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Hochschild cohomology ring of an order of a quaternion algebra

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Introduction

The cohomology theory of associative algebras was initiated by Hochschild [6], Cartan and Eilenberg [1] and MacLane [7]. Let $R$ be a commutative ring and $\Lambda$ an $R$-algebra which is a finitely generated projective $R$-module. If $M$ is a $\Lambda$-bimodule (i.e., a $\Lambda_e = \Lambda \otimes_R \Lambda^{\text{op}}$-module), then the $n$th Hochschild cohomology of $\Lambda$ with coefficients in $M$ is defined by $H^n(\Lambda, M) := \text{Ext}^n_R(\Lambda, M)$. We set $HH^n(\Lambda) = H^n(\Lambda, \Lambda)$. The cup product gives $HH^*(\Lambda) := \bigoplus_{n \geq 0} HH^n(\Lambda)$ a graded ring structure with $1 \in Z\Lambda \simeq HH^0(\Lambda)$ where $Z\Lambda$ denotes the center of $\Lambda$. $HH^*(\Lambda)$ is called the Hochschild cohomology ring of $\Lambda$. It is known that the cup product coincides with the Yoneda product on the Ext-algebra. Note that the Hochschild cohomology ring $HH^*(\Lambda)$ is graded-commutative, that is, for $\alpha \in HH^p(\Lambda)$ and $\beta \in HH^q(\Lambda)$ we have $\alpha \beta = (-1)^{pq} \beta \alpha$. The Hochschild cohomology is an important invariant of algebras, however the Hochschild cohomology ring is difficult to compute in general.

Let $G$ denote the generalized quaternion 2-group of order $2^{r+2}$ for $r \geq 1$:

$$Q_{2^r} = \langle x, y | x^{2^{r+1}} = 1, x^{2^r} = y^2, yxy^{-1} = x^{-1} \rangle.$$

We set $e = (1 - x^{2^r})/2 \in \mathbb{Q}G$ and denote $xe$ by $\zeta$, a primitive $2^{r+1}$-th root of $e$. Then $e$ is a centrally primitive idempotent of $\mathbb{Q}G$. The simple component $\mathbb{Q}Ge$ is just the ordinary quaternion algebra over the field $K := \mathbb{Q}(\zeta + \zeta^{-1})$ with identity $e$, that is, $\mathbb{Q}Ge = K \otimes K \otimes K_j \otimes K_{ij}$ where we set $i = x^{2^{r-1}}e$ and $j = ye$ (see [2, (7.40)]). Note that $\zeta^k j = j \zeta^{-k}$ and $\zeta^{2^r} = -e$ hold. In the following we set $R = \mathbb{Z}[\zeta + \zeta^{-1}]$, the ring of integers of $K$, and we set $\Gamma = \mathbb{Z}Ge = R \oplus R\zeta \oplus Rj \oplus R\zeta j$. Note that $R$ is a commuting parameter ring, because $y$ commutes with $x + x^{-1}$. Then $\Gamma$ is an $R$-order of $\mathbb{Q}Ge$. In particular if $r = 1$, $\Gamma = \mathbb{Z}e \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z}ij$ is just the ordinary quaternion algebra over $\mathbb{Z}$ with identity $e$.

We will give an efficient bimodule projective resolution of $\Gamma$, and we will determine the ring structure of the Hochschild cohomology $HH^*(\Gamma)$ by calculating the Yoneda products using this bimodule projective resolution. This paper is a summary of [3].

1 A bimodule projective resolution of $\Gamma$

In this section, we state a $\Gamma^e$-projective resolution of $\Gamma$.

In general, $\Gamma \otimes \Gamma$ is a left $\Gamma_e$-module (i.e., a $\Gamma$-bimodule) by putting

$$(a \otimes b^o) \cdot (\gamma_1 \otimes \gamma_2) := a\gamma_1 \otimes \gamma_2 b$$
for all \( a, b, \gamma_1, \gamma_2 \in \Gamma \). For each \( q \geq 0 \), let \( Y_q \) be a direct sum of \( q + 1 \) copies of \( \Gamma \otimes \Gamma \). As elements of \( Y_q \), we set
\[
c_q^s = \begin{cases} (0, \ldots, 0, e \otimes e_0, 0, \ldots, 0) & \text{if } 1 \leq s \leq q + 1, \\ 0 & \text{(otherwise)}. \end{cases}
\]

Then we have \( Y_q = \bigoplus_{k=1}^{q+1} \Gamma c_q^k \Gamma \). Let \( t = 2^r \). Define left \( \Gamma^e \)-homomorphisms \( \pi : Y_0 \to \Gamma; c_0^1 \mapsto e \) and \( \delta_q : Y_q \to Y_{q-1} \) \((q > 0)\) given by
\[
\delta_q(c_q^s) = \begin{cases} -\zeta c_{q-1}^s + c_{q-1}^s \zeta + (-1)^{(q-s)/2} \zeta j c_{q-1}^{s-1} j \zeta - c_{q-1}^{s-1} & \text{for } q \text{ even, } s \text{ even,} \\ \sum_{l=0}^{t-1} \zeta^{t-1-l} c_{q-1}^s \zeta^l + (-1)^{(q-s-1)/2} j c_{q-1}^{s-1} j + c_{q-1}^{s-1} & \text{for } q \text{ even, } s \text{ odd,} \\ -\sum_{l=0}^{t-1} \zeta^{t-1-l} c_{q-1}^s \zeta^l + (-1)^{(q-s-1)/2} j c_{q-1}^{s-1} j - c_{q-1}^{s-1} & \text{for } q \text{ odd, } s \text{ even,} \\ \zeta c_{q-1}^s - c_{q-1}^s \zeta + (-1)^{(q-s)/2} \zeta j c_{q-1}^{s-1} j \zeta + c_{q-1}^{s-1} & \text{for } q \text{ odd, } s \text{ odd.} \end{cases}
\]

**Theorem 1.** The above \( (Y, \pi, \delta) \) is a \( \Gamma^e \)-projective resolution of \( \Gamma \).

**Proof.** By the direct calculations, we have \( \pi \cdot \delta_1 = 0 \) and \( \delta_q \cdot \delta_{q+1} = 0 \) \((q \geq 1)\).

To see that the complex \( (Y, \pi, \delta) \) is acyclic, we state a contracting homotopy. In general, it suffices to define the homotopy as an abelian group homomorphism. However, we can see that there exists a homotopy as a right \( \Gamma \)-module, which permits us to cut down the number of cases. We define right \( \Gamma \)-homomorphisms \( T_{-1} : \Gamma \to Y_0 \) and \( T_q : Y_q \to Y_{q+1} \) \((q \geq 0)\) as follows:
\[
T_{-1}(\gamma) = c_0^1 \gamma \quad \text{(for } \gamma \in \Gamma) .
\]

If \( q \geq 0 \) is even, then
\[
T_q(\zeta^k c_q^s) = \begin{cases} 0 & (k = 0, s = 1), \\ \sum_{l=0}^{k-1} \zeta^k \zeta^l c_{q+1}^1 j & (1 \leq k < t, s = 1), \\ 0 & (s \geq 2 \text{ even}), \\ -\zeta^k c_{q+1}^{s+1} & (s \geq 3 \text{ odd}), \end{cases}
\]

\[
T_q(\zeta^k j c_q^s) = \begin{cases} (-1)^{q/2} c_{q+1}^2 j & (k = 0, s = 1), \\ (-1)^{q/2} \left( \sum_{l=0}^{k-1} \zeta^{k-l} c_{q+1}^1 j + \zeta^k c_{q+1}^2 j \right) & (1 \leq k < t, s = 1), \\ \zeta^k j c_{q+1}^{s+1} & (s \geq 2 \text{ even}), \\ 0 & (s \geq 3 \text{ odd}). \end{cases}
\]
If \( q \geq 1 \) is odd, then

\[
T_q(\zeta^k c_q^s) = \begin{cases} 
0 & (0 \leq k \leq t-2, \ s = 1), \\
c_{q+1}^1 & (k = t-1, \ s = 1), \\
0 & (s \geq 2 \text{ even}), \\
-\zeta^k c_{q+1}^{s+1} & (s \geq 3 \text{ odd}),
\end{cases}
\]

\[
T_q(\zeta^k j c_q^s) = \begin{cases} 
(-1)^{(q-1)/2}(c_{q+1}^1 j \zeta + \zeta^{t-1} c_{q+1}^2 j \zeta) & (k = 0, 1 \leq k < t, \ s = 1), \\
\zeta^k j c_{q+1}^{s+1} & (s \geq 2 \text{ even}), \\
0 & (s \geq 3 \text{ odd}).
\end{cases}
\]

Then by the direct calculations, we have

\[
\delta_{q+1} T_q + T_{q-1}\delta_q = \text{id}_{Y_q}
\]

for \( q \geq 0 \). Hence \((Y, \pi, \delta)\) is a \( \Gamma^e \)-projective resolution of \( \Gamma \).

## 2 Hochschild cohomology \( HH^*(\Gamma) \)

### 2.1 Module structure

In this section, we give the module structure of \( HH^*(\Gamma) \). This is obtained by using the \( \Gamma^e \)-projective resolution \((Y, \pi, \delta)\) of \( \Gamma \) stated in Theorem 1. In the following we denote a direct sum of \( q \) copies of a module \( M \) by \( M^q \).

First, we state the following lemma:

**Lemma 1.** Let \( \zeta \) be a primitive \( 2^{r+1} \)-th root of 1 for any positive integer \( r \geq 2 \) and \( K \) the maximal real subfield \( \mathbb{Q}(\zeta + \zeta^{-1}) \) of \( \mathbb{Q}(\zeta) \). Then \((\zeta + \zeta^{-1})^2\) divides 2 in \( R \), where \( R \) denotes \( \mathbb{Z}[\zeta + \zeta^{-1}] \), the ring of integers of \( K \).

**Proof.** See [4, Lemma 1]. Note that \( \zeta^{2k} + \zeta^{-2k} \) divides 2 in \( R \) for \( 0 \leq k \leq r - 2 \). □

If \( r \geq 2 \), we set \( \eta_k = 2e/(\zeta^{2k} + \zeta^{-2k}) \) for \( 0 \leq k \leq r - 2 \) in the following. Let \( \eta = \eta_0 \).

In the following, we show that \( e - \eta^2 \) is an unit in \( R \). If \( r = 2 \), then we have \( e - \eta^2 = -e \).

If \( r \geq 3 \), then we have

\[
-(e - \eta^2) \prod_{k=1}^{r-2}(e + \eta_k)^2 = -(e - \eta_{r-2}^2) = e,
\]

because the equation \((e - \eta_k^2)(e + \eta_k)^2 = e - \eta_k^2 \) holds for \( 1 \leq k \leq r - 2 \). Therefore \( e - \eta^2 \) is an unit in \( R \).

As elements of \( \Gamma^{q+1} \), we set

\[
\iota_q^s = \begin{cases} 
(0, \ldots, 0, \delta, 0, \ldots, 0) & (\text{if } 1 \leq s \leq q + 1), \\
0 & (\text{otherwise}).
\end{cases}
\]
Then we have $\Gamma^{q+1} = \bigoplus_{k=1}^{q+1} \Gamma^k$.

Applying the functor $\text{Hom}_{\Gamma}(\cdot, \Gamma)$ to the resolution $(Y, \pi, \delta)$, we have the following complex, where we identify $\text{Hom}_{\Gamma}(Y_{q}, \Gamma)$ with $\Gamma^{q+1}$ using an isomorphism $\text{Hom}_{\Gamma^G}(Y_{q}, \Gamma) \rightarrow \Gamma^{q+1}; f \mapsto \sum_{k=1}^{q+1} f(c_{q}^{k})\iota_{q}^{k}$:

\[
\text{(Hom}_{\Gamma}(Y, \Gamma), \delta^\#) : 0 \rightarrow \Gamma \xrightarrow{\delta^\#} \Gamma^2 \xrightarrow{\delta_2^\#} \Gamma^3 \xrightarrow{\delta_3^\#} \Gamma^4 \xrightarrow{\delta_4^\#} \Gamma^5 \rightarrow \cdots
\]

\[
\delta_{q+1}^\#(\gamma \iota_{q}^s) = \begin{cases} 
- \sum_{l=0}^{q-1} \zeta^{q-l}\gamma \iota_{q}^l + ((-1)^{(q-s)/2}\zeta j \gamma j + \gamma)\iota_{q}^{s+1} & \text{for } q \text{ even, } s \text{ even}, \\
(-\zeta \gamma - \gamma \zeta)\iota_{q}^{s+1} + ((-1)^{(q-s-1)/2}\gamma j j - \gamma)\iota_{q}^{s+1} & \text{for } q \text{ even, } s \text{ odd}, \\
-\zeta \gamma - \gamma \zeta)\iota_{q}^{s+1} + ((-1)^{(q-s-1)/2}\gamma j j + \gamma)\iota_{q}^{s+1} & \text{for } q \text{ odd, } s \text{ even}, \\
\sum_{l=0}^{t-1} \zeta^{t-1-l}\gamma \iota_{q}^l \iota_{q}^{s+1} + ((-1)^{(q-s)/2}\zeta j j \gamma j - \gamma)\iota_{q}^{s+1} & \text{for } q \text{ odd, } s \text{ odd}.
\end{cases}
\]

In the above, note that

\[
\gamma \iota_{q}^s = \begin{cases} 
(0, \ldots, 0, \check{\gamma}, 0, \ldots, 0) & \text{if } 1 \leq s \leq q + 1, \\
0 & \text{otherwise},
\end{cases}
\]

for $\gamma \in \Gamma$, and so on.

**Theorem 2.** (1) If $r = 1$, the $\mathbb{Z}$-module structure of $HH^n(\Gamma)$ is given as follows:

(i) If $n = 0$, then $HH^{0}(\Gamma) = \mathbb{Z}$.

(ii) If $n = 1$, then $HH^{1}(\Gamma) = (\mathbb{Z}/2\mathbb{Z})^3$ with generators $\zeta j \iota_{1}^1, j \iota_{1}^1 + \zeta j \iota_{1}^2, \zeta \iota_{1}^2$.

(iii) If $n = 2$, then $HH^{2}(\Gamma) = (\mathbb{Z}/2\mathbb{Z})^5$ with generators $\zeta \iota_{2}^1, \iota_{2}^1 + \zeta \iota_{2}^2, j \iota_{2}^2, \zeta \iota_{2}^2 - j \iota_{2}^3, \iota_{2}^3$.

(iv) If $n = 3$, then $HH^{3}(\Gamma) = (\mathbb{Z}/2\mathbb{Z})^7$ with generators $ji_{3}^1, \zeta ji_{3}^1 - ji_{3}^2, i_{3}^3, \zeta i_{3}^3 - i_{3}^3, ji_{3}^3, ji_{3}^3$.

(v) If $n = 4k \ (k \neq 0)$, then $HH^{n}(\Gamma) = (\mathbb{Z}/2\mathbb{Z})^{2n+1}$ with generators

\[
\iota_{n}^{4l+1}, \zeta \iota_{n}^{4l+1} - \iota_{n}^{4l+2}, j \iota_{n}^{4l+2}, j \iota_{n}^{4l+2} + \zeta j \iota_{n}^{4l+3}, \zeta j \iota_{n}^{4l+3} + \zeta j \iota_{n}^{4l+4}, j \iota_{n}^{4l+4}, \zeta j \iota_{n}^{4l+4} - j \iota_{n}^{4l+4}, \iota_{n}^{4l+4},
\]

where $l = 0, 1, 2, \ldots, k - 1$.

(vi) If $n = 4k + 1 \ (k \neq 0)$, then $HH^{n}(\Gamma) = (\mathbb{Z}/2\mathbb{Z})^{2n+1}$ with generators

\[
\zeta j \iota_{n}^{4l+1}, j \iota_{n}^{4l+1} + \zeta j \iota_{n}^{4l+2}, \zeta j \iota_{n}^{4l+2} + \zeta j \iota_{n}^{4l+3}, \zeta j \iota_{n}^{4l+3} + ji_{n}^{4m+3}, ji_{n}^{4m+3}, \zeta ji_{n}^{4m+3} - ji_{n}^{4m+4}, ji_{n}^{4m+4}, \zeta ji_{n}^{4m+4} - ji_{n}^{4m+5},
\]

where $l = 0, 1, 2, \ldots, k$ and $m = 0, 1, 2, \ldots, k - 1$. 

(vii) If $n = 4k + 2$ ($k \neq 0$), then $HH^n(\Gamma) = (\mathbb{Z}/2\mathbb{Z})^{2n+1}$ with generators
\[
\begin{align*}
\zeta \iota_n^{4l+1}, & \quad \iota_n^{4l+1} + \zeta \iota_n^{4l+2}, \quad j \iota_n^{4l+2}, \quad \zeta j \iota_n^{4l+2} - j \iota_n^{4l+3}, \quad \iota_n^{4l+3}, \\
\zeta \iota_n^{4m+3} - \iota_n^{4m+4}, & \quad \zeta j \iota_n^{4m+4}, \quad j \iota_n^{4m+4} + \zeta j \iota_n^{4m+5},
\end{align*}
\]
where $l = 0, 1, 2, \ldots, k$ and $m = 0, 1, 2, \ldots, k - 1$.

(viii) If $n = 4k + 3$ ($k \neq 0$), then $HH^n(\Gamma) = (\mathbb{Z}/2\mathbb{Z})^{2n+1}$ with generators
\[
\begin{align*}
j \iota_n^{4l+1}, & \quad \zeta j \iota_n^{4l+1} - j \iota_n^{4l+2}, \quad \iota_n^{4l+2} + \zeta \iota_n^{4l+3}, \quad \zeta j \iota_n^{4l+3}, \\
\zeta \iota_n^{4l+3} + j \iota_n^{4l+4}, & \quad \zeta \iota_n^{4l+4}, \quad \iota_n^{4m+4} + \zeta j \iota_n^{4m+5},
\end{align*}
\]
where $l = 0, 1, 2, \ldots, k$ and $m = 0, 1, 2, \ldots, k - 1$.

(2) If $r \geq 2$, the $R$-module structure of $HH^n(\Gamma)$ is as follows:

(i) If $n = 0$, then $HH^0(\Gamma) = R$.

(ii) If $n = 1$, then $HH^1(\Gamma) = (R/(\zeta + \zeta^{-1})R)^3$ with generators $(j - \eta \zeta j) \iota_1^1$, $(\zeta j - \eta j) \iota_1^2$, $(e - \eta \zeta) \iota_1^3$.

(iii) If $n = 2$, then $HH^2(\Gamma) = R/2^r R \oplus (R/(\zeta + \zeta^{-1})R)^4$, where the $R/2^r R$ summand is generated by $(e - \eta \zeta) \iota_2^1$ and the $(R/(\zeta + \zeta^{-1})R)^4$ summands are generated by $2^{-r-1} \eta \iota_2^1 + \zeta \iota_2^2, \ i_2^2, \ j \iota_2^2 - j \iota_3^2, \ i_3^2$.

(iv) If $n = 3$, then $HH^3(\Gamma) = (R/(\zeta + \zeta^{-1})R)^7$ with generators $j \iota_3^1, \ j \iota_3^1 - j \iota_3^2, \ i_3^2, \ 2^{-r-1} \eta \zeta \iota_3^2 + (\zeta - \eta) \iota_3^3, \ (j - \eta \zeta j) \iota_3^4, \ (j - \zeta j) \iota_3^5, \ (\zeta - \eta j) \iota_3^6, \ (e - \zeta \iota_3^7$.

(v) If $n = 4k$ ($k \neq 0$), then $HH^n(\Gamma) = R/2^r R \oplus (R/(\zeta + \zeta^{-1})R)^{2n}$, where the $R/2^r R$ summand is generated by $\iota_n^1$ and the $(R/(\zeta + \zeta^{-1})R)^{2n}$ summands are generated by
\[
\begin{align*}
2^{-r-1} \eta \iota_n^{4l+1} + (\zeta - \eta) \iota_n^{4l+2}, & \quad (j - \eta \zeta j) \iota_n^{4l+2}, \quad (\zeta j - \eta j) \iota_n^{4l+3}, \quad (j - \eta \zeta j) \iota_n^{4l+3}, \\
(\zeta - \eta \zeta j) \iota_n^{4l+4}, & \quad j \iota_n^{4l+4}, \quad \zeta j \iota_n^{4l+4} - j \iota_n^{4l+5}, \quad \iota_n^{4l+5},
\end{align*}
\]
where $l = 0, 1, 2, \ldots, k$.

(vi) If $n = 4k + 1$ ($k \neq 0$), then $HH^n(\Gamma) = (R/(\zeta + \zeta^{-1})R)^{2n+1}$ with generators
\[
\begin{align*}
(j - \eta \zeta j) \iota_n^{4l+1}, & \quad (\zeta j - \eta j) \iota_n^{4l+1} + (j - \eta \zeta j) \iota_n^{4l+2}, \quad (e - \eta \zeta) \iota_n^{4l+2}, \\
2^{-r-1} \eta \iota_n^{4m+2} + \zeta \iota_n^{4m+3}, & \quad j \iota_n^{4m+3}, \quad \zeta j \iota_n^{4m+3} - j \iota_n^{4m+4}, \quad \iota_n^{4m+4}, \\
2^{-r-1} \eta \iota_n^{4m+4} + (\zeta - \eta) \iota_n^{4m+5},
\end{align*}
\]
where $l = 0, 1, 2, \ldots, k$ and $m = 0, 1, 2, \ldots, k - 1$.

(vii) If $n = 4k + 2$ ($k \neq 0$), then $HH^n(\Gamma) = R/2^r R \oplus (R/(\zeta + \zeta^{-1})R)^{2n}$, where the $R/2^r R$ summand is generated by $(e - \eta \zeta) \iota_n^1$ and the $(R/(\zeta + \zeta^{-1})R)^{2n}$ summands are generated by
\[
\begin{align*}
2^{-r-1} \eta \iota_n^{4l+1} + \zeta \iota_n^{4l+2}, & \quad j \iota_n^{4l+2}, \quad \zeta j \iota_n^{4l+2} - j \iota_n^{4l+3}, \quad \iota_n^{4l+3}, \\
2^{-r-1} \eta \zeta \iota_n^{4m+3} + (\zeta - \eta) \iota_n^{4m+4}, & \quad (j - \eta \zeta j) \iota_n^{4m+4}, \quad (j - \zeta j) \iota_n^{4m+4} + (j - \eta \zeta j) \iota_n^{4m+5}, \quad (e - \zeta \iota_n^{4m+5},
\end{align*}
\]
where $l = 0, 1, 2, \ldots, k$ and $m = 0, 1, 2, \ldots, k - 1$. 

\[\]
(viii) If $n = 4k + 3$ ($k \neq 0$), then $HH^n(\Gamma) = (R/(\zeta + \zeta^{-1})R)^{2n+1}$ with generators
$$j^i_n, \zeta j^i_n, j^{i+2}_n, \zeta j^{i+2}_n, 2^{r-1}\mu_n, (\zeta - \eta j^i_n), (\zeta - \eta j^{i+2}_n), (j - \eta j^i_n), (j - \eta j^{i+2}_n),$$
$$\zeta^i_n, j^i_n + (j - \eta j^i_n), \zeta^i_n + (j - \eta j^i_n),$$
where $l = 0, 1, 2, \ldots, k$ and $m = 0, 1, 2, \ldots, k - 1$.

Proof. The proof is straightforward. However it is complicated.

2.2 Ring structure

In this subsection, we will determine the ring structure of the Hochschild cohomology ring $HH^*(\Gamma)$.

Recall the Yoneda product in $HH^*(\Gamma)$. Let $\alpha \in HH^n(\Gamma)$ and $\beta \in HH^m(\Gamma)$, where $\alpha$ and $\beta$ are represented by cocycles $f_{\alpha} : Y_n \to \Gamma$ and $f_{\beta} : Y_m \to \Gamma$, respectively. There exists the commutative diagram of $\Gamma$-modules:

$$
\ldots \xrightarrow{j} Y_{n+m} \xrightarrow{j} Y_{n+m} \xrightarrow{j} Y_{n+m} \xrightarrow{j} Y_{n+m} \xrightarrow{j} Y_{n+m} \xrightarrow{j} \Gamma \\
\mu_n \downarrow \mu_{n+1} \quad \mu_n \downarrow \mu_n \downarrow \mu_n \downarrow \mu_n \downarrow \mu_n \downarrow \\
\ldots \xrightarrow{j} Y_n \xrightarrow{j} Y_n \xrightarrow{j} Y_n \xrightarrow{j} Y_n \xrightarrow{j} \Gamma
$$

where $\mu_l$ ($0 \leq l \leq n$) are liftings of $f_{\beta}$. We define the product $\alpha \cdot \beta \in HH^{n+m}(\Gamma)$ by the cohomology class of $f_{\alpha} \mu_n$. This product is independent of the choice of representatives $f_{\alpha}$ and $f_{\beta}$, and liftings $\mu_l$ ($0 \leq l \leq n$).

First, we consider the case $r = 1$. Note the Hochschild cohomology ring $HH^*(\Gamma)$ is graded-commutative. From Theorem 2 (1), $HH^*(\Gamma)$ is a commutative ring in this case. We take generators of $HH^1(\Gamma)$ as follows:

$$A = \zeta j^1, \quad B = \zeta j^1, \quad C = j^1 + \zeta j^1.$$

Then we have $2A = 2B = 2C = 0$. We calculate the Yoneda products. Then $HH^n(\Gamma)$ ($n \geq 2$) is multiplicatively generated by $A, B$ and $C$, and the equation $A^2 + B^2 + C^2 = 0$ holds. Moreover the relations are enough. Thus we can determine the ring structure of $HH^*(\Gamma)$ in the case $r = 1$ (see [3, Section 3.1] for details).

Next, we consider the case $r \geq 2$. The computation is similar to the case where $r = 1$, however it is more complicated. By Theorem 2 (2), we take generators of $HH^1(\Gamma)$ as follows:

$$A = (e - \eta j^1), \quad B = (j - \eta j^1), \quad C = (\zeta j - \eta) j^1 + (j - \eta j^1).$$

Then we have $(\zeta + \zeta^{-1})A = (\zeta + \zeta^{-1})B = (\zeta + \zeta^{-1})C = 0$. Note that products of $A, B, C$ and $X \in HH^n(\Gamma)$ ($n \geq 0$) are commutative, because $HH^*(\Gamma)$ is graded-commutative and the equations $2A = 2B = 2C = 0$ hold. By calculating the Yoneda products we have the following proposition.
Proposition 2. If $r \geq 2$, then the following equations hold in $HH^2(\Gamma)$:

\[ A^2 = \iota_2^3, \quad AB = \iota_2^2, \quad AC = \zeta \iota_2^2 - \iota_2^3, \quad B^2 = 2^{r-1} \eta \zeta \iota_2^2 + \iota_2^2, \]
\[ BC = 2^{r-1} \eta (\epsilon - \eta \zeta) \iota_2^1, \quad C^2 = 2^{r-1} \eta \zeta \iota_2^1 + \zeta \iota_2^2 + \iota_2^3. \]

In particular, generators of $HH^2(\Gamma)$ except $(\epsilon - \eta \zeta) \iota_2^1$ are generated by the products of $A, B$ and $C$, and the equation $A^2 + B^2 + C^2 = 0$ holds.

In the following, we put $D = (\epsilon - \eta \zeta) \iota_2^1$ which is a generator of $HH^2(\Gamma)$, and then we have $2^r D = 0$ and $BC = 2^{r-1} \eta D$. Similarly, we calculate the Yoneda products. Then $HH^n(\Gamma)$ $(n \geq 3)$ is multiplicatively generated by $A, B, C$ and $D$, and the relations are enough. Thus we can determine the ring structure of $HH^*(\Gamma)$ in the case $r \geq 2$ (see [3, Section 3.2] for details).

We state the ring structure of the Hochschild cohomology ring $HH^*(\Gamma)$ by summarizing these computations.

Theorem 3. (1) If $r = 1$, then the Hochschild cohomology ring $HH^*(\Gamma)$ is isomorphic to

\[ \mathbb{Z}[A, B, C]/(2A, 2B, 2C, A^2 + B^2 + C^2), \]

where $\deg A = \deg B = \deg C = 1$.

(2) If $r \geq 2$, then the Hochschild cohomology ring $HH^*(\Gamma)$ is isomorphic to

\[ R[A, B, C, D]/((\zeta + \zeta^{-1}) A, (\zeta + \zeta^{-1}) B, (\zeta + \zeta^{-1}) C, 2^r D, \]
\[ A^2 + B^2 + C^2, BC - 2^{r-1} \eta D), \]

where $R = \mathbb{Z}[\zeta + \zeta^{-1}]$, $\deg A = \deg B = \deg C = 1$ and $\deg D = 2$.

Remark. In the case $r = 1$, this cohomology ring is already known by Sanada [8, Section 3.4]. In [8], he treats the Hochschild cohomology of crossed products over a commutative ring and its product structure using a spectral sequence of a double complex. As a special case, he determines the Hochschild cohomology ring of the quaternion algebra over $\mathbb{Z}$.

References


