 & 5 & 6 \\ 5 & 2 & 1 & 11 & 10 \\ 9 & 8 & 7 & 3 & 1 \\ 4 & 2 & 8 & 10 & 3 \\ 6 & 5 & 9 & 9 & 11 \\ 12 & 13 & 12 & 1 & 10 \end{pmatrix} \]

In this case, \( V_1 = \{4, 4, 2, 4\} \) and \( \max V_1 = 4 = \left\lceil \frac{13}{4} \right\rceil = \left\lceil \frac{g(\alpha)}{d(\alpha)} \right\rceil + 1 \). Thus we obtain:

**Theorem 2.** Let \( S_n \) be the semigroup of all singular mappings from \( X \) into \( X \) where \( X = \{1, 2, \ldots, n\} \), and let \( E \) be the set of idempotents of \( S_n \). For each \( \alpha \in S_n \), let \( k(\alpha) \) be the unique positive integer for which \( \alpha \in E^{k(\alpha)} \), \( \alpha \notin E^{k(\alpha)+1} \), and \( g(\alpha) \) the gravity of \( \alpha \) and \( d(\alpha) \) the defect of \( \alpha \). Then \( k(\alpha) = \left\lceil \frac{g(\alpha)}{d(\alpha)} \right\rceil \) or \( \left\lceil \frac{g(\alpha)}{d(\alpha)} \right\rceil + 1 \), and equals to \( \left\lceil \frac{g(\alpha)}{d(\alpha)} \right\rceil \) if \( g(\alpha) = 1 \pmod{d(\alpha)} \), where \( \lfloor x \rfloor \) for any real number \( x \) denotes the least integer \( m \) for which \( m \geq x \).

**References**


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