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The evolution of harmonic mappings with free boundaries

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Abstract: We establish the existence of a global, partially regular weak solution to the evolution problem for harmonic maps with free boundaries on a suitable support hypersurface.

1. Let \((M,g)\) be a \(m\)-dimensional manifold with boundary \(\partial M\) and let \(N\) be a compact \(l\)-dimensional manifold, which for convenience we may regard as isometrically embedded in some Euclidean space \(\mathbb{R}^n\). Also let \(\Sigma\) be a \(k\)-dimensional sub-manifold of \(\mathbb{R}^n\), \(S = \Sigma \cap N\). Finally, let \(u_0 = (u_0^1, \ldots, u_0^n) : M \rightarrow N\) with \(u_0(\partial M) \subset S\) be given.

We study the existence of harmonic maps \(u : M \rightarrow N \hookrightarrow \mathbb{R}^n\) solving the free boundary problem

\[
\begin{align*}
-\Delta u &= \Gamma(u)(\nabla u, \nabla u)\perp T_u(N), \\
u(\partial M) &\subset S,
\end{align*}
\]

\[
\frac{\partial}{\partial n}u \perp T_uS \quad \text{on} \quad \partial M,
\]

where \(n\) denotes a unit normal vector field along \(\partial M\), \(\Delta = \Delta_M\) is the Laplace-Beltrami operator on \(M\), and \(\Gamma\) denotes a bilinear form related to the second fundamental form of the embedding \(N \hookrightarrow \mathbb{R}^n\). Finally, \(T_pN\) denotes the tangent space (in \(\mathbb{R}^n\)) of \(N\) at \(p\), and \(\perp\) means orthogonal (in \(\mathbb{R}^n\)). That is, we look for critical points of the energy

\[
E(u) = \frac{1}{2} \int_M |\nabla u|^2 dM
\]

on the space of maps

\[
H^{1,2}_S(M; N) = \{u \in H^{1,2}(M; \mathbb{R}^n); u(M) \subset N, u(\partial M) \subset S\}.
\]

Here, \(H^{1,2}(M; \mathbb{R}^n)\) is the Sobolev space of \(L^2\)-maps \(u : M \rightarrow \mathbb{R}^n\) with \(\nabla u \in L^2\); the norm \(|\nabla u|^2\) is computed in the metric on \(M\).
As in [13] for a related problem, we approach (1.1)-(1.3) by means of the evolution problem

\begin{equation}
(1.5) \quad u_t - \Delta u = \Gamma(u)(\nabla u, \nabla u) \quad \text{on } M \times \mathbb{R},
\end{equation}

\begin{equation}
(1.6) \quad u(x, t) \subset S, \text{ for } x \in \partial M, t > 0,
\end{equation}

\begin{equation}
(1.7) \quad \frac{\partial}{\partial n} u(x, t) \perp T_{u(x,t)}S, \text{ for } x \in \partial M, t > 0,
\end{equation}

\begin{equation}
(1.8) \quad u(\cdot, 0) = u_0 \quad \text{on } M.
\end{equation}

If \( m = 2 \) this strategy has been sucessfully implemented by Ma Li [10]. See also Dierkes-Hildebrandt-Wohlrab [5] and Hildebrandt-Nitsche [7] for further material on the two-dimensional case. Here we confront the higher dimensional case \( m \geq 3 \).

Assume all data are smooth. For simplicity, we consider only the case

\[ M = B = B_1(0) = \{x \in \mathbb{R}^m; |x| < 1\} \]

Moreover, we make the following assumption about \( \Sigma \), the global "extension" of \( S \) to the ambient Euclidean space:

\begin{equation}
(1.9) \quad \text{There exists a ball } U \subset \mathbb{R}^n \text{ containing } N, \text{ whose boundary } \partial U \text{ intersects } \Sigma \text{ orthogonally in the sense that the normal } \nu_U \text{ to } \partial U \text{ at a point } p \in \Sigma \text{ lies in } T_p\Sigma.
\end{equation}

In addition assume that the nearest neighbor projection \( \pi_\Sigma : U \to \Sigma \cap U \) is well-defined and smooth in \( U \), and

\begin{equation}
(1.10) \quad |D^2 \pi_\Sigma| \cdot \text{diam}(U) < 1/2.
\end{equation}

Let \( R_\Sigma(p) = 2\pi_\Sigma(p) - p \) be the reflection of a point \( p \in U \) in \( \Sigma \). Also we suppose \( \Sigma \) is oriented by a smooth normal frame \( \nu = (\nu_1, \ldots, \nu_{n-1}) \). An example of a configuration \((N, \Sigma)\) satisfying (1.9-10) is \( N = S^{n-1} \subset \mathbb{R}^n, \Sigma = \mathbb{R}^k \times \{0\}, k \leq n-1 \), or a perturbation of \( \mathbb{R}^k \times \{0\} \) by a diffeomorphism \( \Phi = id + \epsilon \tau \), with a smooth map
$\tau : \mathbb{R}^n \rightarrow \mathbb{R}^n$ having compact support, and $|\epsilon| < \epsilon_0 = \epsilon_0(\tau)$. Then we obtain the following result reminiscent of the results in [2] for the evolution of harmonic maps on closed domains, that is, with $\partial M = \emptyset$.

**Theorem 1.1:** Suppose $M = B, N, S, u_0$ are as above and $S$ satisfies conditions (1.9-10). Then there exists a global weak solution $u$ of problem (1.5-8) satisfying the energy inequality

$$
\int_0^T \int_B |u_t|^2 \, dx \, dt + E(u(T)) \leq E(u_0),
$$

for all $T \geq 0$, and smooth off a singular set of codimension $\geq 2$. As $t \rightarrow \infty$ suitably, $u(t)$ converges weakly in $H^{1,2}(B; N)$ to a weak solution $u_\infty$ of (1.1-3) which is smooth off a set of codimension $\geq 2$.

**Remark 1.1:** (i) If the range $u(B \times [0, \infty[)$ lies in a convex neighborhood of a point $p$ on $N$, $u$ is globally smooth and converges uniformly on $\overline{B}$ to a smooth solution $u_\infty$ of (1.1-3) homotopic to $u_0$.

(ii) Conversely, for instance in the case of a sphere as target manifold, it is known that solutions to (1.5) may develop singularities in finite time, see [4], [1].

(iii) A result like Theorem 1.1 should also hold without the hypotheses (1.9-10) on $S$; however, for a general support manifold $S$ - already in the Euclidean case $N = \mathbb{R}^n$ and in contrast to the two-dimensional case - in higher dimensions $m \geq 3$ the problem of boundary regularity for (1.5) poses considerable difficulties and the construction of global, partially regular solutions to (1.5-8) or (2.1), (1.6-8) below is not yet within reach.

(iv) Similar results should hold on a general compact domain with boundary. In fact, much of what follows is true for such general domains and we keep the notation $M$ in that case.

2. Let $U_\delta(N)$ be the $\delta$-tubular neighborhood of $N$ in $\mathbb{R}^n$. We may choose $\delta > 0$ such that $U_\delta(N) \subset U$, see (1.9), and such that the nearest neighbor projection $\pi : U_\delta(N) \rightarrow N$ is well-defined and smooth in $U_\delta(N)$. Let $\chi \in C^\infty_0(\mathbb{R})$ be a non-decreasing function satisfying $\chi(s) = s$ for $0 < s < \frac{\delta^2}{2}$, $\chi(s) = \delta^2$ for $s \geq \delta^2$. 
Following the approach of [2], we approximate (1.5-8) by the following evolution problem for maps with range in $\mathbb{R}^n$:

\begin{equation}
(2.1) \quad u_t - \Delta u + K\chi'\left(\text{dist}^2(u, N)\right) \frac{d}{du} \left( \frac{\text{dist}^2(u, N)}{2} \right) = 0
\end{equation}

in $M \times [0, \infty[$, with boundary and initial conditions (1.6-8). (2.1) is the evolution equation for the functional

\begin{equation}
(2.2) \quad E_K(u) = \frac{1}{2} \int_M \left[ |\nabla u|^2 + K\chi(\text{dist}^2(u, N)) \right] dM.
\end{equation}

for maps $u : M \to \mathbb{R}^n$.

**Lemma 2.1:** Let $u$ be a smooth solution to (2.1), (1.6-8). Then we have

$$
\int_0^T \int_M |u_t|^2 dM \, dt + E_K(u(T)) \leq E_K(u(0)) = E(u_0)
$$

for all $T \geq 0$.

**Proof:** Multiply (2.1) by $u_t$ and integrate by parts. The boundary term vanishes on account of (1.6-7). \hfill \Box

For the following result hypotheses (1.9-10) on $S$ are essential.

**Lemma 2.2:** Suppose $u \in C^1(M \times [0, T]; \mathbb{R}^n)$ is a smooth solution to (2.1), (1.6-8) on $\overline{M} \times [0, T]$; then $u$ and its first spatial derivatives are uniformly bounded and $u$ extends to a smooth solution of (2.1), (1.6-8) on $\overline{M} \times [0, T]$.

**Proof:** The interior estimates easily follow from the energy estimate Lemma 2.1 and the interior regularity estimates for the heat equation; see for instance [9]. To obtain the estimates at the boundary we argue as follows. Note that by the maximum principle for the heat equation and (1.6-7), (1.9) the image of $u$ satisfies $u(x, t) \in U$ for all $(x, t)$, and by (1.10) the reflection of $u$ in $\Sigma$ is defined. Thus, in the special case $M = B$, for $z \in \mathbb{R}^m$, $t \geq 0$ we may let

$$
\tilde{u}(x, t) = \begin{cases} u(x, t), & \text{if } |x| < 1, \\ R_\Sigma(u(x/|x|^2)), & \text{if } |x| > 1. \end{cases}
$$
Then $\tilde{u}$ is of class $C^1$ on $\mathbb{R}^m \times [0, T]$ and satisfies

\begin{equation}
|\tilde{u}_t + A\tilde{u}| \leq \begin{cases} CK & \text{if } |x| < 1 \\ CK + \Gamma_\Sigma(\tilde{u})(\nabla\overline{u}, \nabla\overline{u}) & \text{if } |x| > 1, \end{cases}
\end{equation}

where $A$ is an elliptic operator in divergence form with Lipschitz coefficients, $A = -\Delta$ for $|x| < 1$, and where $\Gamma_\Sigma$ is a bilinear form related to the second fundamental form of $\Sigma \subset \mathbb{R}^n$.

In fact, from

\[
(\tilde{u}_t + A\tilde{u}) \left( \frac{x}{|x|^2} \right) := \left( 2(\partial_t - \Delta)\pi_\Sigma(u) - (\partial_t - \Delta)u \right)(x,t) =
\]

\[
= \left( 2[D\pi_\Sigma(u) - id][(\partial_t - \Delta)u] - 2D^2\pi_\Sigma(u)(\nabla u, \nabla u) \right)(x,t),
\]

we can read off the precise form of $A$ and $\Gamma_\Sigma$. (2.3) is a parabolic system of the type

\[
u_t + Au = f(\cdot, u, \nabla u),
\]

on any ball $B_\rho = B_\rho(0)$, where

\[
|f(\cdot, u, p)| \leq a|p|^2 + b
\]

with constants $a, b \in \mathbb{R}$. Moreover, by (1.10), for $\rho > 1$ sufficiently close to 1 there holds

\[
a \cdot \sup |u| < \lambda,
\]

where $\lambda > 0$ denotes the ellipticity constant of the operator $A$ on $B_\rho$. By the results of [6] for such systems, $\tilde{u}$ is locally Hölder continuous on $B_\rho \times [0, T]$. Higher regularity $|\nabla^2 \tilde{u}| \in L^{p}_{loc}(B_\rho \times [0, T]), |\nabla \tilde{u}| \in L^{4}_{loc}(B_\rho \times [0, T])$ then follows as in [9]. Finally, by [9; p. 593f.] we also obtain uniform bounds for $\nabla \tilde{u}$ in $L^{2p}_{loc}$ and hence $\tilde{u}_t$ and $\nabla^2 \tilde{u}$ in $L^p_{loc}$ for all $p < \infty$. By the Sobolev embedding theorem [9; Lemma II. 3.3] this then implies the desired bound.

\[\square\]
The a-priori bounds of Lemma 2.2 now yield the following global existence result.

**Proposition 2.1:** Under the hypotheses of Theorem 1.1, for any $K \in \mathbb{N}$ there exists a global solution $u = u_K \in C^1(\overline{B} \times [0, \infty[; \mathbb{R}^n)$ to (2.1), (1.6-8). The solution $u$ is smooth in $\overline{B} \times [0, \infty[$ and satisfies the energy inequality Lemma 2.1.

**Proof:** Local existence follows from a fixed point argument as in [13]. For completeness we sketch the argument. Extend $u_0$ to $\mathbb{R}^m$ by letting

$$u_0(x) = R_\Sigma \left( u \left( \frac{x}{|x|^2} \right) \right)$$

for $x \not\in \overline{B}$, and fix $\rho > 0$, $T > 0$ sufficiently small. Let

$$V_\rho(T) = \left\{ u \in C^{1,1/2}(\overline{B}_\rho \times [0, T]; \mathbb{R}^n); u(0) = u_0 \right\},$$

where $C^{1,1/2}(\ldots)$ is the space of functions $u$ which are continuously differentiable in the spatial variable $x$ and uniformly Hölder continuous in time with Hölder exponent $\frac{1}{2}$. A norm is given by the Hölder constant and $||\nabla u||_{L^\infty}$. In [9;p.7f.] this space is introduced as $H^{1,1/2}$.

For $u \in V_\rho(T)$ let $v$ solve

$$v_t + Av = \begin{cases} K \chi'(\text{dist}^2(u, N)) \frac{d}{du} \left( \frac{\text{dist}^2(u, N)}{2} \right), & \text{if } |x| < 1 \\ K \chi'(... \frac{d}{du}(...)) + \Gamma_\Sigma(u)(\nabla u \nabla u), & \text{if } |x| > 1, \end{cases}$$

on $B_\rho \times [0, T]$ with boundary and initial data $u$. By the interior estimates for the heat equation we can bound $v$ and its first and second derivatives in Hölder norm on $\partial B_{1/\rho} \times [0, T]$ in terms of the $C^{1,1/2}$-norm of $u$ on $B_\rho \times [0, T]$ and $u_0$. Define new $C^2$-Dirichlet data by letting

$$w(x, t) = R_\Sigma \left( v \left( \frac{x}{|x|^2}, t \right) \right), x \in \partial B_\rho,$$

and let $\bar{u}$ solve (2.5) with initial data $u_0$ and boundary data $w$. By (2.4) $w$ and $u_0$ are compatible. Moreover, by the linear estimates for the heat equation (see [7; Theorem IV. 9.1]) the map $F : u \mapsto \bar{u}$ is bounded from $C^{1,1/2}(\overline{B}_\rho \times [0, T])$ into the space

$$W_p^{2,1} = \left\{ u \in L^p(B_\rho \times [0, T]); u_t, \nabla^2 u \in L^p \right\}$$
for all $p < \infty$, which for $p > m + 2$ is compactly embedded into $C^{1,\frac{1}{2}}(\overline{B_\rho} \times [0,T])$; see [9; Lemma II.3.3]. Finally, if $T > 0$ is sufficiently small, $F$ maps a convex $C^{1,\frac{1}{2}}$-neighborhood of the function $u(t) \equiv u_0$ to itself. Hence $F$ has a fixed point $u = F(u)$, satisfying (2.5) and the condition

$$u(x, t) = w(x, t) = R_{\Sigma}(v(x/|x|^2, t))$$

on $\partial B_\rho \times [0, T]$. But then also $u_1(x, t) = R_{\Sigma}(u(\frac{x}{|x|^2}, t))$ is a solution of (2.5) in $\{(x, t); 1/\rho < |x| < \rho\}$ with the same initial and boundary data. It follows that $u = u_1$, and thus $u$ satisfies (2.1), (1.6-8). The local solution can be continued globally on account of Lemma 2.2.

\[\square\]

To derive uniform interior estimates independent of $K$ we need the following analogue of the monotonicity formula from [14]. Fix $z_0 = (x_0, t_0) \in \overline{M \times ]0, \infty[}$. Let

$$G(x, t) = \frac{1}{\sqrt{4\pi |t|^m}} \exp\left(-\frac{|x|^2}{4|t|}\right)$$

be the fundamental solution to the heat equation. Then let

$$\Phi_{z_0}(R) = \Phi_{z_0}(R; u, K) = \frac{1}{2} R^2 \int [\nabla u|^2 + K \chi(\text{dist}^2(u, N))] G(\cdot - z_0) \, dx,$$

where we integrate over $B \times \{t_0 - R^2\}$. On a general domain we would need to localize $\Phi$ in coordinate charts via suitable cut-off functions, as in [2].

**Lemma 2.3:** There exist constants depending only on $M$ and $N$ such that for all $z_0 = (x_0, t_0)$ and $0 \leq R \leq R_0 \leq \sqrt{t_0}$ there holds

$$\Phi_{z_0}(R) \leq \exp(c(R_0 - R)) \Phi_{z_0}(R) + c E(u_0)(R_0 - R).$$

**Proof:** At interior points this result was obtained in [2; Lemma 4.2]. At the boundary, for simplicity we present the proof only for a half-space $M = \mathbb{R}^{m}_+$, where

$$\mathbb{R}^{m}_+ = \{x = (x', x_m) \in \mathbb{R}^m; x_m > 0\},$$

and $z_0 = (0, 0)$. (The general case then follows as in [2].) Consider the family of scaled maps

$$u_R(x, t) = u(Rx, R^2 t).$$
Note that $u_R$ satisfies (2.1) with $R^2 K$ instead of $K$, and also satisfies (1.6), (1.7). Moreover,

$$
\Phi_0(R;u,K) = \Phi_0(1;u_R,R^2 K),
$$
whence (at $R = 1$, say)

$$
\frac{d}{dR} \Phi_0(R;u,K) = \frac{d}{dR} \Phi_0(1;u_R,R^2 K)
$$

$$
= \int_{S_+} \left\{ \nabla u \nabla \left( \frac{d}{dR} u_R \right) + K \chi(\text{dist}^2(u,N))
+ K \chi'(... \frac{d}{du} \left( \frac{\text{dist}^2(u,N)}{2} \right) \frac{d}{dR} u_R \right\} G dx,
$$

where $S_+ = \mathbb{R}_+^m \times \{-1\}$. Integrating by parts in the first term, on account of (2.1) and the fact that $\nabla G = \frac{x}{2t} G$,

$$
= \int_{S_+} \frac{|x \cdot \nabla u + 2tu_u|^2}{2t} G dx + \int_{S_+} K \chi(\text{dist}^2(u,N)) G dx \geq 0
$$
as desired. Note that by (1.6-7) no boundary terms appear.

\[ \square \]

Denote by

$$
e_K(u) = \frac{1}{2} \left\{ |\nabla u|^2 + K \chi(\text{dist}^2(u,N)) \right\}
$$

the energy density for the penalized equation. For a point $z_0 = (x_0, t_0) \in \mathbb{R}^m \times \mathbb{R}$, $\rho > 0$ also denote

$$
P_\rho(z_0) = \{z = (x,t); |x - x_0| < \rho, \ t_0 - \rho^2 < t < t_0 \}
$$

the parabolic cylinder of radius $\rho$ centered at $z_0$, $P_\rho = P_\rho(0)$ for brevity, and let

$$
P_\rho^+(z_0) = P_\rho(z_0) \cap \{x_m > 0\},
$$

$$
P_\rho^-(z_0) = P_\rho(z_0) \cap \{x_m < 0\},
$$

respectively.

**Lemma 2.4:** There exists a constant $\epsilon_0 > 0$ depending only on $M$ and $N$ with the following property: If for some $z_0 = (x_0, t_0) \in \overline{M} \times ]0, \infty[$ and $R < \epsilon_0$ the inequality

$$
\Phi_{z_0}(R;u_K,K) < \epsilon_0
$$
is satisfied, then
\[ \sup_{P_{SR}(z_{0})} e_{K}(u_{K}) \leq c(\delta R)^{-2}, \]
with constants \( c \) depending only on \( M \) and \( N \) and \( \delta > 0 \) possibly depending also on \( E(u_{0}) \) and \( \min \{ R, 1 \} \).

**Proof:** The proof for interior points \( x_{0} \in M \) is the same as that of Lemmas 2.4, 4.4 of [2]. We sketch the modifications at a boundary point \( x_{0} \). Again assume for simplicity that \( M = \mathbb{R}_{+}^{m} \) and shift \( z_{0} \) to 0. By reflection we may extend \( u \) to a solution \( \tilde{u} \) of

\[ \tilde{u}_{t} - \Delta \tilde{u} = \left\{ \begin{array}{ll}
K \chi'(\text{dist}^{2}(\tilde{u}, N)) \frac{d}{du} \left( \frac{\text{dist}^{2}(\tilde{u}, N)}{2} \right), & \text{if } x_{m} > 0 \\
K \chi'(\ldots) \frac{d}{du}(\ldots) + \Gamma_{\Sigma}(\tilde{u})(\nabla \tilde{u}, \nabla \tilde{u}), & \text{if } x_{m} < 0
\end{array} \right. \]
on a full neighborhood of \( x_{0} \). Scaling as in [2; p. 92], we obtain a solution \( v \) of problem (2.6) for some \( \overline{K} = \frac{K}{e_{0}} \) on \( P_{1} \), satisfying
\[ e_{\overline{K}}(v) \leq 4 \]
and
\[ e_{\overline{K}}(v)(0) = 1. \]
Moreover, we have the differential inequality
\[ (\partial_{t} - \Delta)e_{K}(v) + |\nabla^{2}v|^{2} \leq Ce_{K}(v), \]
separately in \( P_{1}^{+} \) and \( P_{1}^{-} \). (The proof of this Bochner-type estimate can be conveyed very easily from [2; p. 90].) Let us for brevity write \( e_{\overline{K}}(v) = e(v) \) in the sequel. Our aim is to extend (2.7) to \( P_{1} \).

Due to the structure of (2.6), \( \Delta e(v) \) may have a singular component on the hypersurface \( \{x_{m} = 0\} \) - in our old coordinates. As in [13], we may control this component in the following way.

Given \( \varphi \in C_{0}^{\infty}(B), -1 < t < 0 \), we have
\[ -\int \Delta e(v) \varphi^{2} dx = \int_{\{x_{m} = 0\}} [\partial_{x_{m}} e(v)]^{+} \varphi^{2} dx' + 2 \int \nabla e(v) \nabla \varphi \varphi dx', \]
where \( \int \ldots \) denotes integration over \( B \times \{t\} \), and where we denote
\[ [f(x', 0)]^{+} = \lim_{x_{m} \nearrow 0} f(x', x_{m}) - \lim_{x_{m} \searrow 0} f(x', x_{m}) \]
for any function $f$.

To estimate the boundary integral we decompose

$$
[\partial_{x_{m}}e(v)]^{+}_{-} = \frac{1}{2} [\partial_{x_{m}}(|\nabla v|^{2})]^{+}_{-} + \frac{K}{2} [\partial_{x_{m}}\chi(\text{dist}^{2}(v,N))]^{+}_{-}
$$

$$
= \frac{1}{2} [\partial_{x_{m}}(|\nabla v|^{2})]^{+}_{-}
$$

$$
= [\partial_{x_{m}}^{2}v \partial_{x_{m}}v]^{+}_{-} + [\partial_{x_{m}}(\nabla_{x'}v) \nabla_{x'}v]^{+}_{-}
$$

$$
= [\Delta_{x'}v \partial_{x_{m}}v]^{+}_{-} - 2[\Delta_{x'}v \partial_{x_{m}}v]^{+}_{-} + [\nabla_{x'} \cdot (\partial_{x_{m}}v \nabla_{x'}v)]^{+}_{-}.
$$

But by (1.6), (1.7)

$$
\partial_{x_{m}}v \nabla_{x'}v = 0.
$$

Hence, and on account of (2.6), (1.6), we have

$$
[\partial_{x_{m}}e(v)]^{+}_{-} = \langle \Gamma_{\Sigma}(v)(\nabla v, \nabla v), \partial_{x_{m}}v \rangle - 2[\Delta_{x'}v \partial_{x_{m}}v]^{+}_{-},
$$

where for clarity we now denote $<\cdot,\cdot>$ the scalar product in $\mathbb{R}^{n}$. Using the normal frame $\nu = (\nu_{1}, \ldots, \nu_{n-k})$ for $\Sigma$, the last term by (1.7) may be more conveniently written

$$
\Delta_{x'}v \partial_{x_{m}}v = \sum_{j} \langle \Delta_{x'}v, \nu_{j}(v) \rangle \langle \nu_{j}(v), \partial_{x_{m}}v \rangle
$$

$$
= - \sum_{j} \langle \nabla_{x'}v, \nabla_{x'}(\nu_{j}(v)) \rangle \langle \nu_{j}(v), \partial_{x_{m}}v \rangle.
$$

Smoothly extend $\nu_{j}$ to $\mathbb{R}^{n}$. Then by the divergence theorem

$$
\int_{\{x_{m}=0\}} [\partial_{x_{m}}e(v)]^{+}_{-} \varphi^{2} dx' = \int_{P_{1}^{+}} \text{div} \left( \langle \Gamma_{\Sigma}(v)(\nabla v, \nabla v), \nabla v \rangle \varphi^{2} \right) dx
$$

$$
\mp \sum_{j} \int_{P_{1}^{+}} \text{div} \left( \langle \nabla_{x'}v, \nabla_{x'}(\nu_{j}(v)) \rangle \langle \nu_{j}(v), \nabla v \rangle \varphi^{2} \right) dx
$$

$$
\leq C \int_{P_{1}} (|\nabla^{2}v| |\nabla v|^{2} + |\nabla v|^{4}) \varphi^{2} dx + C \int_{P_{1}} |\nabla v|^{3} |\nabla \varphi| |\varphi| dx
$$

$$
\leq \epsilon \int_{P_{1}} |\nabla^{2}v|^{2} \varphi^{2} dx + C(\epsilon) \int_{P_{1}} |\nabla v|^{4} \varphi^{2} dx
$$

$$
+ C(\epsilon) \int_{P_{1}} |\nabla v|^{2} |\nabla \varphi|^{2} dx,
$$
and choosing \( \epsilon > 0 \) sufficiently small - it follows that the inequality (2.7) - up to a factor - holds on \( P_1 \) in the distribution sense. But then the remainder of the proof of [2] applies also in this case.

As in [2], we may now pass to the limit \( K \to \infty \). Let \( u_K \) be a sequence of smooth solutions to (2.1), (1.6-8). We may assume that \( u_K \) converges weakly to \( u \) in the sense

\[
\nabla u_K \rightharpoonup \nabla u \quad \text{weakly in } L^\infty([0, \infty); L^2(M)),
\]

\[
\frac{\partial}{\partial t} u_K \rightharpoonup \frac{\partial}{\partial t} u \quad \text{weakly in } L^2(M \times [0, \infty]),
\]

\[
u_K \rightharpoonup u \quad \text{strongly in } L^2_{loc}(M \times [0, \infty]),
\]

and almost everywhere, where \( u : \overline{M} \times [0, \infty[ \to N \).

**Proposition 2.2:** The limit \( u \) weakly solves problem (1.5-8). Moreover, \( u \) is smooth and solves (1.5) classically on a dense relatively open set \( Q_0 \subset \overline{M} \times [0, \infty[ \) whose complement \( Q' \) has locally finite \((m - 2)\)-dimensional Hausdorff measure on each time slice \( \overline{M} \times \{t = \text{const.}\} \). Moreover, \( u \) satisfies the energy inequality

\[
\int_0^T \int_M |u_t|^2 \, dM \, dt + E(u(T)) \leq E(u_0),
\]

for all \( T > 0 \). Finally, as \( t \to \infty \) suitably, a sequence \( u(\cdot, t) \) converges weakly in \( H^{1,2}(M; \mathbb{R}) \) to a solution \( u_\infty \) of (1.1-3) with \( E(u_\infty) \leq E(u_0) \) and smooth away from a closed set \( Q'' \) of finite \((m - 2)\)-dimensional Hausdorff measure.

**Proof:** All proofs except (1.3), (1.7) are identical with those of [12; Theorem 6.1], resp. [2; Theorem 1.5] in the case of harmonic maps on domains without boundary. See [3] for an estimate of \( H^{m-2}(Q' \cap \{t = \text{const.}\}) \). To see (1.3), (1.7) in the case of a half-plane we extend \( u_K \) by reflection to solutions \( \tilde{u}_K \) of equations (2.6), converging weakly locally to a function \( \tilde{u} \). On \( Q_0 \), as in [2; p. 94], we have \( C^1 \)-convergence \( u_K \to u \), and (1.7) holds on \( Q_0 \). Moreover, there holds \( K \cdot \text{dist} (u, N) \to \lambda \) weakly in \( L^2_{loc}(Q_0) \), whence

\[
\tilde{u}_t - \Delta \tilde{u} \in L^2_{loc}(Q_0).
\]
Now let $\varphi$ be an arbitrary testing function and let $\eta \in H^{1,\infty}, 0 \leq \eta \leq 1, \eta = 0$ in a neighborhood of $Q'$, as in [2; p. 95]. Multiplying (2.7) by $\varphi \eta$, we obtain that

$$
\int_{0}^{\infty} \int_{\mathbb{R}^m} (\tilde{u}_t - \Delta \tilde{u}) \varphi \eta \, dx \, dt = \int \int \{\tilde{u}_t \varphi + \nabla \tilde{u} \nabla \varphi\} \eta \, dx \, dt + F,
$$

where

$$
|F| \leq \int |\nabla u| |\nabla \eta| |\varphi| \, dx \, dt \leq C(\eta) \left( \int_{\text{supp}(\nabla \eta)} |\nabla u|^2 \varphi^2 \, dx \, dt \right)^{1/2}.
$$

As in [2] we may choose a sequence of maps $\eta$ as above with a uniform constant $C(\eta) = C$ such that $\eta \to 1$ almost everywhere and $(\text{supp}(\nabla \eta)) \to 0$ in measure. By absolute continuity of the Lebesgue integral, thus $F \to 0$, and (1.7) also holds in the distribution sense. The proof of (1.3) is similar.

\[ \square \]

Theorem 1.1 is an immediate consequence of Proposition 2.2. Remark 1.1 follows by adapting the argument of [8] to our problem. Since this technique is by now well-known we may omit the details.

REFERENCES


