Some properties of Julia sets of transcendental entire functions with multiply-connected wandering domains (Complex Dynamics and Related Topics)

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Some properties of Julia sets of transcendental entire functions with multiply-connected wandering domains

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Abstract

We study Julia components of transcendental entire functions with multiply-connected wandering domains. Under the assumption that the post singular set is contained in the Fatou set, it is shown that every repelling periodic point $p$ satisfies either

1. $C(p) \supset \partial U$, where $C(p)$ is the Julia component containing $p$ and $U$ is an immediate attractive basin.

2. $C(p) = \{p\}$ and this is a buried singleton component of $J(f)$.

§1 Introduction

Let $f$ be a transcendental entire function, $F(f)$ its Fatou set and $J(f)$ its Julia set. The following are some fundamental results on the connectivity of $J(f)$:

Proposition 1 If every Fatou component is bounded and simply connected, then $J(f) \subset \mathbb{C}$ is connected.

So it follows that if $J(f) \subset \mathbb{C}$ is disconnected, then either

(a) $f$ has an unbounded Fatou component or

(b) $f$ has a multiply-connected Fatou component.

For the case (a), the following holds. Note that an unbounded Fatou component $U$ is always simply connected (see [Ba1]) and so we can consider a Riemann map $\varphi : \mathbb{D} \to U$ of $U$.

Theorem 2 ([K, p.192, Main Theorem]) Suppose there exists an unbounded invariant Fatou component $U$ and let us consider the following conditions:
\( \infty \in \partial U \) is accessible in \( U \).

(B) There exist a finite point \( q \in \partial U \) with \( q \notin P(f) \), \( m_0 \in \mathbb{N} \) and a continuous curve \( C(t) \subset U \ (0 \leq t < 1) \) with \( C(1) = q \) which satisfies \( f^{m_0}(C) \supset C \), where

\[
P(f) = \bigcup_{n=0}^{\infty} f^n(\text{sing}(f^{-1}))
\]

is the post-singular set of \( f \).

(1) If \( U \) is either an attractive basin with (A) and (B), or a parabolic basin with (A) and (B), or a Siegel disk with (A), then the set

\[
\Theta_\infty := \{e^{i\theta} \mid \varphi(e^{i\theta}) := \lim_{r \nearrow 1} \varphi(re^{i\theta}) = \infty\} \subset \partial D
\]

is dense in \( \partial D \). In particular, \( J(f) \subset \mathbb{C} \) is disconnected.

(2) If \( U \) is a Baker domain with (B) and \( f|U \) is not univalent, then \( \Theta_\infty \) is dense in \( \partial D \) or at least its closure \( \bar{\Theta}_\infty \) contains a certain perfect set in \( \partial D \). In particular, \( J(f) \subset \mathbb{C} \) is disconnected.

Next result is a generalization of the above result:

**Theorem 3** [BD1, p.439, Theorem 1.1, 1.2, Corollary 1.3] Theorem 2 holds without the assumption (B).

On the other hand, \( J(f) \subset \mathbb{C} \) can be connected nevertheless \( f \) has an unbounded Fatou component. For example,

\[
f(z) = 2 - \log 2 + 2z - e^z
\]

has a Baker domain but \( J(f) \) is connected ([K, p.194, Theorem 4]).

For the case (b), it is known that if \( f \) has a multiply-connected Fatou component \( U \), then \( U \) is a wandering domain and bounded (see, [Ba2, Theorem 3.1]) and therefore \( J(f) \subset \mathbb{C} \) is always disconnected. Furthermore \( J(f) \cup \{\infty\} \subset \hat{\mathbb{C}} \) is also disconnected and actually this is the only case where \( J(f) \cup \{\infty\} \subset \hat{\mathbb{C}} \) can be disconnected as follows:

**Proposition 4** ([K, p.191, Theorem 1]) \( J(f) \cup \{\infty\} \subset \hat{\mathbb{C}} \) is disconnected if and only if \( f \) has a multiply-connected wandering domain.

In what follows, we will concentrate on the case (b), that is, the case where \( f \) has a multiply-connected wandering domain \( U \) and investigate some properties of connected components of the Julia set, which we call Julia components. We note the following fact (see, [Ba2, p.565, Theorem 3.1]):

**Proposition 5** If \( U \) is a multiply-connected wandering domain, then \( f^n|U \to \infty \).
Definition 6  (1) We call a connected component of $J(f)$ a Julia component.

(2) $z \in J(f)$ is called a buried point if $z$ satisfies $z \notin \partial U$ for any Fatou component $U$.

(3) We call the set
$$J_0(f) := \{ z \in J(f) \mid z \text{ is a buried point} \}$$
the residual Julia set of $f$.

(4) A Julia component $C$ is called a buried component if $C \subset J_0(f)$.

For rational cases, the following are known:

Example 7 ([Mc]) Let $f(z) = z^2 + \frac{\lambda}{z^3}$, where $\lambda > 0$ is small. Then $J(f)$ is a Cantor set of nested quasi-circles. So there are buried components. In particular, $J_0(f) \neq \emptyset$.

Theorem 8 ([Mo, p.208, Theorem 3]) Let $f$ be a hyperbolic rational function. Then $J_0(f) \neq \emptyset$ if and only if

(1) $F(f)$ has a completely invariant component, or

(2) $F(f)$ consists of only two components.

Example 9 ([Mo, p.209]) Let $f(z) = \frac{-2z + 1}{(z - 1)^2}$, then the following hold:

(1) The set $\{0, 1, \infty\}$ is a super-attracting cycle.

(2) $f$ is hyperbolic.

(3) Any Fatou component is a preimage of the super-attractive basin above.

(4) $J(f)$ is connected.

So by Theorem 8, we have $J_0(f) \neq \emptyset$. But since $J(f)$ is connected, there is no buried component.

Example 10 ([U]) There exists a rational function $f$ whose Julia set is homeomorphic to a Sierpinski gasket. So $J_0(f) \neq \emptyset$, but again there is no buried component.

Here are some fundamental properties for buried points and residual Julia sets. Note that $f$ need not be rational and these hold also for transcendental entire functions and even for meromorphic functions.

Proposition 11  (1) If $F(f)$ has a completely invariant component, then $J_0(f) = \emptyset$.

(2) If there exists a buried component of $J(f)$, then $J(f)$ is disconnected.

(3) If $J_0(f) \neq \emptyset$, then $J_0(f)$ is completely invariant, dense in $J(f)$, and uncountable.

More information on residual Julia sets, see [DF].
§2 Results

Main result of this paper is as follows:

Theorem A Let $f$ be a transcendental entire function. Assume that

(a) $P(f) = \bigcup_{n=0}^{\infty} f^n(\text{sing}(f^{-1})) \subset F(f),$

(b) $f$ has a multiply-connected wandering domain.

Then every repelling periodic point $p$ satisfies either one of the following:

1. $C(p) \supset \partial U$, where $C(p)$ is the Julia component containing $p$ and $U$ is an immediate attractive basin.
2. $\{p\}$ is a buried singleton component of $J(f)$.

Corollary B Let $f$ be a transcendental entire function. Assume the above conditions (a), (b) and also

(c) $f^n(z) \to \infty$ for any $z \in F(f)$.

Then every repelling periodic point $p$ is a buried singleton component of $J(f)$.

Remark $f$ is called hyperbolic if

$$\text{dist}_C(P(f), J(f)) > 0,$$

where $\text{dist}_C$ is the Euclidean distance on $\mathbb{C}$. So the condition (a) in Theorem A is slightly weaker than hyperbolicity.

(Outline of the Proof): Let $p$ be a repelling periodic point. For simplicity, we assume that $p$ is a fixed point. Suppose that $p$ does not satisfy (1). Let $C(p) \subset J(f)$ be the Julia component containing $p$. Then $f(C(p)) = C(p)$ and we can show that $C(p)$ is bounded. If there exists a Fatou component $U \subset F(f)$ such that $C(p) \cap \partial U \neq \emptyset$, then it follows that $U$ is a wandering domain which satisfies $f^n(U) \to \infty (n \to \infty)$. Then this contradicts the fact that $C(p)$ is bounded. Hence $C(p)$ is a buried component.

Next we can show that the complement of $C(p)$ has no bounded component. Then since $P(f) \subset F(f)$ and $C(p)$ is bounded, we have

$$\text{dist}_C(C(p), P(f)) > 0.$$

Then there exists a simply connected domain $W$ such that $C(p) \subset W$ and there exists a branch $g_n$ of $f^{-n}$ which satisfies $g_n(p) = p$. It is well-known that $\{g_n\}_{n=1}^{\infty}$ is a normal family and hence there exists a subsequence $g_{n_i}$ converging to a constant function which must be the point $p$. On the other hand, we have $g_n(C(p)) = C(p)$, so we conclude that $C(p) = \{p\}$. This completes the proof of Theorem A. Corollary B is an immediate consequence of Theorem A. $\square$
§3 Examples

Example 12 ([BD2, p.375, Theorem G]) There exists an $f(z)$ with the following form
\begin{equation*}
f(z) = k \prod_{n=1}^{\infty} \left(1 + \frac{z}{r_n}\right), \quad 0 < r_1 < r_2 < \cdots, \quad k > 0
\end{equation*}
such that for every repelling periodic point $p$ is a buried singleton component of $J(f)$.

Example 13 ([KS]) There exists a transcendental entire function $f$ with doubly-connected wandering domains, which satisfies the following: Every critical point $c$ satisfies $f^2(c) = 0$ and $0$ is a super-attracting fixed point. This implies that this $f$ satisfies the assumptions of Theorem A. Therefore every repelling periodic point $p$ satisfies either $C(p) \supset \partial U$ for the immediate attractive basin $U$ of the super-attractive fixed point $0$ or \{p\} is a buried singleton component of $J(f)$.

Example 14 ([Be]) By using the similar method as in Example 13, Bergweiler constructed an example of transcendental entire function $f$ which has both a simply connected and a multiply connected wandering domain. Critical points of $f$ satisfy the following:

1. $c_0 = 0 < c_1 < c_2 < \cdots \rightarrow \infty$,
2. $f(0) = 0$, $f(c_i) = c_{i+1}$, $i = 1, 2, \ldots$
3. $c_i$ is contained in a simply connected wandering domain $U_i$ which satisfies
\begin{equation*}
f(U_i) = U_{i+1}, \quad f^n|_{U_i} \rightarrow \infty (n \rightarrow \infty).
\end{equation*}

So this $f$ also satisfies the assumptions (a) and (b) of Theorem A.

Example C We can construct an $f$ which satisfies the assumptions (a), (b) and (c) by using the similar method as in Example 13. Hence every repelling periodic point $p$ is a buried singleton component of $J(f)$ from Corollary B. We omit the details.

References


