Existence of invariant manifolds at an indeterminate point

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Existence of invariant manifolds at an indeterminate point

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Abstract

In this note, consider dynamics of a rational mapping $F$ on 2-dimensional complex projective space $\mathbb{P}^2$ which has a periodic indeterminate point $p$. By using a symbol sequence $j \in \{1,2\}^\infty$, we will define some family $\{V_j\}_{j \in J}$ which consists of locally invariant holomorphic curves at $p$ by $F$, algebraically.

1. Introduction.

In this note, we consider a local dynamical structure of a rational mapping $F$ of $\mathbb{P}^2$ near a periodic indeterminate point $p$. Using a blow up, we construct a family $\{V_j\}_{j \in J}$ which consists of locally invariant curves at $p$ by $F$, where $J$ is a subset of the Cantor set $\{1,2\}^\infty$.

Here, prepare some notation and terminology. Let $f_i(x, y, t)(i = 0, 1, 2)$ be homogeneous polynomials with degree $d$, $F : [x : y : t] \mapsto [f_0 : f_1 : f_2]$ a rational mapping on $\mathbb{P}^2$ and $\tilde{F} : (x, y, t) \mapsto (f_0, f_1, f_2)$ a polynomial mapping on $\mathbb{C}^3$. Then, we have $\pi \circ \tilde{F} = F \circ \pi$ on $\mathbb{C}^3$ except some analytic sets, where $\pi : \mathbb{C}^3 \setminus \{(0,0,0)\} \rightarrow \mathbb{P}^2$ is the canonical projection. A point $p \in \mathbb{P}^2$ is said to be an indeterminate point of $F$ if $\tilde{F}(\tilde{p}) = (0,0,0)$ for some point $\tilde{p} \in \pi^{-1}(p)$. In general, if $p$ is an indeterminate point, then $\bigcap_{U_p} F(U_p \setminus \{p\})$ is not a single point, where the intersection is taken over all open neighborhoods $U_p$ of $p$. So, no definition of the image $F(p)$ makes the mapping $F$ be continuous. Moreover, if $p \in \bigcap_{U_p} F(U_p \setminus \{p\})$, it is called a periodic indeterminate point. It can be seen from the definition that a periodic indeterminate point has a certain recurrent property, hence we expect a local dynamical structure at this point.

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In this note, we assume that $F : \mathbb{P}^2 \to \mathbb{P}^2$ is a rational mapping with an indeterminate point $p = [0 : 0 : 1]$. We often identified $\mathbb{C}^2$ with the affine chart of $\mathbb{P}^2$ which is defined by $\{[x : y : t] \in \mathbb{P}^2 \mid t \neq 0\}$, and put $p = (0, 0)$. Let

$$X := \{(x, y) \times [u : v] \in \mathbb{C}^2 \times \mathbb{P}^1 \mid xv - yu = 0\}$$

be a subset of $\mathbb{C}^2 \times \mathbb{P}^1$. Then, $X$ is a subvariety of $\mathbb{C}^2 \times \mathbb{P}^1$ and covered by the following two coordinate neighborhoods $\{(U^j, \mu^j)\}_{j=0,1}$,

$$U^0 := \{(x, y) \times [u : v] \in X \mid y = \frac{v}{u}x\}, \quad \mu^0 : U^0 \ni (x, y) \times [u : v] \mapsto (x, \frac{v}{u}) \in \mathbb{C}^2,$$

$$U^1 := \{(x, y) \times [u : v] \in X \mid x = \frac{u}{v}y\}, \quad \mu^1 : U^1 \ni (x, y) \times [u : v] \mapsto (\frac{u}{v}, y) \in \mathbb{C}^2.$$

**Definition 1** (see [4]). The mapping $\pi : X \to \mathbb{C}^2$ defined by restricting the first projection $\mathbb{C}^2 \times \mathbb{P}^1 \to \mathbb{C}^2$ is called the blow up of $\mathbb{C}^2$ centered at $p = (0, 0)$ and $E := \pi^{-1}(0, 0) = (0, 0) \times \mathbb{P}^1$ is called the exceptional curve.

It is remarked here that $\pi : X \setminus E \to \mathbb{C}^2 \setminus \{(0, 0)\}$ is a biholomorphic mapping, and by replacing $\mathbb{C}^2$ with the affine chart $\{[x : y : t] \in \mathbb{P}^2 \mid t \neq 0\}$ of $\mathbb{P}^2$, it can be naturally extended as the blow up of $\mathbb{P}^2$ centered at $p = [0 : 0 : 1]$. To simplify the notation, we call it blow up of $\mathbb{P}^2$ centered at $p$, too.

The study of local dynamics of a periodic indeterminate point was started by Y. Yamagishi (see [8] and [9]). Here, introduce his results. Set $\tilde{F} := F \circ \pi : X \to \mathbb{P}^2$. Assume that $F$ satisfies the following assumption.

$$\begin{aligned}
&A.0 \left\{ 
\begin{array}{l}
(1) \text{$\tilde{F}$ is a holomorphic mapping on some open neighborhood of $E$,} \\
(2) \text{$\tilde{F}^{-1}(p) \cap E = \{p_1, p_2\}$ and,} \\
(3) \text{there exist open neighborhoods $N_i$ of $p_i$ such that $\tilde{F}|_{N_i}$ is a biholomorphic mapping for $i = 1, 2$.}
\end{array}
\right.
\end{aligned}$$

Here, we remark that $p$ is a periodic indeterminate point of $F$. Moreover, he assumed that $\tilde{F}$ is contracting in the horizontal direction on $N_i$. Then, it has been proved that there exists a family of local stable manifolds of $p$ which is indexed by the Cantor set

$$\{1, 2\}^N := \{j = (j_1, j_2, \ldots) \mid j_n = 1, 2 \text{ for } n \in \mathbb{N}\}.$$
It is called the Cantor bouquet (for detail, see [8] and [9]).

In this note, we consider the following family of curves which is a generalization of the Cantor bouquet.

**Definition 2.** A family \( \{W_{\lambda}\}_{\lambda \in \Lambda} \) of curves is locally invariant at \( p \) by \( F \) if

1. every \( W_{\lambda} \) is given by a graph of some continuous function
   \[ \phi_{\lambda} : \Delta_{\rho_{\lambda}} \ni x \mapsto y = \phi_{\lambda}(x) \in \mathbb{C} \]
   with \( \phi_{\lambda}(0) = 0 \), where \( \Delta_{\rho_{\lambda}} := \{ x \in \mathbb{C} \mid |x| < \rho_{\lambda} \} \), and
2. for any \( W_{\lambda} \) there is a \( \lambda' \in \Lambda \) and some open neighborhood \( N_{\lambda'} \) of \( p \) such that
   \[ \lim_{x \to 0} F(x, \phi_{\lambda}(x)) = p \]
   and
   \[ F(x, \phi_{\lambda}(x)) \cap N_{\lambda'} \subset W_{\lambda'} \text{ for } x \in \Delta_{\rho_{\lambda}} \setminus \{0\} \].

Especially, if every \( \phi_{\lambda} \) is a holomorphic function, then \( \{W_{\lambda}\} \) is called a family of holomorphic curves.

**Remark.** Let be a mapping \( \Phi_{\lambda} : \Delta_{\rho_{\lambda}} \to \mathbb{C}^2 \) by \( \Phi_{\lambda}(x) = (x, \phi_{\lambda}(x)) \). Assume that \( \Phi_{\lambda} \) is a holomorphic mapping. Then, \( F \circ \Phi_{\lambda} \) is well-defined on \( \Delta_{\rho_{\lambda}} \), even if \( p \) is an indeterminate point of \( F \), that is, there is a unique holomorphic mapping \( g : \Delta_{\rho_{\lambda}} \to \mathbb{C}^2 \) such that \( g(z) = F \circ \Phi_{\lambda}(z) \) for \( z \in \Delta_{\rho_{\lambda}} \setminus \{0\} \) (for detail, see [1]).

Now, we state our Main theorems. In the reminder of this note, denote \( j_{n} = 1, 2 \) for every \( n \in \mathbb{N} \). Assume that \( F \) satisfies the condition (A.0). Then, the following claim (A.1) holds.

\[
\begin{align*}
(1) \quad & F_0 := \pi^{-1} \circ \tilde{F} \text{ is a meromorphic mapping on } N(E) \text{ and } \{p_1, p_2\} \text{ are } \\
& \text{indeterminate points of } F_0, \text{ where } N(E) \text{ is an open neighborhood of } E. \\
& \text{Let } \pi_{j_1} : X_{j_1} \to X \text{ be the blow up of } X \text{ centered at } p_{j_1} \text{ and } \\
& \tilde{F}_{j_1} := F_0 \circ \pi_{j_1} : X_{j_1} \to X. \text{ Then,} \\
(2) \quad & \tilde{F}_{j_1|E_{j_1}} : E_{j_1} \to E \text{ is bijective, and one can set } p_{j_1j_2} := \tilde{F}_{j_1}^{-1}(p_{j_2}) \in E_{j_1}. \\
(3) \quad & \text{There is an open neighborhood } N_{j_1j_2} \text{ of } p_{j_1j_2} \text{ such that } \\
& \tilde{F}_{j_1|N_{j_1j_2}} \text{ is a biholomorphic mapping.}
\end{align*}
\]

**Theorem 1** (see [5]). We can repeat this process inductively for all \( n \in \mathbb{N} \) and symbol sequences \( j = (j_1, \ldots) \in \{1, 2\}^\mathbb{N} \), and succeed with infinitely many times of
In addition to the condition (A.0), we suppose the following condition (B).

\[(B) \ p_{j_{1} \ldots j_{n}} \in U_{j_{1} \ldots j_{n-1}}^{0} \text{ for every } n \in \mathbb{N}, \]

where $U_{j_{1} \ldots j_{n-1}}^{0}$ is the coordinate neighborhood of $X_{j_{1} \ldots j_{n-1}}$ analogue to that defined for $X$. Then, we can set $p_{j_{1} \ldots j_{n}} := (0, \alpha_{j_{1} \ldots j_{n}})$ by using the local coordinates system of $U_{j_{1} \ldots j_{n-1}}^{0}$. Finally, for all symbol sequences $j \in \{1, 2\}^{N}$ with $j = (j_{1}, j_{2}, \ldots)$, define a formal power series

$$y = \phi_{j}(x) := \alpha_{j_{1}}x + \alpha_{j_{1}j_{2}}x^{2} + \cdots,$$

$$J := \{j \in \{1, 2\}^{N} \mid \phi_{j}(x) \text{ has a positive convergent radius } \rho_{j} > 0\},$$

$$V_{j} := \{(x, y) \in N_{j} \mid y = \phi_{j}(x) \text{ on } \Delta_{\rho_{j}}\} \text{ for all } j \in J.$$ 

Then, we have the following Theorem 2.

**Theorem 2** (see [5]). $\{V_{j}\}_{j \in J}$ is a family of locally invariant holomorphic curves at $p$ by $F$. In particular, every family $\{W_{\lambda}\}_{\lambda \in \Lambda}$ of locally invariant holomorphic curves at $p$ by $F$ must be a subfamily of $\{V_{j}\}_{j \in J}$.

As applications, consider the following rational mappings of $\mathbb{C}^{2}$.

\[\ (*)_{1} \quad F(x, y) = \left(ax, \frac{y(y-x)}{x^2}\right), \quad |a| > 4,\]

\[\ (*)_{2} \quad F(x, y) = \left(x + ax^2, \frac{y(2y-x)}{x^2}\right), \quad |a| \neq 0.\]

**Theorem 3** (see [6]). Suppose that $F$ is the rational mapping in $\ (*)_{1}$. For all symbol sequences $j = (j_{1}, j_{2}, \ldots) \in \{1, 2\}^{N}$, one of the following claims holds.

1. If there exists an integer $n_{0}$ such that $j_{n} = 1$ for any $n \geq n_{0}$, then $V_{j} \neq \emptyset$ and $V_{j} \subset F^{-n_{0}}(V_{11}) = F^{-n_{0}}(\{y = 0\})$. Especially, $V_{j}$ are unstable manifolds of $p$.
2. If there exist infinitely many $n_{0} \in \mathbb{N}$ with $j_{n_{0}} = 2$, then $V_{j} = \emptyset$.

For the rational mapping $F$ in $\ (*)_{2}$, the following theorems 4 and 5 hold.
Theorem 4. For every symbol sequence $j \in \{1, 2\}^N$ there exists a continuous function $y = \psi_j(x)$ on $\Delta_\delta$. Put

$$W_j := \{(x, y) \in C^2 \mid y = \psi_j(x) \text{ on } \Delta_\delta\}.$$ 

In particular, $\{W_j\}_{j \in \{1, 2\}^N}$ is a family of curves which is locally invariant at $p$ by $F$.

Theorem 5. For any symbol sequence $j \in \{1, 2\}^N$, there exists $j' = (j'_1, j'_2, \ldots) \in \{1, 2\}^N$ such that the formal power series $\phi_{j'}(x) = \sum \alpha_{j'_1 \ldots j'_{n}} x^n$ is the asymptotic expansion of $\psi_j(x)$. That is, for all $n \in \mathbb{N}$, there exist positive constants $\delta_n$ and $M_n$ such that

$$|\psi_j(x) - \alpha_{j'_1} x - \cdots - \alpha_{j'_1 \ldots j'_{n-1}} x^{n-1}| \leq M_n|x|^n,$$

for any $x \in \Delta_{\delta_n}$.

Remark. Although $\phi_{j'}$ may not be a convergent power series, for any fixed $n \in \mathbb{N}$, $\psi_j(0)$ is approximated by the polynomial $\alpha_{j'_1} x + \cdots + \alpha_{j'_{n-1}} x^{n-1}$ with the order $O(|x|^n)$ by taking the limit as $x \to 0$.

Theorems 1, 2 and 3 have been obtained by [5] and [6]. In this note, we will give an outline of proof of Theorems 4 and 5.


Put $q(x) := x + ax^2$. This is the first component of $F$ in (*2). We begin with basic facts on dynamics of the polynomial $q(x)$ at $x = 0$ (for detail, see [7]). For the polynomial $q(x)$, $x = 0$ is a rationally indifferent fixed point and there exist an attracting petal $P$ and a repelling petal $R$ such that

1. $q(P) \subset P \cup \{0\}$, $\bigcap_{n=1}^\infty (P) = \{0\}$,

2. $(q|_R)^{-1}(\overline{R}) \subset R \cup \{0\}$, $\bigcap_{n=1}^\infty (q|_R)^{-n}(\overline{R}) = \{0\}$,

3. $\{0\} \cup P \cup R$ is an open neighborhood of 0.

Now, let us start the proof of Theorem 4. In the following part, we shall give a
proof which is based on an argument by Hadamard-Perron Theorem in [3] and the construction of the Cantor bouquet in [8].

For $\alpha_j \in C$, $p = (0, 0) \in C^2$ and $p_j := (0, \alpha_j) \in C^2$, define the following sets:

$$\Delta_r(\alpha_j) := \{x \in C | |x - \alpha_j| < r\}, \quad \Delta_r := \Delta_r(0),$$

$$\Delta_r^2(p) := \Delta_r \times \Delta_r, \quad \Delta_r^2(p_j) := \Delta_r \times \Delta_r(\alpha_j).$$

From some easy calculation, one can check that our $F$ satisfies the conditions $(A.0)$ and $(B)$. Hence, Theorems 1 and 2 hold and for any infinite symbol sequence $j \in \{1, 2\}^\mathbb{N}$, there exists the sequence of points $\{\alpha_{j_1 \ldots j_n}\}_{n \geq 1}$.

In the reminder of this note, denote $k, l = 1, 2$. From $(A.0)$, $\tilde{F}$ is a locally biholomorphic mapping on some neighborhoods of $p_l$, and there are positive constants $r$ and $r'$ and branches $G_l : \Delta_r^2(p) \to \Delta_r^2(p_l)$ of $\tilde{F}$. Let $\rho : C^2 \to [0, 1]$ be a $C^1$-function such that

$$\rho(x, y) = \begin{cases} 
1 & \text{on } \Delta_r^2(p_k), \\
0 & \text{on } \Delta_{2r}^2(p_k)^c.
\end{cases}$$

By using this $C^1$-function $\rho$, define a $C^1$-mapping $f_{kl} : C^2 \to C^2$ such that

$$f_{kl} = \begin{cases} 
G_l \circ \pi & \text{on } \Delta_r^2(p_k), \\
J(G_l \circ \pi)_{p_k} & \text{on } \Delta_{2r}^2(p_k)^c,
\end{cases}$$

where $J(G_l \circ \pi)_{p_k}$ is the Jacobian matrix of $G_l \circ \pi$ at the point $p_k$. Set

$$C_{\gamma}^{p_k} := \{\phi : C \to C, \text{Lipshitz ft. with Lipshitz constant } \gamma \text{ and } \phi(0) = \alpha_k\},$$

$$C_{\gamma} := C_{\gamma}^{p_1} \cup C_{\gamma}^{p_2}.$$ 

Then, $C_{\gamma}$ is a complete metric space with respect to the metric $d$ defined as follows;

$$d(\phi, \psi) := \begin{cases} 
\sup_{x \in C \setminus \{0\}} \frac{|\phi(x) - \psi(x)|}{|x|} & \text{if } \phi, \psi \in C_{\gamma}^{p_k}, \\
3 & \text{if } \phi \in C_{\gamma}^{p_k} \text{ and } \psi \in C_{\gamma}^{p_l} (k \neq l).
\end{cases}$$

It can be seen that for any $\phi \in C_{\gamma}^{p_k}$ there exists $\psi \in C_{\gamma}^{p_l}$ such that

$$f_{kl}(\text{graph } \phi) = \text{graph } \psi.$$ 

By using this fact, one can define the action of $g_l$ on $C_{\gamma}$ by

$$g_l(\text{graph } \phi) := \text{graph } ((f_{kl})_{*} \phi), \text{ if } \phi \in C_{\gamma}^{p_k}.$$
and know that $g_{l} : C_{\gamma} \rightarrow C^{p}_{\gamma}$ is a contraction mapping.

Let $S$ be the space of non-empty compact subsets of $C_{\gamma}$. Then, $S$ is a complete metric space with respect to the Hausdorff metric. Setting a mapping

$$G : S \rightarrow S, \text{ by } A \mapsto G(A) := g_{1}(A) \cup g_{2}(A)$$

we can show that $G$ is contraction on $S$, since $g_{l}$ is a contraction mapping.

Thus, it follows from Banach's contraction mapping theorem that $G$ has the unique fixed point $E \in S$, and $G^{n}(A)$ converges to $E$ for any $A \in S$. Here, we choose a subset $A$ of $S$ satisfying $g_{l}(A) \subset A$ for $l = 1, 2$. Then

$$\bigcap_{n=0}^{\infty} G^{n}(A) = E.$$  

Consequently, since $g_{1}(A) \cap g_{2}(A) = \emptyset$, there exists the unique point $\tilde{\phi}_{j} \in C_{\gamma}$ such that $g_{j_{1}} \circ \cdots \circ g_{j_{n}}(A) \rightarrow \tilde{\phi}_{j}$ ($n \rightarrow \infty$) for every symbol sequence $j \in \{1, 2\}^{N}$.

By using $\tilde{\phi}_{j}$, let us set

$$\tilde{W}_{j} := \{(x, y) \in C^{2} \mid y = \tilde{\phi}_{j}(x)\}.$$  

Then, it implies that $g_{l}(\tilde{W}_{j}) = \tilde{W}_{\sigma(j)}$, where $\sigma$ is the shift mapping on $\{1, 2\}^{N}$. Take a small positive constant $\delta$ with $0 < \delta < r$, and put

$$\tilde{W}_{j}^{\delta} := \tilde{W}_{j} \cap \Delta_{\delta} \times C \quad \text{and} \quad W_{j} := \pi(\tilde{W}_{j}^{\delta}).$$

Finally, we can prove that

$$W_{j} = \{(x, y) \in C^{2} \mid y = x\tilde{\phi}_{j}(x)\}$$

and $\{W_{j}\}_{j \in \{1, 2\}^{N}}$ is a family of curves which is locally invariant at $p$ by $F$. This is required.

Remark. Unfortunately, $\tilde{\phi}_{j}$ depends on the construction of an extension mapping $f_{kl}$ and does not have uniqueness. However, $\tilde{\phi}_{j}(x)$ is determined uniquely for any $x \in P$, where $P$ is an attracting petal of $q(x)$ at 0, and

$$F^{n}(x, y) \rightarrow p \quad \text{as} \quad n \rightarrow \infty \quad \text{for any} \quad (x, y) \in W_{j} \cap \{P \times C\} \quad \text{with} \quad x \neq 0.$$
3. Proof of Theorem 5.

To prove Theorem 5, we need the following Lemmas 1 and 2.

Lemma 1. For every symbol sequence \( j \in \{1, 2\}^N \) the following claims hold;
(1) there exists a point \( p_{j_1} \in \{p_1, p_2\} \) such that \( \pi^{-1}(W_j \setminus \{p\}) \cap E = \{p_{j_1}\} \), and put
\[
(W_j)_{j_1} := \pi^{-1}(W_j \setminus \{p\}),
\]
(2) there exists a continuous function \( \phi_{j_1} \) on \( \Delta_\delta \) such that
\[
(W_j)_{j_1} = \{(x, y) \in \Delta_\delta \times \mathbb{C} | y = \phi_{j_1}(x) \text{ on } \Delta_\delta \}.
\]

Since \( \{W_j\}_{j \in \{1, 2\}^N} \) is a family of curves which is locally invariant at \( p \) by \( F \), for every \( W_j \) there exists a symbol sequence \( i = (i_1, i_2, \ldots) \in \{1, 2\}^N \) and an open neighborhood \( N_i \) of \( p \) such that \( F(W_j \setminus \{p\}) \cap N_i \subset W_i \). From Lemma 1 (1), there is a point \( p_{i_1} \in \{p_1, p_2\} \) such that \( \pi^{-1}(W_i \setminus \{p\}) \cap E = \{p_{i_1}\} \). Put
\[
(W_i)_{i_1} := \pi^{-1}(W_i \setminus \{p\}) \quad \text{and} \quad F_0 := \pi^{-1} \circ \tilde{F}.
\]

Then, we have the following lemma.

Lemma 2.
(1) There exists an open neighborhood \( (N_i)_{i_1} \) of \( p_{i_1} \) such that
\[
\lim_{x \to 0} F_0(x, \phi_{j_1}(x)) = p_{i_1} \quad \text{and} \quad F_0((W_j)_{j_1} \setminus \{p_{j_1}\}) \cap (N_i)_{i_1} \subset (W_i)_{i_1}
\]
(2) There exist positive constants \( \delta_{j_1} \) and \( M_{j_1} \) such that
\[
|\psi_j(x) - \alpha_{j_1} x| \leq M_{j_1} \quad \text{for} \quad x \in \Delta_{\delta_{j_1}}.
\]

We can repeat this process inductively for every \( n \in \mathbb{N} \) and prove Theorem 5.

References


