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Extensions of the results on powers of $p$-hyponormal operators to class wF$(p,r,q)$ operators (Inequalities on Linear Operators and its Applications)

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Extensions of the results on powers of $p$-hyponormal operators to class $wF(p, r, q)$ operators

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This report is based on “M.Ito, Parallel results to that on powers of $p$-hyponormal, log-hyponormal and class $A$ operators, to appear in Acta Sci. Math. (Szeged).”

Abstract

In this report, we shall show that inequalities
\[(T^{n+1}T^{n+1})^{\frac{n+p}{n+1}} \geq (T^{n}T^{n})^{\frac{n+p}{n}}\]
and
\[(T^{n}T^{n})^{\frac{n+r}{n}} \geq (T^{n+1}T^{n+1})^{\frac{n+r}{n+1}}\]
for $0 < p \leq 1$ and all positive integer $n$ hold for weaker conditions than $p$-hyponormality, that is, class $F(p, r, q)$ defined by Fujii-Nakamoto or class $wF(p, r, q)$ defined by Yang-Yuan under appropriate conditions of $p$, $r$ and $q$.

1 Introduction

In this report, a capital letter means a bounded linear operator on a complex Hilbert space $\mathcal{H}$. An operator $T$ is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in \mathcal{H}$, and also an operator $T$ is said to be strictly positive (denoted by $T > 0$) if $T$ is positive and invertible.

As an extension of hyponormal operators, i.e., $T^{*}T \geq TT^{*}$, it is well known that $p$-hyponormal operators for $p > 0$ are defined by $(T^{*}T)^{p} \geq (TT^{*})^{p}$, and also an operator $T$ is said to be $p$-quasihyponormal for $p > 0$ if $T^{*}(T^{*}T)^{p} - (TT^{*})^{p}T \geq 0$. It is easily obtained that every $p$-hyponormal operator is $q$-hyponormal for $p > q > 0$ by Löwner-Heinz theorem “$A \geq B \geq 0$ ensures $A^{\alpha} \geq B^{\alpha}$ for any $\alpha \in [0, 1]$. “

On powers of $p$-hyponormal operators, Aluthge-Wang [1] showed that “If $T$ is a $p$-hyponormal operator for $0 < p \leq 1$, then $T^{n}$ is $\frac{p}{n}$-hyponormal for any positive integer $n$.” As a more precise result than theirs, Furuta-Yanagida [8] obtained the following.

Theorem 1.A ([8]). Let $T$ be a $p$-hyponormal operator for $0 < p \leq 1$. Then
\[(T^{n}T)^{\frac{p+1}{n}} \geq \cdots \geq (T^{2}T^{2})^{\frac{p+1}{2}} \geq (T^{*}T)^{p+1},\]
that is,
\[|T^{n}|^{2(p+1)} \geq \cdots \geq |T^{2}|^{p+1} \geq |T|^{2(p+1)}\]
and
\[(TT^{*})^{p+1} \geq (T^{2}T^{2})^{\frac{p+1}{2}} \geq \cdots \geq (T^{n}T^{n})^{\frac{p+1}{n}},\]
that is,
\[|T^{*}|^{2(p+1)} \geq |T^{2}|^{p+1} \geq \cdots \geq |T^{n}|^{2(p+1)}\]
hold for all positive integer $n$. 
Recently, Gao-Yang [9] obtained the results on comparison of $n$th power and $(n+1)$th power of $p$-hyponormal operators for $0 < p \leq 1$.

**Theorem 1.B** ([9]). Let $T$ be a $p$-hyponormal operator for $0 < p \leq 1$. Then

\[(T^{n+1}T^{n+1})^{\frac{n+2}{n+1}} \geq (T^{n}T^{n})^{\frac{n+2}{n+1}}, \quad \text{that is,} \quad |T^{n+1}|^{\frac{2(p+n)}{n+1}} \geq |T^{n}|^{\frac{2(p+n)}{n+1}}\]

and

\[(T^{n}T^{n})^{\frac{n+2}{n+1}} \geq (T^{n+1}T^{n+1})^{\frac{n+2}{n+1}}, \quad \text{that is,} \quad |T^{n+1}|^{\frac{2(p+n)}{n+1}} \geq |T^{n+1}|^{\frac{2(p+n)}{n+1}}\]

hold for all positive integer $n$.

As an extension of hyponormal operators, it is also well known that invertible log-hyponormal operators are defined by $\log T^*T \geq \log TT^*$ for an invertible operator $T$. We remark that we treat only invertible log-hyponormal operators in this paper (see also [17]). It is easily obtained that every invertible $p$-hyponormal operator for $p > 0$ is log-hyponormal since $\log t$ is an operator monotone function. We note that log-hyponormality is sometimes regarded as $0$-hyponormality since $\frac{X^p-I}{p} \rightarrow \log X$ as $p \rightarrow +0$ for $X > 0$. An operator $T$ is paranormal if $\|T^2x\| \geq \|Tx\|^p$ for every unit vector $x \in \mathcal{H}$. Ando [2] showed that every $p$-hyponormal operator for $p > 0$ and invertible log-hyponormal operator is paranormal. (Invertibility of a log-hyponormal operator is not necessarily required.)

Yamazaki [18] showed that “If $T$ is an invertible log-hyponormal operator, then $T^n$ is also log-hyponormal for any positive integer $n”, and also he obtained the following results.

**Theorem 1.C** ([18]). Let $T$ be an invertible log-hyponormal operator. Then

\[(T^nT^n)^{\frac{1}{n}} \geq \cdots \geq (T^2T^2)^{\frac{1}{2}} \geq T^*T, \quad \text{that is,} \quad |T^n|^\frac{2}{n} \geq \cdots \geq |T^2| \geq |T|^2\]

and

\[TT^* \geq (T^2T^2)^{\frac{1}{2}} \geq \cdots \geq (T^nT^n)^{\frac{1}{n}}, \quad \text{that is,} \quad |T^*|^2 \geq |T^2| \geq \cdots \geq |T^n|^\frac{2}{n}\]

hold for all positive integer $n$.

**Theorem 1.D** ([18]). Let $T$ be an invertible log-hyponormal operator. Then

\[(T^{n+1}T^{n+1})^{\frac{n+2}{n+1}} \geq T^nT^n, \quad \text{that is,} \quad |T^{n+1}|^{\frac{2(p+1)}{n+1}} \geq |T^n|^2\]

and

\[T^nT^n \geq (T^{n+1}T^{n+1})^{\frac{n+2}{n+1}}, \quad \text{that is,} \quad |T^n|^2 \geq |T^{n+1}|^{\frac{2(p+1)}{n+1}}\]

hold for all positive integer $n$. 
We remark that Theorems 1.C and 1.D correspond to Theorems 1.A and 1.B, respectively. On powers of $p$-hyponormal and log-hyponormal operators, related results are obtained in [7], [13], [22], [24] and so on.

On the other hand, in [6], we introduced class A defined by $|T^2| \geq |T|^2$ where $|T| = (T^*T)^{1/2}$, and we showed that every invertible log-hyponormal operator belongs to class A and every class A operator is paranormal. We remark that class A is defined by an operator inequality and paranormality is defined by a norm inequality, and their definitions appear to be similar forms.

As we have pointed out in [14], we have the following result by combining [20, Theorem 1] and [15, Theorem 3] as a result on powers of class A operators. We remark that Theorem 1.E in case of invertible operators was shown in [11].

**Theorem 1.E ([20][15][14]).** If $T$ is a class A operator, then

(i) $|T^{n+1}|^{\frac{2n}{n+1}} \geq |T^n|^2$ and $|T^*|^2 \geq |T^{n+1}|^{\frac{2n}{n+1}}$ hold for all positive integer $n$.

(ii) $|T^n|^\frac{2}{n} \geq \cdots \geq |T^2| \geq |T|^2$ and $|T^*|^2 \geq |T^{2*}| \geq \cdots \geq |T^n|^\frac{2}{n}$ hold for all positive integer $n$.

(i) (resp. (ii)) of Theorem 1.E is an extension of Theorem 1.D (resp. Theorem 1.C) since every invertible log-hyponormal operator belongs to class A.

As generalizations of class A and paranormality, Fujii-Jung-S.H.Lee-M.Y.Lee-Nakamoto [3] introduced class $A(p, r)$, Yamazaki-Yanagida [19] introduced absolute-$(p, r)$-paranormality, and Fujii-Nakamoto [4] introduced class $F(p, r, q)$ and $(p, r, q)$-paranormality as follows:

**Definition.**

(i) For each $p > 0$ and $r > 0$, an operator $T$ belongs to class $A(p, r)$ if

$$(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{r}{p+r}} \geq |T^*|^{2r}.$$ 

(ii) For each $p > 0$ and $r > 0$, an operator $T$ is absolute-$(p, r)$-paranormal if

$$|||T^p|T^*|^r x||^r \geq |||T^*|^r x||^{p+r}$$

for every unit vector $x \in H$.

(iii) For each $p > 0$, $r \geq 0$ and $q > 0$, an operator $T$ belongs to class $F(p, r, q)$ if

$$(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{1}{q}} \geq |T^*|^{\frac{2(p+q)}{q}}.$$
(iv) For each $p > 0$, $r \geq 0$ and $q > 0$, an operator $T$ is $(p, r, q)$-paranormal if

$$
\|\|T^p U|T|^r x\|^{\frac{1}{q}} \geq \|\|T^{\frac{p+r}{q}} x\|
$$

(1.1)

for every unit vector $x \in H$, where $T = U|T|$ is the polar decomposition of $T$. In particular, if $r > 0$ and $q \geq 1$, then (1.1) is equivalent to

$$
\|\|T^p |T^*|^r x\|^{\frac{1}{q}} \geq \|\|T^*^{\frac{p+r}{q}} x\|
$$

for every unit vector $x \in H$ ([12]).

We remark that class $F(p, r, \frac{p+r}{r})$ equals class $A(p, r)$ and also class $F(1, 1, 2)$ (i.e., class $A(1, 1)$) equals class $A$. Similarly $(p, r, \frac{p+r}{r})$-paranormality equals absolute-$(p, r)$-paranormality and also $(1, 1, 2)$-paranormality (i.e., absolute-$(1, 1)$-paranormality) equals paranormality.

Inclusion relations among these classes were shown in [3], [4], [12], [14], [15], [19] and so on (see also Theorems 3.A and 3.B). The following Figure 1 represents the inclusion relations among the families of class $F(p, r, q)$ and $(p, r, q)$-paranormality.

![Inclusion Relations among Paranormal Classes](image)

**Figure 1**

We can pick up inclusion relations among classes discussed in this report as follows: For $0 < \delta < p < 1$ and $0 < r < 1$,.
\[ \delta \text{-hyponormal} \subset \text{class } F(p, r, \frac{p+1}{\delta+1}) \cap \subset \text{class } F(1, 1, \frac{2}{\delta+1}) \]
\[ \log \text{-hyponormal} \subset \text{class } A(p, r) \subset \text{class } A \]

We remark that we assume invertibility on log-hyponormal operators.

In this report, as a parallel result to Theorem 1.E, we shall show that inequalities in Theorems 1.A and 1.B hold for weaker conditions than \( p \)-hyponormality, that is, class \( F(p, r, q) \) defined by Fujii-Nakamoto or class \( wF(p, r, q) \) recently defined by Yang-Yuan [23][21] (see Section 3) under appropriate conditions of \( p, r \) and \( q \).

\section{Main results}

In this section, we shall show our main results.

\textbf{Theorem 2.1.} If \( (|T^*||T|^2|T^*|)^{\delta+1} \geq |T|^2(\delta+1) \) (i.e., \( T \) belongs to class \( F(1, 1, \frac{2}{\delta+1}) \)) for some \( 0 \leq \delta \leq 1 \), then

(i) \[ |T^{n+1}|^{\frac{2(\delta+n)}{n}} \geq |T^n|^{\frac{2(\delta+n)}{n}} \] holds for all positive integer \( n \).

(ii) \[ |T^n|^{\frac{2(\delta+1)}{n}} \geq \cdots \geq |T^2|^{\delta+1} \geq |T|^{2(\delta+1)} \] holds for all positive integer \( n \).

\textbf{Theorem 2.2.} If \( |T|^{2(\gamma+1)} \geq (|T||T|^2|T|)^{\iota_{\frac{+1}{2}}} \) for some \( 0 \leq \gamma \leq 1 \) holds and either

(a) \( (|T^*||T|^2|T^*|)^{\frac{1}{2}} \geq |T^*|^2 \) (i.e., \( T \) belongs to class \( A \)) or

(b) \( N(|T|) \subseteq N(|T^*|) \)

holds, then

(i) \[ |T^n|^{\frac{2(\gamma+n)}{n}} \geq |T^{n+1}|^{\frac{2(\gamma+n)}{n+1}} \] holds for all positive integer \( n \).

(ii) \[ |T^*|^{2(\gamma+1)} \geq |T^2|^\gamma \geq \cdots \geq |T^n|^{\frac{2(\gamma+1)}{n}} \] holds for all positive integer \( n \).

We need the following results in order to prove Theorems 2.1 and 2.2.

\textbf{Theorem 2.A ([15])}. Let \( A \) and \( B \) be positive operators. Then for each \( p \geq 0 \) and \( r \geq 0 \),

(i) If \( (B^{\frac{r}{2}}A^{\frac{p}{2}}B^{\frac{r}{2}})^{\frac{r}{r+p}} \geq B^r \), then \( A^p \geq (A^{\frac{r}{2}}B^rA^{\frac{r}{2}})^{\frac{r}{r+p}} \).

(ii) If \( A^p \geq (A^{\frac{r}{2}}B^rA^{\frac{r}{2}})^{\frac{r}{r+p}} \) and \( N(A) \subseteq N(B) \), then \( (B^{\frac{r}{2}}A^{\frac{p}{2}}B^{\frac{r}{2}})^{\frac{r}{r+p}} \geq B^r \).
Theorem 2.B ([20]). Let $A$ and $B$ be positive operators. Then

(i) If $(B^\beta A^\alpha B^\alpha)^{\frac{\gamma}{\alpha_0+\beta_0}} \geq B^\beta_0$ holds for fixed $\alpha_0 > 0$ and $\beta_0 > 0$, then

\[ (B^\beta A^\alpha B^\alpha)^{\frac{\gamma}{\alpha_0+\beta}} \geq B^\beta \]

holds for any $\beta \geq \beta_0$. Moreover,

\[ A^{\alpha_0} B^\beta A^{\alpha_0} \geq (A^{\alpha_0} B^\beta A^{\alpha_0})^{\frac{\alpha_0+\beta_1}{\alpha_0+\beta}} \]

holds for any $\beta_1$ and $\beta_2$ such that $\beta_2 \geq \beta_1 \geq \beta_0$.

(ii) If $A^\alpha \geq (A^{\alpha_0} B^\beta A^{\alpha_0})^{\frac{\alpha_0+\beta_0}{\alpha_0+\beta}}$ holds for fixed $\alpha_0 > 0$ and $\beta_0 > 0$, then

\[ A^\alpha \geq (A^{\alpha_0} B^\beta A^{\alpha_0})^{\frac{\alpha_0+\beta_1}{\alpha_0+\beta_0}} \]

holds for any $\alpha \geq \alpha_0$. Moreover,

\[ (B^\beta A^\alpha B^\alpha)^{\frac{\alpha_0+\beta_0}{\alpha_0+\beta_0}} \geq B^\beta A^{\alpha_1} B^\beta \]

holds for any $\alpha_1$ and $\alpha_2$ such that $\alpha_2 \geq \alpha_1 \geq \alpha_0$.

Lemma 2.C ([20][16]). Let $A$, $B$ and $C$ be positive operators. Then for $p > 0$ and $0 < r \leq 1$,

(i) If $(B^p A^p B^p)^{\frac{1}{p+r}} \geq B^p$ and $B \geq C$, then $(C^p A^p C^p)^{\frac{1}{p+r}} \geq C^p$.

(ii) If $A \geq B$, $B^r \geq (B^p C^p B^p)^{\frac{1}{p+r}}$ and $N(A) = N(B)$, then $A^r \geq (A^p C^p A^p)^{\frac{1}{p+r}}$.

Lemma 2.D ([5]). Let $A > 0$ and $B$ be an invertible operator. Then

\[ (BAB^*)^\lambda = BA^{\frac{1}{4}} (A^{\frac{1}{4}} B^* B A^{\frac{1}{4}})^{\lambda-1} A^{\frac{1}{4}} B^* \]

holds for any real number $\lambda$.

We remark that Lemma 2.D holds without invertibility of $A$ and $B$ when $\lambda \geq 1$.

Proof of Theorem 2.1. Let $T = U |T|$ be the polar decomposition of $T$, and put $A_k = (T^k T^{*k})^{\frac{1}{2}} = |T^k|^{\frac{1}{2}}$ and $B_k = (T^k T^{*k})^{\frac{1}{2}} = |T^k|^{\frac{1}{2}}$ for a positive integer $k$. We remark that $T^* = U^* |T^*|$ is also the polar decomposition of $T^*$. 

Firstly we shall show $|T^2|^{\delta+1} \geq |T|^{2(\delta+1)}$. By the hypothesis $$(|T^*||T^2||T^*|)^{\delta+1} \geq |T^*|^{2(\delta+1)}$$ for some $0 \leq \delta \leq 1$, we have

$$|T^2|^{\delta+1} = (U^*|T^*||T^2||T^*|U)^{\frac{\delta+1}{2}}$$
$$= U^*|T^*|^{\frac{\delta+1}{2}}$$
$$\geq U^*|T^*|^{2(\delta+1)}U$$
$$= |T|^{2(\delta+1)}.$$

Next we assume that $|T^{n+1}|^{\frac{2(\delta+n)}{k+1}} \geq |T^n|^{\frac{2(\delta+n)}{k}}$, that is, $A_{n+1}^{\delta+n} \geq A_n^{\delta+n}$ (2.1) holds for $n = 1, 2, \ldots, k$. By (2.1) and Löwner-Heinz theorem, we have

$$A_{k+1} \geq A_k \geq \cdots \geq A_2 \geq A_1$$
(2.2)

since $\frac{1}{\delta+n} \in (0, 1)$ in (2.1). The hypothesis $$(|T^*||T^2||T^*|)^{\frac{\delta+1}{2}} \geq |T^*|^{2(\delta+1)}$$ can be rewritten by $(B_1^{\frac{1}{2}}A_k^\frac{1}{2}B_1^{\frac{1}{2}})^{\frac{\delta+1}{2}} \geq B_1^{\delta+1}$, and also this yields $A_1 \geq (A_1^{\frac{1}{2}}B_1A_1^{\frac{1}{2}})^{\frac{1}{2}}$ by Löwner-Heinz theorem and (i) of Theorem 2.A. (2.2) and $A_1 \geq (A_1^{\frac{1}{2}}B_1A_1^{\frac{1}{2}})^{\frac{1}{2}}$ ensure

$$A_k \geq (A_k^{\frac{1}{2}}B_1A_k^{\frac{1}{2}})^{\frac{1}{2}}$$
(2.3)

by (ii) of Lemma 2.C since $N(A_k) = N(A_1)$ holds. We remark that $N(A_k) \subseteq N(A_1)$ holds by (2.2) and $N(A_k) = N(T^k) \supseteq N(T) = N(A_1)$ always holds. Then we get

$$A_k^\frac{1}{2} \geq (A_k^{\frac{1}{2}}B_1A_k^{\frac{1}{2}})^{\frac{1}{2}}$$
(2.4)

by (2.3) and (ii) of Theorem 2.B. Similarly, (2.2) and $A_1 \geq (A_1^{\frac{1}{2}}B_1A_1^{\frac{1}{2}})^{\frac{1}{2}}$ ensure

$$A_{k+1} \geq (A_{k+1}^{\frac{1}{2}}B_1A_{k+1}^{\frac{1}{2}})^{\frac{1}{2}}.$$  
(2.5)

Therefore we have

$$|T^{k+1}|^{\frac{2(\delta+k+1)}{k+1}} = (U^*|T^*||T^k||T^*|U)^{\frac{\delta+k+1}{k+1}}$$
$$= U^*(B_1^{\frac{1}{2}}A_k^\frac{1}{2}B_1^{\frac{1}{2}})^{\frac{\delta+k+1}{k+1}}$$
$$= U^*B_1^{\frac{1}{2}}A_k^\frac{1}{2}(A_k^\frac{1}{2}B_kA_k^\frac{1}{2})^{\frac{\delta+k+1}{k+1}}A_k^\frac{1}{2}B_1^{\frac{1}{2}}U$$
by Lemma 2.D

$$\leq U^*B_1^{\frac{1}{2}}A_k^\frac{1}{2}A_k^\frac{1}{2}A_k^\frac{1}{2}B_1^{\frac{1}{2}}U$$
by (2.4) and Löwner-Heinz theorem

$$= U^*B_1^{\frac{1}{2}}A_k^{\delta+k}B_1^{\frac{1}{2}}U$$
$$\leq U^*B_1^{\frac{1}{2}}A_k^{\delta+k}B_1^{\frac{1}{2}}U$$
by (2.1)

$$\leq U^*(B_1^{\frac{1}{2}}A_k^{k+1}B_1^{\frac{1}{2}})^{\frac{\delta+k+1}{k+1}}U$$
$$= (U^*|T^*||T^{k+1}||T^*|U)^{\frac{\delta+k+1}{k+1}}$$
$$= |T^{k+2}|^{\frac{2(\delta+k+1)}{k+2}}.$$
We remark that the last inequality holds by (ii) of Theorem 2.B since (2.5) holds and $k + 1 \geq \delta + k \geq 1$.

Consequently the proof of (i) is complete. We can easily obtain (ii) by (i) and Löwner-Heinz theorem, so we omit its proof. \(\square\)

**Proof of Theorem 2.2.** Let $T = U|T|$ be the polar decomposition of $T$, and put $A_k = (T^k T^k)^{1\over 2}$ and $B_k = (T^k T^k)^{1\over 2} = |T^k|^1$ for a positive integer $k$. We remark that $T^* = U^*|T^*|$ is also the polar decomposition of $T^*$.

$|T||T^*|^2 |T|^1 \geq (|T||T^*|^2 |T|)^{1\over 2}$ and condition (b) ensure condition (a) by Löwner-Heinz theorem and (ii) of Theorem 2.A, so that we have only to prove the case where condition (a) holds.

Firstly we shall show $|T^*|^2(|T^*|^2 |T|^1 \geq |T^2|^1$. By the hypothesis $|T|^2(|T^*|^2 |T|^1 \geq (|T||T^*|^2 |T|)^{1\over 2}$ for some $0 \leq \gamma \leq 1$, we have

$$
|T^2|^1 = (U|T||T^*|^2 |T| U^*)^{1\over 2} = U(|T||T^*|^2 |T| U^*)^{1\over 2} \leq |T|^{2(\gamma+1)} U^* = |T^*|^2(\gamma+1).
$$

Next we assume that

$$
|T^n|^2(\gamma+1) \geq |T^n|^2(\gamma+1), \quad \text{that is,} \quad B_n^{\gamma+n} \geq B_{n+1}^{\gamma+n} \quad (2.6)
$$

holds for $n = 1, 2, \ldots, k$. By (2.6) and Löwner-Heinz theorem, we have

$$
B_1 \geq B_2 \geq \cdots \geq B_k \geq B_{k+1} \quad (2.7)
$$

since $1 \over \gamma+n \in (0, 1]$ in (2.6). Condition (a) can be rewritten by $(B_1^{1\over 2} A_1 B_1^{1\over 2})^{1\over 2} \geq B_1$. (2.7) and $(B_1^{1\over 2} A_1 B_1^{1\over 2})^{1\over 2} \geq B_1$ ensure

$$
(B_k^{1\over 2} A_1 B_k^{1\over 2})^{1\over 2} \geq B_k. \quad (2.8)
$$

by (i) of Lemma 2.C Then we get

$$
(B_k^{1\over 2} A_k B_k^{1\over 2})^{1\over 2} \geq B_k. \quad (2.9)
$$

by (2.8) and (i) of Theorem 2.B. Similarly, (2.7) and $(B_1^{1\over 2} A_1 B_1^{1\over 2})^{1\over 2} \geq B_1$ ensure

$$
(B_{k+1}^{1\over 2} A_1 B_{k+1}^{1\over 2})^{1\over 2} \geq B_{k+1}. \quad (2.10)
$$
Therefore we have
\[
|T^{k+1}|^{\frac{2(k+1)}{k+1}} = (U|T||T^k|^{2}|T|^*U^{*})^{\frac{k+1}{k+1}} \\
= U(A^{\frac{1}{2}}B_k A^{\frac{1}{2}})^{\frac{k+1}{k+1}} U^{*} \tag{2.9}
\]
by (2.6) and Löwner-Heinz theorem
\[
= UA^{\frac{1}{2}}B_{k+1} A_{\gamma+k} A^{\frac{1}{2}} U^{*} \tag{2.10}
\]
by (i) of Theorem 2.A.

We remark that the last inequality holds by (i) of Theorem 2.B since (2.10) holds and
\[
k + 1 \geq \gamma + k \geq 1.
\]

Consequently the proof of (i) is complete. We can easily obtain (ii) by (i) and Löwner-Heinz theorem, so we omit its proof.

\[\square\]

\textbf{Remark.} By putting $\delta = 0$ in Theorem 2.1 and $\gamma = 0$ in Theorem 2.2, we get Theorem 1.E since $\frac{1}{2} \geq |T^*|^2$ (i.e., $T$ belongs to class A) ensures $|T|^2 \geq (|T^*|^2|T|)^{\frac{1}{2}}$ by (i) of Theorem 2.A.

\section*{3 Classes $F(p, r, q)$ and $wF(p, r, q)$ operators}

Recently, in order to continue the study of class $F(p, r, q)$, Yang-Yuan \cite{23}\cite{21} introduced class $wF(p, r, q)$ operators as follows: For each $p \geq 0$, $r \geq 0$ and $q \geq 1$ with $(p, r) \neq (0, 0)$ and $(p, q) \neq (0, 1)$, an operator $T$ belongs to class $wF(p, r, q)$ if
\begin{equation}
(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{1}{q}} \geq |T^*|^2 \tag{3.1}
\end{equation}
and
\begin{equation}
|T|^{2(p+r)(1-\frac{1}{q})} \geq (|T^*|^r|T|^2r|T^*|^r)^{1-\frac{1}{q}}, \tag{3.2}
\end{equation}
denoting $(1-q^{-1})^{-1}$ by $q^*$ when $q > 1$ because $q$ and $(1-q^{-1})^{-1}$ are a couple of conjugate exponents. On discussions of class $wF(p, r, q)$ (or class $F(p, r, q)$), we frequently consider class $wF(p, r, \frac{q+r}{q+r})$ (or class $F(p, r, \frac{q+r}{q+r})$) by putting $q = \frac{r+r}{q+r}$ as follows: For $p \geq 0$, $r \geq 0$ and $-r < \delta \leq p$ with $(p, r) \neq (0, 0)$ and $(p, \delta) \neq (0, 0)$, an operator $T$ belongs to class $wF(p, r, \frac{q+r}{q+r})$ if
\begin{equation}
(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{q+r}{q+r}} \geq |T^*|^2(\delta+r) \tag{3.3}
\end{equation}
and
\[ |T|^{2(-\delta+p)} \geq (|T|^{p}|T^{*}|^{2r}|T|^{p})^{\frac{\delta+r}{p+r}}. \] (3.4)

We remark that (3.1) is the definition of class \( F(p, r, q) \). We also remark that class \( wF(p, r, \frac{p+r}{r}) \) equals class \( wA(p, r) \) defined in [10], and also it was shown in [15] that class \( wA(p, r) \) (i.e., class \( wF(p, r, \frac{p+r}{r}) \)) coincides with class \( A(p, r) \). On inclusion relations of classes \( A(p, r), F(p, r, q) \) and \( wF(p, r, q) \), the following results were obtained.

**Theorem 3.A.**

(i) For invertible operator \( T \), \( T \) is log-hyponormal if and only if \( T \) belongs to class \( A(p, r) \) for all \( p > 0 \) and \( r > 0 \) ([3]).

(ii) If \( T \) belongs to class \( A(p_0, r_0) \) for \( p_0 > 0, r_0 > 0 \), then \( T \) belongs to class \( A(p, r) \) for any \( p \geq p_0 \) and \( r \geq r_0 \) ([15]).

We note that log-hyponormality can be regarded as class \( A(0, 0) \) by Theorem 3.A.

**Theorem 3.B.**

(i) For a fixed \( \delta > 0 \), \( T \) is \( \delta \)-hyponormal if and only if \( T \) belongs to class \( F(2\delta p, 2\delta r, q) \) for all \( p > 0, r \geq 0 \) and \( q \geq 1 \) with \( (1 + 2r)q \geq 2(p + r) \), i.e., \( T \) belongs to class \( F(p, r, q) \) for all \( p > 0, r \geq 0 \) and \( q \geq 1 \) with \( (\delta + r)q \geq p + r \) ([4]).

(ii) For each \( p > 0 \) and \( r > 0 \), \( T \) is \( p \)-quasihyponormal if and only if \( T \) belongs to class \( F(p, r, 1) \) ([12]).

(iii) If \( T \) belongs to class \( F(p_0, r_0, q_0) \) for \( p_0 > 0, r_0 \geq 0 \) and \( q_0 \geq 1 \), then \( T \) belongs to class \( F(p_0, r_0, q) \) for any \( q \geq q_0 \) ([4]).

(iv) If \( T \) belongs to class \( F(p_0, r_0, \frac{p_0+r_0}{\delta+r_0}) \) for \( p_0 > 0, r_0 \geq 0 \) and \( 0 \leq \delta \leq p_0 \), then \( T \) belongs to class \( F(p, r, \frac{p+r_0}{\delta+r_0}) \) for any \( p \geq p_0 \) and \( r \geq r_0 \) ([14]).

(v) If \( T \) belongs to class \( F(p_0, r_0, \frac{p_0+r_0}{\delta+r_0}) \) for \( p_0 > 0, r_0 \geq 0 \) and \( -r_0 < \delta \leq p_0 \), then \( T \) belongs to class \( F(p_0, r_0, \frac{p_0+r_0}{\delta+r}) \) for any \( r \geq r_0 \) ([12]).

**Theorem 3.C ([23]).**

(i) If \( T \) belongs to class \( wF(p_0, r_0, q_0) \) for \( p_0 > 0, r_0 \geq 0 \) and \( q_0 \geq 1 \), then \( T \) belongs to class \( wF(p_0, r_0, q) \) for any \( q \geq q_0 \) with \( r_0q \leq p_0 + r_0 \).

(ii) If \( T \) belongs to class \( wF(p_0, r_0, q_0) \) for \( p_0 > 0, r_0 \geq 0, q_0 \geq 1 \) and \( N(T) \subseteq N(T^{*}) \), then \( T \) belongs to class \( wF(p_0, r_0, q) \) for any \( q \) such that \( q^* \geq q_0^* \) with \( p_0q^* \leq p_0 + r_0 \).
If $T$ belongs to class $wF(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$ for $p_0 > 0$, $r_0 \geq 0$ and $-r < \delta \leq p_0$, then $T$ belongs to class $wF(p, r, \frac{p+r}{\delta+r})$ for any $p \geq p_0$ and $r \geq r_0$.

If $p > 0$, $r \geq 0$, $q \geq 1$ with $rq \leq p + r$, then class $wF(p, r, q)$ coincides with class $F(p, r, q)$. In other words, if $p > 0$, $r \geq 0$, $0 \leq \delta \leq p$ and $\delta + r \neq 0$, then class $wF(p, r, \frac{p+r}{\delta+r})$ coincides with class $F(p, r, \frac{p+r}{\delta+r})$.

In this section, firstly we shall get a relation between $p$-hyponormality and class $wF(p, r, q)$ (or class $F(p, r, q)$). We remark that Theorem 3.1 is a parallel result to (i) of Theorem 3.A.

**Theorem 3.1.**

(i) For a fixed $\delta > 0$, $T$ is $\delta$-hyponormal (i.e., $T$ belongs to class $F(p_0, 0, \frac{p_0}{\delta})$ for some $p_0 \geq \delta$) if and only if $T$ belongs to class $F(p, r, \frac{p+r}{\delta+r})$ for all $p \geq \delta$ and $r \geq 0$.

(ii) For a fixed $\delta < 0$, $T$ is $(-\delta)$-hyponormal (i.e., $T$ belongs to class $wF(0, r_0, \frac{r_0}{\delta+r_0})$ for some $r_0 > -\delta$) if and only if $T$ belongs to class $wF(p, r, \frac{p+r}{\delta+r})$ for all $p \geq 0$ and $r > -\delta$.

For $0 < \delta < p < 1$ and $0 < -\delta' < r < 1$, inclusion relations among class $wF(p, r, q)$ and other classes can be expressed as the following diagram. We remark that we assume invertibility on log-hyponormal operators, and also $N(T) \subseteq N(T^*)$ is required in ($*$).

\[
\begin{align*}
\delta\text{-hyponormal} & \subset \text{class } F(p, r, \frac{p+r}{\delta+r}) \subset \text{class } F(1, 1, \frac{2}{\delta+1}) \\
\cap & \cap \\
\text{log-hyponormal} & \subset \text{class } A(p, r) \subset \text{class } A \\
\cup & \cup \text{(*)} \\
(-\delta')\text{-hyponormal} & \subset \text{class } wF(p, r, \frac{p+r}{\delta+r}) \subset \text{class } wF(1, 1, \frac{2}{\delta+1}) \cup \text{(*)}
\end{align*}
\]

Next we shall obtain the following corollaries led by Theorems 2.1 and 2.2, and also Theorems 1.A and 1.B follow from these corollaries.

**Corollary 3.2.** If $T$ belongs to class $F(p, r, \frac{p+r}{\delta+r})$ for some $0 \leq \delta \leq 1$, $0 < p \leq 1$ and $0 \leq r \leq 1$ such that $-r < \delta \leq p$, then

(i) $|T^{n+1}|^{\frac{2(d+n)}{n+1}} \geq |T^n|^{\frac{2(d+n)}{n}}$ holds for all positive integer $n$.

(ii) $|T^n|^{\frac{2(d+1)}{n}} \geq \cdots \geq |T^{2(d+1)}| \geq |T|^{2(d+1)}$ holds for all positive integer $n$. 
Corollary 3.3. If $T$ belongs to class $wF(p, r, \frac{p+r}{p+r}, \delta, E, \frac{r}{p+r})$ for some $-1 \leq \delta \leq 0$, $0 \leq p \leq 1$ and $0 \leq r \leq 1$ such that $-r < \delta < p$, and $T$ satisfies $N(T) \subseteq N(T^*)$, then

(i) $|T^n|^\frac{2(-\delta+n)}{n} \geq |T^{n+1}|^\frac{2(-\delta+n)}{n+1}$ holds for all positive integer $n$.

(ii) $|T^n|^{2(-\delta+1)} \geq |T^{2}|^{-\delta+1} \geq \cdots \geq |T^{n}|^\frac{2(-\delta+1)}{n}$ holds for all positive integer $n$.

We omit proofs of the results in this section.

References


[15] M.Ito and T.Yamazaki, *Relations between two inequalities* \((B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}})^{\frac{p}{p+r}} \geq B^{r}\) and \(A^{p} \geq (A^{\frac{r}{2}}B^{r}A^{\frac{r}{2}})^{\frac{p}{p+r}}\) and their applications, Integral Equations and Operator Theory, 44 (2002), 442–450.


