Linear evolution equations in a reflexive Banach space

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§1. INTRODUCTION

In this paper we discuss the construction of an evolution system associated with the well posed problem in the sense of Hadamard for the time-dependent differential equation in a Banach space $X$

\[(DE) \begin{cases} \frac{d}{dt}u(t) = A(t)u(t) \text{ for } t \in [s, T] \\ u(s) = x, \end{cases}\]

where $s \in [0, T)$, $u(\cdot)$ stands for an $X$-valued unknown function on the interval $[s, T]$ and $\{A(t) : t \in [0, T]\}$ is a given family of linear operators in $X$.

Assume for the moment that there exist a dense subspace $Y$ of $X$ and an injective bounded linear operator $C_1$ on $X$ such that $Y \subset D(A(t))$ for $t \in [0, T]$ and the following conditions hold:

1) For $s \in [0, T]$ and $x \in C_1(Y)$, there exists a unique solution $u(t; s, x)$ such that $u(t; s, x) \in Y$ for $t \in [s, T]$.

2) For $x \in C_1(Y)$, $u(t; s, x)$ is continuous for $0 \leq s \leq t \leq T$.

3) If $\{u(t; s, x_n)\}$ is a sequence of solutions with $x_n \to 0$ in the $C_1^{-1}$-graph norm as $n \to \infty$ then $u(t; s, x_n)$ converges to zero uniformly with respect to $t$ and $s$.

Here we note that in the special case where $A(t) = A$, $s = 0$, $Y = D(A)$ and $C_1 = R(c : A)^n$ ($n \in \mathbb{N} \cup \{0\}$ and $c \in \rho(A)$), the concept of the above well posed problem is equal to that of the well posed problem in the sense of Hadamard in the autonomous case (see [5,8]), which several authors [1,4,9,10,11,12] recently have studied via the theory of integrated semigroups or $C$-semigroups.
Now we turn to the above well posed problem. We define a linear subspace $D(s)$ of $X$ and a linear operator $U(t, s)$ on $D(s)$ by

\[
\begin{align*}
D(s) &= \{ x \in X : \text{(DE)}_* \text{ has a unique solution } u(t; s, x) \} \\
U(t, s)x &= u(t; s, x) \text{ for } x \in D(s).
\end{align*}
\]

Then, from the uniqueness of the solutions it follows that $U(t, s) : D(s) \to D(t)$ and $U(t, r)U(r, s) = U(t, s)$ on $D(s)$ for $0 \leq s \leq r \leq t \leq T$. Formally, the two parameter family $\{U(t, s) : 0 \leq s \leq t \leq T\}$ may have the properties

(1.1) \quad \frac{\partial}{\partial t}U(t, s) = A(t)U(t, s)

(this property is useful to show the existence of the solutions),

(1.2) \quad \frac{\partial}{\partial s}U(t, s) = -U(t, s)A(s)

(this property is useful to show the uniqueness of the solutions).

We define $\{V_1(t, s) : 0 \leq s \leq t \leq T\}$ by

\[
V_1(t, s)y = U(t, s)C_1y \quad (= u(t; s, C_1y)) \quad \text{for } y \in Y.
\]

Since $Y$ is dense in $X$ one can see by the condition 3) that $V_1(t, s)$ is extended to a bounded linear operator on $X$, which we denote by the same symbol. Then, the two parameter family $\{V_1(t, s) : 0 \leq s \leq t \leq T\}$ has the properties

(i) for $x \in X$, $(t, s) \to V_1(t, s)x$ is continuous for $0 \leq s \leq t \leq T$,

(ii) $V_1(t, s)(Y) \subset Y$ for $0 \leq s \leq t \leq T$,

(iii) $\frac{\partial}{\partial t}V_1(t, s)y = A(t)V_1(t, s)y$ for $y \in Y$, and $V_1(s, s) = C_1$.

We also consider the following important property to show the uniqueness of the solutions:

(iv) $\frac{\partial}{\partial s}V_2(t, s)y = -V_2(t, s)A(s)y$ for $y \in Y$, and $V_2(s, s) = C_2$.

Multiplying (1.2) by the injective bounded linear operator $C_2$ from the left-hand side, and then defining $V_2(t, s)$ by $C_2U(t, s)$ we obtain the property (iv).
Moreover, the following relation between $V_1(t, s)$ and $V_2(t, s)$ holds:

\[(v) \quad C_2 V_1(t, s) = V_2(t, s) C_1 \quad \text{for} \quad 0 \leq s \leq t \leq T.\]

In §2 we will construct a pair of evolution systems $\{V_1(t, s)\}, \{V_2(t, s)\}$ having the properties (i) - (v) in order to investigate the well posed problem in the sense of Hadamard for the time-dependent differential equation $(DE)_\varepsilon$.

As an application we also consider the second order differential equation in a reflexive Banach space $X$

\[ (DE)_2^s \begin{cases} u''(t) = Au(t) + B(t)u(t) & \text{for} \quad t \in [s, T] \\ u(s) = x, \ u'(s) = y, \end{cases} \]

where $A$ is the infinitesimal generator of a cosine family and $\{B(t) : t \in [0, T]\}$ is a given family of linear operators in $X$.

\section{Construction of evolution systems}

Let $X$ and $Y$ be Banach spaces with norm $\| \cdot \|$ and $\| \cdot \|_Y$ respectively. We write $B(Y, X)$ for the set of all bounded linear operators on $Y$ to $X$ and denote $B(X, X)$ by $B(X)$. For each $i = 1, 2$, let $C_i$ be an injective operator in $B(X)$.

Throughout this paper we will assume that

\begin{itemize}
  \item [(H1)] $Y$ is reflexive,
  \item [(H2)] $Y$ is densely and continuously imbedded in $X$, that is, $Y$ is a dense subspace of $X$ and there is a constant $L$ such that $\|y\| \leq L\|y\|_Y$ for $y \in Y$,
  \item [(H3)] $C_1(Y) \subset Y$ and $C_1(Y)$ is $\| \cdot \|_Y$-dense in $Y$.
\end{itemize}

We will make the following assumptions on a family $\{A(t) : t \in [0, T]\}$ of closed linear operators in $X$:

\begin{itemize}
  \item [(A1)] There are constants $M_1 \geq 0$ and $\omega_1 \geq 0$ such that
    \[ (\omega_1, \infty) \subset \rho(A(t)) \quad \text{for} \quad t \in [0, T] \quad \text{and} \quad \left\| \lambda^n \left( \prod_{i=1}^m R(\lambda : A(t_i)) \right) C_1 \right\| \leq M_1 \quad \text{for} \quad \lambda > \omega_1 \]
\end{itemize}
and every finite sequence \(\{t_i\}_{i=1}^{m}\) such that \(0 \leq t_1 \leq \cdots \leq t_m \leq T\) and \(m\) with \(0 \leq m/\lambda \leq T\).

\((A_2)\) There are constants \(M_2 \geq 0\) and \(\omega_2 \geq \omega_1\) such that
\[
\left(\prod_{i=1}^{m} R(\lambda : A(t_i))\right) C_1(Y) \subset Y \quad \text{and}
\]
\[
\|\lambda^m \left(\prod_{i=1}^{m} R(\lambda : A(t_i))\right) C_1\|_Y \leq M_2 \quad \text{for } \lambda > \omega_2
\]
and every finite sequence \(\{t_i\}_{i=1}^{m}\) such that \(0 \leq t_1 \leq \cdots \leq t_m \leq T\) and \(m\) with \(0 \leq m/\lambda \leq T\).

\((A_3)\) There are constants \(M_3 \geq 0\) and \(\omega_3 \geq \omega_1\) such that
\[
\|C_2 \left(\lambda^m \left(\prod_{i=1}^{m} R(\lambda : A(t_i))\right)\right)\|_Y \leq M_3 \quad \text{for } \lambda > \omega_3
\]
and every finite sequence \(\{t_i\}_{i=1}^{m}\) such that \(0 \leq t_1 \leq \cdots \leq t_m \leq T\) and \(m\) with \(0 \leq m/\lambda \leq T\).

\((A_4)\) For \(t \in [0, T]\), \(D(A(t)) \supset Y\) and \(D(C_1^{-1}A(t)C_1) \supset Y\), and the function \(t \to A(t)\) is continuous in the \(B(Y,X)\) norm \(\| \cdot \|_{Y \to X}\) and \(M_4 = \sup\{\|C_1^{-1}A(t)C_1\|_{Y \to X} : t \in [0, T]\} < \infty\).

The main result of this paper is given by

**Theorem 2.1.** If the family \(\{A(t) : t \in [0, T]\}\) of closed linear operators in \(X\) satisfies \((A_1)-(A_4)\) then there exists a unique pair \((\{V_1(t,s)\}, \{V_2(t,s)\})\) of strongly continuous families of bounded linear operators defined on the triangle \(\Delta = \{(t,s) : 0 \leq s \leq t \leq T\}\) with the following properties:

(a) For \(i = 1,2\), \(V_i(s,s) = C_i\) on \([0, T]\) and \(C_2V_1(t,s) = V_2(t,s)C_1\) on \(\Delta\).

(b) \(V_1(t,s)(Y) \subset Y\) for \(0 \leq s \leq t \leq T\).

(c) For \(y \in Y\) and \(y^* \in Y^*\), \((t,s) \to \langle y^*, V_1(t,s)y \rangle\) is continuous on \(\Delta\).

(d) \[\langle x^*, V_1(t,s)y - V_1(r,s)y \rangle = \int_r^t \langle x^*, A(\tau)V_1(\tau,s)y \rangle d\tau\]
for $y \in Y, x^* \in X^*$ and $0 \leq s \leq r \leq t \leq T$. In particular, $(\partial/\partial t)V_1(t, s)y$ exists for almost every $t \in [s, T]$ and equals $A(t)V_1(t, s)y$.

(e) \[ V_2(t, r)y - V_2(t, s)y = -\int_s^r V_2(t, \tau)A(\tau)y d\tau \]

for $y \in Y$ and $0 \leq s \leq r \leq t \leq T$.

Remarks. 1) In the case where $A(t) \subset C_1^{-1}A(t)C_1$ for $t \in [0, T]$, the condition $(A_3)$ is automatically satisfied with $C_2 = C_1$ if the condition $(A_1)$ is satisfied.

2) In the case where $C_1 = C_2 = I$ (the identity operator on $X$), Theorem 2.1 is [6, Theorem 5.1].

Before proving Theorem 2.1 we prepare three lemmas. Let $s \in [0, T)$ and let $\lambda > 0$ be such that $\lambda \omega_3 < 1$. Set

$$P_{\lambda, k}(s) = \prod_{i=1}^{k} J_{\lambda}(s+i\lambda) \quad for \quad 0 \leq k \leq [(T-s)/\lambda],$$

where $[\ ]$ denotes the Gaussian bracket and $J_{\lambda}(t) = (1-\lambda A(t))^{-1} = \lambda^{-1}R(\lambda^{-1}: A(t))$ for $t \in [0, T]$.

Now we define $A_{k,l}$ and $B_{k,l}$ by

$$\begin{cases}
A_{k,l}x = P_{\lambda, k}(s)C_1x - P_{\mu, l}(s)C_1x \quad for \quad x \in X, \\
B_{k,l}y = \mu(\lambda A(s+k\lambda) - \lambda A(s+l\mu))P_{\mu, l}(s)C_1y \quad for \quad y \in Y.
\end{cases}$$

Here we note by the conditions $(A_2)$ and $(A_4)$ that $B_{k,l}$ is well defined because $P_{\mu, l}(s)C_1(Y) \subset Y \subset D(A(t))$ for $t \in [0, T]$.

Using the resolvent identity we obtain by a standard argument

**LEMMA 2.2.** Let $s \in [0, T)$ and $\lambda, \mu > 0$ be such that $\lambda \omega_3, \mu \omega_3 < 1$. Then, for $y \in Y$ we have

$$A_{k,l}y = J_{\mu}(s+k\lambda)(\alpha A_{k-1,l-1}y + \beta A_{k,l-1}y + B_{k,l}y)$$
for $0 \leq k \leq [(T - s)/\lambda]$ and $0 \leq l \leq [(T - s)/\mu]$, where $\alpha = \frac{k}{\lambda}$ and $\beta = \frac{\lambda - \mu}{\lambda}$.

Let $s \in [0, T)$ and $\lambda, \mu > 0$ be such that $\lambda \omega_3, \mu \omega_3 < 1$. Let $k$ and $j$ be nonnegative integers. We denote by $H(m, k)$ the set of all operators $Q$ obtained by multiplying $k$ operators $J_\mu(t_i) (i = 1, \cdots, k)$ in the family $\{J_\mu(s + i\lambda) : i = 1, \cdots, m\}$ such that $Q = \prod_{i=1}^{k} J_\mu(t_i)$ for $0 \leq s + \lambda \leq t_1 \leq \cdots \leq t_k \leq s + m\lambda \leq T$; $H(m, 0) = H(0, k) = \{\text{the identity operator}\}$. By $H(m, k, j)$ we denote the set of all sums of $j$ operators $Q_i (i = 1, \cdots, j)$ in $H(m, k)$, where in $j$ operators $Q_1, \cdots, Q_j$, same operators are allowed to appear repeatedly.

Using the relation (2.1) and then taking account of the definition $H(\cdot, \cdot, \cdot)$ we obtain by a routine calculation the following crucial estimate:

**Lemma 2.3.** Let $s \in [0, T)$ and let $\lambda, \mu > 0$ such that $\lambda \omega_3, \mu \omega_3 < 1$. Then, for $y \in Y$ we have

$$
A_{m,n}y \in \sum_{i=0}^{(m-1)\wedge n} \alpha^i \beta^{n-i} H\left(m, n, \binom{n}{i}\right) A_{m-i,0}y + \sum_{i=m}^{n} \alpha^{m} \beta^{i-m} H\left(m,i, \binom{i-1}{m-1}\right) A_{0,n-i}y + \sum_{j=0}^{n-1} \sum_{i=0}^{(m-1)\wedge j} \alpha^i \beta^{j-i} H\left(m,j+1, \binom{j}{i}\right) B_{m-i,n-j}y
$$

for $0 \leq m \leq [(T - s)/\lambda]$ and $0 \leq n \leq [(T - s)/\mu]$, where $\alpha = \frac{k}{\lambda}$, $\beta = \frac{\lambda - \mu}{\lambda}$, $l \wedge k = \min(l, k)$ and $\binom{j}{i}$ is the binomial coefficient.

**Lemma 2.4.** (I) Let $s \in [0, T)$ and let $\lambda > \mu > 0$ be such that $\lambda \omega_3 < 1$. Then, there exists a positive constant $K$, depending only on $M_i (i = 1, 2, 3, 4)$, such that

$$
\| C_2 P_{\lambda,m}(s) C_1 y - C_2 P_{\mu,n}(s) C_1 y \| \leq K \| y \| Y \left\{ 2((n\mu - m\lambda)^2 + T(\lambda - \mu))^{1/2} + T\left(\rho(\|n\mu - m\lambda\|) + \rho(\delta)\right) + \frac{T^2}{\delta^2} \rho(T)(\lambda - \mu) \right\}
$$

for $1 \leq m \leq [(T - s)/\lambda]$, $1 \leq n \leq [(T - s)/\mu]$, $y \in Y$ and $\delta > 0$, where $\rho(r) = \sup\{\|A(t) - A(s)\|_{Y \rightarrow X} : t, s \in [0, T], |t - s| \leq r\}$ for $r \geq 0$. 
(II) Let $0 \leq r \leq s \leq T$ and let $\lambda > 0$ be such that $\lambda \omega_3 < 1$. Then there exists a positive constant $K$, depending only on $M_i(i = 2, 3)$, such that

$$(2.3) \quad \|C_2P_{\lambda,m}(s)C_1y - C_2P_{\lambda,m}(r)C_1y\| \leq KT\|y\|_Y \rho(s - r)$$

for $1 \leq m \leq [(T - s)/\lambda]$ and $y \in Y$.

**Proof:** By virtue of Lemma 2.3 we can show (2.2) in the same way as in the proof of [2, Theorem 2.1]. To prove (2.3), let $0 \leq r \leq s \leq T$ and let $\lambda > 0$ be such that $\lambda \omega_3 < 1$. For $1 \leq k \leq [(T - s)/\lambda]$ we define $A_k$ and $B_k$ by

$$\begin{align*}
A_kx &= P_{\lambda,k}(s)C_1x - P_{\lambda,k}(r)C_1x \quad \text{for } x \in X, \\
B_ky &= \lambda(A(s + k\lambda) - A(r + k\lambda))P_{\lambda,k}(s)C_1y \quad \text{for } y \in Y.
\end{align*}$$

Then, by a simple computation we have

$$A_ky = (J_\lambda(s + k\lambda) - J_\lambda(r + k\lambda))P_{\lambda,k-1}(s)C_1y + J_\lambda(r + k\lambda)(P_{\lambda,k-1}(s)C_1y - P_{\lambda,k-1}(r)C_1y)$$

$$= J_\lambda(r + k\lambda)(A_{k-1}y + B_ky)$$

for $y \in Y$. By solving this we find

$$A_my = \sum_{i=1}^{m} \left( \prod_{k=i}^{m} J_\lambda(r + k\lambda) \right)B_iy$$

for $y \in Y$ and $1 \leq m \leq [(T - s)/\lambda]$. Therefore, we obtain the desired estimate (2.3) by the conditions $(A_2)$ and $(A_3)$. Q.E.D.

**Proof of Theorem 2.1:** Let $s, r \in [0, T)$ and let $\lambda > \mu > 0$ be such that $\lambda \omega_3 < 1$. Let $m$ and $n$ be integers such that $0 \leq s + m\lambda, r + n\mu \leq T$ and let $y \in Y$. If $s \leq r$ then $0 \leq s + n\mu \leq T$, so that $P_{\mu,n}(s)$ is well defined. Similarly, $P_{\lambda,m}(r)$ is well defined if $s \geq r$. Therefore, $C_2P_{\lambda,m}(s)C_1y - C_2P_{\mu,n}(r)C_1y$ can be written as

$$\begin{align*}
\begin{cases}
C_2P_{\lambda,m}(s)C_1y - C_2P_{\mu,n}(s)C_1y + (C_2P_{\mu,n}(s)C_1y - C_2P_{\mu,n}(r)C_1y) & \text{if } s \leq r \\
C_2P_{\lambda,m}(s)C_1y - C_2P_{\lambda,m}(r)C_1y + (C_2P_{\lambda,m}(r)C_1y - C_2P_{\mu,n}(r)C_1y) & \text{if } s \geq r.
\end{cases}
\end{align*}$$

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Applying Lemma 2.4 to this we see that there exists a positive constant $K$, depending only on $M_i (i = 1, 2, 3, 4)$, such that

$$
\|C_2P_{\lambda,m}(s)C_1y - C_2P_{\mu,n}(r)C_1y\|
\leq K\|y\|\left\{2((n\mu - m\lambda)^2 + T(\lambda - \mu))^{1/2} + T(\rho(|n\mu - m\lambda|) + \rho(\delta)) + \frac{T^2}{\delta^2}\rho(T)(\lambda - \mu)\right\}
$$

for $\delta > 0$ and $y \in Y$. Since $C_1(Y)$ is dense in $X$ and $\|C_2P_{\lambda,n}(s_n)\| \leq M_3$ for $n \geq 1$ it follows that

(2.4) $$V_2(t,s)x = \lim_{n\to\infty}C_2\left(\prod_{i=1}^{n}J_{\lambda_n}(s_n+i\lambda_n)\right)x$$

exists for $x \in X$ if $\{s_n\}$ is a sequence of nonnegative numbers with $\lim_{n\to\infty}s_n = s$ and $\{\lambda_n\}$ is a sequence such that $0 \leq s_n + n\lambda_n \leq T$ and $s_n + n\lambda_n \to t - s > 0$ as $n \to \infty$. Here we have used the fact that $\rho(\delta) \to 0$ as $\delta \to 0+$. We note that the limit is independent of $\{s_n\}$ and $\{\lambda_n\}$.

Let $\{s_n\}$ be a sequence of nonnegative numbers such that $\lim_{n\to\infty}s_n = s$ and let $\{\lambda_n\}$ be a sequence such that $0 \leq s_n + n\lambda_n \leq T$ and $s_n + n\lambda_n \to t - s > 0$ as $n \to \infty$. We then define $V_1^{(n)}(t,s)$ on $X$ by

$$V_1^{(n)}(t,s) = \begin{cases} 
C_1 & \text{for } t = s, \\
(\prod_{i=1}^{n}J_{\lambda_n}(s_n+i\lambda_n))C_1 & \text{for } s < t.
\end{cases}$$

Then, by the condition $(A_2)$ we have

$$V_1^{(n)}(t,s)(Y) \subset Y \text{ and } \|V_1^{(n)}(t,s)\|_Y \leq M_2 \text{ for } 0 \leq s \leq t \leq T \text{ and } n \geq 1.$$

We now show that for $y \in Y$ and $y^* \in Y^*$, $\langle y^*, V_1^{(n)}(t,s)y \rangle$ is convergent. Let $\{n_k\}$ be any subsequence of $\{n\}$. Since $Y$ is reflexive there exists a subsequence $\{n_k^*\}$ of $\{n_k\}$ and $y(t,s) \in Y$, depending upon $\{n_k^*\}$, such that

$$\langle y^*, V_1^{(n_k^*)}(t,s)y \rangle \to \langle y^*, y(t,s) \rangle$$

for $y^* \in Y^*$ as $n \to \infty$. In particular, for $x^* \in X^*$ we have

$$\langle C_2^*x^*, V_1^{(n_k^*)}(t,s)y \rangle \to \langle C_2^*x^*, y(t,s) \rangle = \langle x^*, C_2y(t,s) \rangle$$
as \( n \to \infty \), since \( C_{2}^{*}x^{*}|_{Y} \in Y^{*} \). On the other hand, by (2.4) we obtain for \( x^{*} \in X^{*} \),

\[
\langle C_{2}^{*}x^{*}, V_{1}^{(n_k)}(t, s)y \rangle = \langle x^{*}, C_{2}V_1^{(n_k)}(t, s)y \rangle \to \langle x^{*}, V_{2}(t, s)C_{1}y \rangle
\]

as \( n \to \infty \). Hence \( C_{2}y(t, s) = V_{2}(t, s)C_{1}y \), so that \( y(t, s) \) is independent of \( \{n_k\} \). Therefore it is proved that

\[
\lim_{n \to \infty} \langle y^{*}, V_{1}^{(n)}(t, s)y \rangle = \langle y^{*}, C_{2}^{-1}V_{2}(t, s)C_{1}y \rangle
\]

for \( y \in Y \). By this together with the fact that \( x^{*}|_{Y} \in Y^{*} \) we have for \( x^{*} \in X^{*} \),

\[
\langle x^{*}, C_{2}^{-1}V_{2}(t, s)C_{1}y \rangle = \lim_{n \to \infty} \langle x^{*}, V_{1}^{(n)}(t, s)y \rangle \quad \text{for } y \in Y.
\]

Hence

\[
\|C_{2}^{-1}V_{2}(t, s)C_{1}y\| \leq M_{1}\|y\|
\]

for \( y \in Y \) and \( 0 \leq s \leq t \leq T \). Since \( Y \) is dense in \( X \) we see by the closed graph theorem that \( C_{2}^{-1}V_{2}(t, s)C_{1} \in B(X) \) and \( \|C_{2}^{-1}V_{2}(t, s)C_{1}\| \leq M_{1} \) for \( 0 \leq s \leq t \leq T \).

We now define \( V_{1}(t, s) \) on \( X \) by

\[
V_{1}(t, s) = C_{2}^{-1}V_{2}(t, s)C_{1} \quad \text{for } 0 \leq s \leq t \leq T.
\]

Then, it follows from the fact which has been proved above that \( \|V_{1}(t, s)\| \leq M_{1}, V_{1}(t, s)(Y) \subset Y, \|V_{1}(t, s)\|_{Y} \leq M_{2} \) and \( C_{2}V_{1}(t, s) = V_{2}(t, s)C_{1} \) for \( 0 \leq s \leq t \leq T \). Moreover, we have

\[
\lim_{n \to \infty} \langle y^{*}, \left( \prod_{i=1}^{n} J_{\lambda_{n}}(s_{n} + i\lambda_{n}) \right) C_{1}y \rangle = \langle y^{*}, V_{1}(t, s)y \rangle
\]

for \( y \in Y \) and \( y^{*} \in Y^{*} \) if \( \{s_{n}\} \) is a sequence of nonnegative numbers such that \( \lim_{n \to \infty} s_{n} = s \) and \( \{\lambda_{n}\} \) is a sequence such that \( 0 \leq s_{n} + n\lambda_{n} \leq T \) and \( s_{n} + n\lambda_{n} \to t - s > 0 \) as \( n \to \infty \).
To prove that for \( x \in X \), \((t,s) \rightarrow V_1(t,s)x\) is continuous on \( \triangle \), since \( Y \) is dense in \( X \) and \( \| V_1(t,s) \| \leq M_1 \) on \( \triangle \) it suffices to show that

\[
(2.5) \quad \| V_1(t,s)y - V_1(\tau,s)y \| \leq K(t-\tau)\|y\|_Y
\]

for \( y \in Y \) and \( 0 \leq s \leq \tau \leq t \leq T \),

\[
(2.6) \quad \| V_1(t,s+h)y - V_1(t,s)y \| \leq Kh\|y\|_Y
\]

for \( y \in Y \) and \( 0 \leq s \leq s+h \leq t \leq T \).

To prove (2.5), let \( y \in Y \) and \( 0 \leq s \leq \tau \leq t \leq T \) and let \( \lambda > 0 \) be such that \( \lambda \omega_3 < 1 \). If \( n \) and \( m \) be integers such that \( m < n \leq \lfloor (T-s)/\lambda \rfloor \) then

\[
\langle x^*, P_{\lambda,n}(s)C_1y - P_{\lambda,m}(s)C_1y \rangle
\]

\[
= \left\langle x^*, \sum_{k=m}^{n-1} (P_{\lambda,k+1}(s)C_1y - P_{\lambda,k}(s)C_1y) \right\rangle
\]

\[
(2.7) \quad = \left\langle x^*, \lambda \sum_{k=m}^{n-1} A(s+(k+1)\lambda)P_{\lambda,k+1}(s)C_1y \right\rangle \quad \text{for} \quad x^* \in X^*,
\]

from which it follows that

\[
|\langle x^*, P_{\lambda,n}(s)C_1y - P_{\lambda,m}(s)C_1y \rangle|
\]

\[
\leq \|x^*\| \lambda(n-m) \cdot \sup \{\|A(t)\|_{Y \rightarrow X} : t \in [0,T]\} \cdot M_2\|y\|_Y
\]

for \( x^* \in X^* \). Setting \( n = \lfloor (t-s)/\lambda \rfloor \) and \( m = \lfloor (\tau-s)/\lambda \rfloor \), and then letting \( \lambda \rightarrow \infty \) we obtain the desired estimate (2.5).

To prove (2.6) let \( y \in Y \) and \( 0 \leq s < s+h < t \leq T \), and choose a sequence \( \{k(n)\} \) of integers such that \( k(n)h/n \leq t - (s+h) \) and \( k(n)h/n \rightarrow t - (s+h) \) as \( n \rightarrow \infty \). Then, since

\[
(2.8) \quad \sum_{j=1}^{n} \left\{ \left( \prod_{i=j+1}^{n+k(n)} J_{h/n}(s+ihs/n) \right)y - \left( \prod_{i=j}^{n+k(n)} J_{h/n}(s+ihs/n) \right)y \right\}
\]

\[
= -(h/n) \sum_{j=1}^{n} \left( \prod_{i=j}^{n+k(n)} J_{h/n}(s+ihs/n) \right)A(s+jh/n)y,
\]
it follows from the conditions \((A_1)\) and \((A_4)\) that

\[
|\langle x^*, P_{h/n,k(n)}(s+h)C_1y - P_{h/n,n+k(n)}(s)C_1y \rangle| \leq hM_1M_4\|y\|_Y\|x^*\|
\]

for \(x^* \in X^*\). Passing to the limit as \(n \to \infty\) we obtain (2.6).

The strongly continuity of \(V_2(t, s)\) immediately follows from the strongly continuity of \(V_1(t, s)\) and the relation that \(C_2V_1(t, s) = V_2(t, s)C_1\), since \(C_1(X)\) is dense in \(X\) and \(\|V_2(t, s)\| \leq M_3\) on \(\triangle\).

Since \(Y\) is reflexive, using the strongly continuity of \(V_1(t, s)\) together with the facts that \(V_1(t, s)(Y) \subset Y\) and \(\|V_1(t, s)\|_Y \leq M_2\) on \(\triangle\) we see by a standard argument that for \(y \in Y\) and \(y^* \in Y^*\), \((t, s) \to \langle y^*, V_1(t, s)y \rangle\) is continuous for \(0 \leq s \leq t \leq T\).

To prove that \(\{V_1(t, s) : 0 \leq s \leq t \leq T\}\) has the property (d), let \(y \in Y, x^* \in X^*\) and \(0 \leq s \leq r < t \leq T\). Setting \(n = [(t-s)/\lambda]\) and \(m = [(r-s)/\lambda]\) in (2.7) we have

\[
\langle x^*, P_{\lambda,[(t-s)/\lambda]}(s)C_1y - P_{\lambda,[(r-s)/\lambda]}(s)C_1y \rangle
= \sum_{k=[(r-s)/\lambda]}^{[(t-s)/\lambda]-1} \int_{s+k\lambda}^{s+(k+1)\lambda} A(s + ([(\tau - s)/\lambda] + 1)\lambda)P_{\lambda,[(r-s)/\lambda]+1}(s)C_1y \, d\tau
= \int_{s+[(r-s)/\lambda]\lambda}^{(t-s)/\lambda} \int_{s+[(r-s)/\lambda]\lambda}^{(t-s)/\lambda} A(s + ([(\tau - s)/\lambda] + 1)\lambda)x^*, P_{\lambda,[(r-s)/\lambda]+1}(s)C_1y \, d\tau,
\]

where \(\tilde{A}(t)^* : X^* \to Y^*\) denotes the adjoint of the restriction \(\tilde{A}(t)\) of \(A(t)\) to \(Y\). The condition \((A_4)\) implies that \(t \to \tilde{A}(t)^*\) is continuous in the \(B(X^*, Y^*)\) norm; thus passing to the limit as \(\lambda \to \infty\) we see by Lebesgue's convergence theorem that

\[
\langle x^*, V_1(t, s)y - V_1(r, s)y \rangle = \int_r^t \langle \tilde{A}(\tau)^*x^*, V_1(\tau, s)y \rangle \, d\tau.
\]

This shows that the property (d) is satisfied.

We next show that \(\{V_2(t, s) : 0 \leq s \leq t \leq T\}\) has the property (e). Let \(0 \leq s < s + h < t \leq T\) and choose a sequence \(\{k(n)\}\) of integers such that
$k(n)h/n \leq t - (s + h)$ and $k(n)h/n \to t - (s + h)$ as $n \to \infty$. By (2.8) we have

$$C_2 P_{h/n,k(n)}(s + h)y - C_2 P_{h/n,n+k(n)}(s)y$$

$$= - \sum_{j=1}^{n} \int_{+(j-1)h/n}^{s+jh/n} C_2 P_{h/n,n+k(n)-j+1}(s+(j-1)h/n)A(s+jh/n)y \, dr$$

$$= - \int_{s}^{s+h} C_2 P_{h/n,n+k(n)-r(n)}(s+r(n)h/n)A(s+(r(n)+1)h/n)y \, dr$$

for $y \in Y$, where $r(n) = [(r-s)/(h/n)]$. Letting $n \to \infty$ in this equality we see that the property (e) is satisfied.

Suppose that $\{W_1(t,s), W_2(t,s)\}$ is a pair of strongly continuous families of bounded linear operators defined on the triangle $\triangle$ with the properties (a) - (e). Then, by the properties (d) and (e) we see that for $y \in Y$, the function $r \to V_2(t,r)W_1(r,s)y$ is Lipschitz continuous and $\partial/\partial r) V_2(t,r)W_1(r,s)y = 0$ for almost every $r \in [s, T]$. Integrating this from $s$ to $t$ we obtain $C_2 W_1(t,s)y = V_2(t,s)C_1 y$ for $y \in Y$. By the property (a), $W_2(t,s)$ is equal to $V_2(t,s)$ on the dense subspace $C_1(Y)$ of $X$, so that $\{V_1(t,s), V_2(t,s)\}$ is the only pair of strongly continuous families of bounded linear operators defined on the triangle $\triangle$ with the properties (a) - (e). Q.E.D.

**Definition 2.1.** A function $u(\cdot; s, x)$ on $[s, T]$ is a strong solution of $(DE)_s$ if

(i) $u(\cdot; s, x) \in A^{1,1}(s, T; X)$,

(ii) $u(\cdot; s, x)$ satisfies $(DE)_s$ almost everywhere.

Here we denote by $A^{k,p}(a, b; X)$ the space of all absolutely continuous functions $u : [a, b] \to X$ for which $d^ju/dt^j$ exist (and are defined almost everywhere) for $j = 1, 2, \cdots, k$ such that $d^ju/dt^j, j = 1, 2, \cdots, k-1$, are all absolutely continuous and $d^k u/dt^k \in L^p(a, b; X)$.

Existence and uniqueness of the strong solutions of the time-dependent differential equation $(DE)_s$ is provided by
THEOREM 2.5. If the family \{A(t) : t \in [0, T]\} of closed linear operators in X satisfies the conditions (A_1) - (A_4) then, for every initial data \(x \in C_1(Y)\) the \((DE)_s\) has a unique strong solution satisfying \(u(t; s, x) \in Y\) for \(t \in [s, T]\) and \(\sup\{||u(t; s, x)||_Y : t \in [s, T]\} < \infty\).

PROOF: By Theorem 2.1 there exists a unique pair \((\{V_1(t, s)\}, \{V_2(t, s)\})\) of strongly continuous families of bounded linear operators defined on the triangle \(\Delta = \{(t, s) : 0 \leq s \leq t \leq T\}\) with the properties (a) - (e). Let \(x \in C_1(Y)\) and set \(u(t; s, x) = V_1(t, s)C_1^{-1}x\) for \(0 \leq s \leq t \leq T\). Then, it is easy to see that \(u(t; s, x)\) is a strong solution of \((DE)_s\) satisfying \(u(t; s, x) \in Y\) for \(t \in [s, T]\) and \(\sup\{||u(t; s, x)||_Y : t \in [s, T]\} < \infty\). To prove the uniqueness of the solutions, let \(v(t; s, x)\) be a strong solution of \((DE)_s\) satisfying \(v(t; s, x) \in Y\) for \(t \in [s, T]\) and \(\sup\{||v(t; s, x)||_Y : t \in [s, T]\} < \infty\). Then, we deduce from the property (e) that \(r \to V_2(t, r)(u(r; s, x) - v(r; s, x))\) is absolutely continuous on \([s, T]\) and

\[
(\partial/\partial r)V_2(t, r)(u(r; s, x) - v(r; s, x)) = 0
\]

for almost every \(r \in [s, T]\). Integrating this equality from \(s\) to \(t\) we have

\[
C_2(u(t; s, x) - v(t; s, x)) = 0,
\]

which shows that \(u(t; s, x) = v(t; s, x)\) for \(t \in [s, T]\), since \(C_2\) is injective.

Q.E.D.

We next consider the second order differential equation in a reflexive Banach space \(X\)

\[
(\text{DE})^2_s \quad \begin{cases} u''(t) = Au(t) + B(t)u(t) & \text{for } t \in [s, T] \\ u(s) = x, \ u'(s) = y, \end{cases}
\]

where \(A\) is the infinitesimal generator of a cosine family and \(\{B(t) : t \in [0, T]\}\) is a family of linear operators in \(X\) satisfying the following conditions:
(B1) $D(A) \subset D(B(t))$ for $t \in [0, T]$.

(B2) There are constants $M \geq 0$ and $\omega \geq 0$ such that \( \{\lambda^2 : \lambda > \omega\} \subset \rho(A) \), for $t \in [0, T]$ $B(t)R(\lambda^2 : A)$ is strongly infinitely differentiable in $\lambda > \omega$ and satisfies

\[
\|(1/n!)(\lambda - \omega)^{n+1}(d/d\lambda)^nB(t)R(\lambda^2 : A)x\| \leq M\|x\|
\]

for $x \in X$, $\lambda > \omega$ and $n = 0, 1, \cdots$.

(B3) \( \lim_{\ell \to \epsilon} \sup \{\|B(t)x - B(s)x\| : x \in D(A), \|x\| + \|Ax\| \leq 1\} = 0 \).

(B4) There exists $\lambda_0 > \omega$ such that $(\lambda_0^2 - A)B(t)R(\lambda_0^2 : A) = B(t) + P(t)$, where \( \{P(t) : t \in [0, T]\} \) is a strongly continuous family of bounded linear operators on $X$.

Definition 2.2. A function $u(\cdot ; s, x, y)$ on $[s, T]$ is a strong solution of $(DE)_s^2$ if

(i) $u(\cdot ; s, x, y) \in A^{2,1}(s,T;X)$,

(ii) $u(\cdot ; s, x, y)$ satisfies $(DE)_s^2$ almost everywhere.

Without proof we state the existence and uniqueness theorem of the strong solutions of the second order differential equation $(DE)_s^2$ which is obtained by applying Theorem 2.5 with $A(t) = \begin{pmatrix} 0 & 1 \\ A + B(t) & 0 \end{pmatrix}$ and $C_1 = C_2 = \begin{pmatrix} 0 & 1 \\ A - \lambda_0^2 & 0 \end{pmatrix}^{-1}$.

**Theorem 2.6.** Assume that $A$ is the infinitesimal generator of a cosine family and \( \{B(t) : t \in [0, T]\} \) is a family of linear operators in $X$ satisfying the conditions $(B_1)$ - $(B_4)$. Then, for every initial data $x \in D(A)$ and $y \in D(A)$ the $(DE)_s^2$ has a unique strong solution $u(t; s, x, y)$ such that $u(t; s, x, y) \in D(A)$ for $t \in [s, T]$ and \( \sup\{\|Au(t; s, x, y)\| : t \in [s, T]\} < \infty \).

**References**


