<table>
<thead>
<tr>
<th>Title</th>
<th>A Note on the Stability Spectrum of Generic Structures (Model theoretic aspects of the notion of independence and dimension)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>IKEDA, Koichiro</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 2007, 1555: 104-109</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2007-05</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/80981">http://hdl.handle.net/2433/80981</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
A Note on the Stability Spectrum of Generic Structures

Koichiro Ikeda
Faculty of Business Administration, Hosei University

Abstract

Let $L$ be a countable relational language and $K$ a class of finite $L$-structures closed under subgraphs. Let $\overline{K}$ be a class of $L$-structures such that any finite substructure belongs to $K$.

**Definition 1** Let $ABC \in \overline{K}$. Then $B$ and $C$ are said to be free over $A$ (in symbol, $B \perp_A C$), if it satisfies the following:
(i) $B \cap C \subset A$;
(ii) $R^{ABC} = R^{AB} \cup R^{AC}$ for any $R \in L$.

**Remark 2** Let $ABCD \in \overline{K}$. Then
(i) If $A \perp_B C$ and $A \perp_{BC} D$, then $A \perp_{BCD}$.
(ii) If $BC \perp_A D$, then $B \perp_{CAD}$.
(iii) If $BC \perp_A D$, then $B \perp_A C$ if and only if $B \perp_D C$.

**Definition 3** $\delta : K \to \mathbb{R}_{\geq 0}$ is said to be a predimension, if
(i) if $A \cong B \in K$, then $\delta(A) = \delta(B)$;
(ii) $\delta(\emptyset) = 0$;
(iii) for all $AB \in K, \delta(A/B) \leq \delta(A/A \cap B)$;
(iv) there is no infinite chain $A_1 \subset A_2 \subset \ldots$ of $A_i \in K$ with $\delta(A_i) > \delta(A_{i+1})$ for

*Research partially supported by Grants-in-Aid for Scientific Research (no.16540123), Ministry of Education, Science and Culture.
(v) for any $AB \in K$, $A \perp_{A \cap B} B$ if and only if $\delta(A/B) = \delta(A/A \cap B)$;
(vi) for any $ABCD \in K$ with $B \cap ACD = \emptyset$, $\delta(B/AC) - \delta(B/A) \leq \delta(B/DAC) - \delta(B/DA)$,
where $\delta(X/Y)$ means $\delta(XY) - \delta(Y)$.

**Definition 4** (i) For $A \subset B \in K$, we define $A \leq B$, if $\delta(X/A') \geq 0$ for any finite $X \subset B - A$ and $A' \subset A$. For $A \subset B \in K$, define $cl_B(A) = \bigcap \{A' : A \subset A' \leq B\}$. By the definition of a predimension, there exists such a $cl_B(A)$, and moreover if $A$ is finite, then so is $cl_B(A)$.
(ii) Fix $M \in K$. For finite $A \subset M$, define $d_M(A) = \delta(cl_B(A))$. For finite $B \subset M$, $d_M(A/B) = d_M(AB) - d_M(B)$. For infinite $B$, $d_M(A/B) = \inf \{d_M(A/B') : B' \subset B \text{ finite}\}$. For (possibly) infinite $A, B, C \subset M$, $d_M(B/C) = d_M(B/A)$ means $d_M(B'/C) = d_M(B'/A)$ for any finite $B' \subset B$.
(iii) A countable $L$-structure $M$ is said to be $(K, \leq)$-generic, if $A \in K$ for any finite $A \subset M$; if $A \leq B \in K$, then there is $B' \cong_A B$ with $B' \leq M$.

Let $\mathcal{M}$ be a big model. The following facts can be found in [2], [5] and [6].

**Fact 5** Let $B, C \leq \mathcal{M}$ and $A = B \cap C$. Then the following are equivalent:
(i) $d(B/C) = d(B/A)$;
(ii) $B \perp_A C$ and $BC \leq \mathcal{M}$.

**Proof** (i)$\Rightarrow$(ii). First we show that $BC \leq \mathcal{M}$. If not, then there are $\overline{b} \in B, \overline{c} \in C, \overline{e} \in cl(\overline{b}\overline{c}) - BC$ with $\delta(\overline{e}/\overline{b}\overline{c}) = -\gamma < 0$. Take $\overline{a} \leq A$ with $d(\overline{b}/\overline{a}) - d(\overline{b}/A) < \gamma/2$ and $d(\overline{c}/\overline{a}) - d(\overline{c}/A) < \gamma/2$. Let $\overline{b}' = cl(\overline{b}\overline{a})$ and $\overline{c}' = cl(\overline{c}\overline{a})$. Then $d(\overline{b}'\overline{c}'/\overline{a}) = d(\overline{b}\overline{c}/\overline{a}) \geq d(\overline{b}/A) + d(\overline{c}/A) = d(\overline{b}/A) + d(\overline{c}/A) > d(\overline{b}/A) + d(\overline{c}/A) - \gamma = \delta(\overline{b}/\overline{a}) + \delta(\overline{c}/\overline{a}) - \gamma > \delta(\overline{b}'\overline{c}'/\overline{a}) - \gamma$. On the other hand, we have $d(\overline{b}'\overline{c}'/\overline{a}) \leq \delta(e\overline{b}'\overline{c}'/\overline{a}) \leq \delta(\overline{b}'\overline{c}'/\overline{a}) + \delta(e/\overline{b}'\overline{c}') = \delta(\overline{b}'\overline{c}'/\overline{a}) - \gamma$. A contradiction. Next we show that $B \perp_A C$. If not, then there are $\overline{b} \in B, \overline{c} \in C$ with $\delta(\overline{b}/\overline{c}) < \delta(\overline{b}/\overline{a})$ where $\overline{a} = \overline{b} \cap \overline{c}$. Let $\gamma = \delta(\overline{b}/\overline{a}) - \delta(\overline{b}/\overline{c})$. Take $\overline{a}' \leq A$ with $\overline{a} \subset \overline{a}'$ and $d(\overline{b}/\overline{a}') - d(\overline{b}/A) < \gamma$. Let $\overline{b}' = cl(\overline{a}'\overline{b})$ and $\overline{c}' = cl(\overline{a}'\overline{c})$. By remark, we have $\delta(\overline{b}'/\overline{a}') - \delta(\overline{b}'/\overline{a}a') \geq \delta(\overline{b}/\overline{a}) - \delta(\overline{b}/\overline{c}) = \gamma$. Then $\delta(\overline{b}'/\overline{a}a') \geq d(\overline{b}'/\overline{a}a') = d(\overline{b}/A) > d(\overline{b}/\overline{a}') - \gamma = \delta(\overline{b}/\overline{a}') - \gamma \geq \delta(\overline{b}'/\overline{a}')$. A contradiction.
(ii)$\Rightarrow$(i). If not, then there are $\overline{b} \in B, \overline{c} \in C$ with $d(\overline{b}/\overline{c}) < d(\overline{b}/A)$. By (ii), we can take $\overline{b}', \overline{c}'$ such that $\overline{b} \subset \overline{b}' \leq B, \overline{c} \subset \overline{c}' \leq C, \overline{b}' \perp_A \overline{c}'$ and $\overline{b}' \leq \mathcal{M}$ where $\overline{a}' = \overline{b}' \cap \overline{c}'$. Then $d(\overline{b}/\overline{c}) = \delta(\overline{b}'/\overline{c}') = \delta(\overline{b}/\overline{a}') \geq d(b/\overline{a}') \geq d(b/A)$. A contradiction.

**Fact 6** Let $B, C \leq \mathcal{M}$ and $A = B \cap C$ be algebraically closed. Then the following are equivalent:
(i) $tp(B/C)$ does not fork over $A$;
(ii) $B \perp_A C$ and $BC \leq \mathcal{M}$.
Proof (i) ⇒ (ii). Suppose that $B \downarrow_A C$. Take a sufficiently saturated model $N \supset A$ with $BC \downarrow_A N$. Then we have $B \downarrow_N C$ and $B \downarrow_A N$.

Claim 1: $d(B/N) = d(B/NC)$.

Proof: If $d(B/N) > d(B/NC)$, then there are $\overline{b} \in B, \overline{c} \in NC$ with $d(\overline{b}/N) > d(\overline{b}/\overline{c})$. Take countable $A_0 \subset N$ with $\overline{b} \overline{c} \downarrow_{A_0} N$. By the saturation of $N$, we can pick $\overline{c}' \in N$ with $\text{stp}(\overline{c}/A_0) = \text{stp}(\overline{c}'/A_0)$. Since $\overline{b} \overline{c} \downarrow_{A_0} N$ and $\overline{b} \downarrow_N \overline{c}$, we have $\overline{b} \downarrow_{A_0} \overline{c}$ and $\overline{b} \downarrow_{A_0} \overline{c}'$. Hence $\text{tp}(b\overline{c}/A_0) = \text{tp}(b\overline{c}'/A_0)$. Then $d(b/\overline{c}) = d(b/\overline{c}') \geq d(b/N)$. A contradiction.

Claim 2: $d(B/A) = d(B/N)$.

Proof: Let $B^* = \text{acl}(B)$. We can take $A_1$ with $d(B^*/N) = d(B^*/A_1)$ where $A \subset A_1 \subset N$ and $|A_1| = |B| + \aleph_0$. $A_1$ acl?? By the saturation of $N$ there is $A_2 \subset N$ with $\text{tp}(A_2/A) = \text{tp}(A_1/A)$ and $A_1 \downarrow_A A_2$. Note that $A_1 \downarrow_{B^*} A_2$ by $B \downarrow_A N$. Let $B^{*_1}_i = \text{cl}(A_1B^*)$ and $B^{*_2}_i = \text{cl}(A_2B^*)$. Then $B^{*_1}_i \cap B^{*_2}_i = B^*$. By fact 6, we have $B^{*_1}_1N, B^{*_2}_1N \leq \mathcal{M}$ since $d(B^*/N) = d(B^*/A_1) = d(B^*/A_2)$. Hence $B^*N = B^{*_1}_1N \cap B^{*_2}_1N \leq \mathcal{M}$. On the other hand, we have $B^* \downarrow_A N$. (Proof: Suppose that $B^* \downarrow A_1N$. Note that $B^* \downarrow A_1N$ and $B^* \downarrow A_2N$ since $d(B^*/N) = d(B^*/A_1) = d(B^*/A_2)$. So we have $B^* \downarrow A_1N$ and $B^* \downarrow A_2N$. Since $A_1 \downarrow A_2$, we have $A_1 \cap A_2 = A$. A contradiction.) Hence $d(B/N) = d(B/A)$.

By claim 1,2, we have $d(B/A) = d(B/NC)$, and hence $d(B/A) = d(B/C)$.

(ii) ⇒ (i). Take $B'$ such that $\text{tp}(B'/C)$ does not fork over $A$ and $\text{tp}(B'/A) = \text{tp}(B'/A)$. By (i) ⇒ (ii), we have $B' \downarrow_A C$ and $B' \leq \mathcal{M}$. So we have $\text{tp}(B'/A) = \text{tp}(B'C/A)$, and hence $\text{tp}(B'/C)$ does not fork over $A$.

For each $A \leq B \in \mathcal{K}$, $B$ is said to be minimal, if $C = A$ or $B$ for any $C$ with $A \leq C \leq B$.

Lemma 7 Let $A \leq B \in \mathcal{K}$ with $B \leq \mathcal{M}$. Let $B$ be minimal over $A$. If $\text{tp}(B/A)$ is algebraic, then $B \perp_A C$ for any $C \leq \mathcal{M}$ with $B \cap C = A$.

Proof Suppose that $\delta(B/C) < \delta(B/A)$ for some $C \leq BC \in \mathcal{K}$ with $B \cap C = A$.

Claim: There is a set $\{B_i\}_{i<\omega}$ of copies of $B$ over $A$ with the following conditions:

(i) $C \leq CB_j \leq CB_0B_1 \cdots B_i \in \mathcal{K}$ for each $j \leq i < \omega$;

(ii) $B_i \cap B_j = A$ for each $j < i < \omega$;

(iii) $B_i, C$ are free over $A$ for each $i < \omega$.

Proof: Suppose that $\{B_i\}_{i \leq n}$ has been defined. By our assumption, we have $C \leq CB \in \mathcal{K}$, and by (i) we have $C \leq CB_0B_1 \cdots B_n \in \mathcal{K}$. By amalgamation, we can take a copy $B^*$ of $B$ over $C$ such that $CB_0 \cdots B_n, CB^* \leq CB_0 \cdots B_n B^* \in \mathcal{K}$. By (iii) and $\delta(B^*/C) < \delta(B^*/A)$, we have $B_i \neq B^*$ for all $i \leq n$. Since $B$ is minimal over $A$, we have $B^* \cap B_i = A$. Since $\mathcal{K}$ is closed under $L$-subgraphs, there is $B_{n+1} \equiv_{\mathcal{A}B_0} B_1 \cdots B_n B^* \text{ such that } CB_0B_1 \cdots B_n B_{n+1} \in \mathcal{K}$ and $B_{n+1}, C$ are free over $A$. So (ii) and (iii) hold. It is not difficult to check that $CB_j \leq CB_0B_1 \cdots B_{n+1} \in \mathcal{K}$ for each $j \leq n + 1$. So (i) holds. (End of Proof of Claim)
By claim, we have $AB_j \leq AB_0...B_i \in K$ for each $j \leq i < \omega$. We can assume that $AB_0...B_i \leq M$. Thus we have $\text{tp}(B_j/A) = \text{tp}(B/A)$ for each $j \leq i$. By (ii) of claim, $B_j$'s are pairwise distinct. Hence $\text{tp}(B/A)$ is not algebraic.

**Lemma 8** Let $A \leq B \in K$ with $B \leq M$. Let $B$ be minimal over $A$. If $\text{tp}(B/A)$ is algebraic, then $BC \leq M$ for any $C \leq M$ with $B \cap C = A$.

**Proof** Suppose by way of contradiction that $BC \not\leq M$ for some $C \leq M$ with $B \cap C = A$. Then there is finite $X \subset M - BC$ such that $\delta(X/BC) < 0$.

Claim 1: There is a set $\{B_i\}_{i<\omega}$ of copies of $B$ with the following conditions:
(i) $B_i \cong_{CB_0...B_{i-1}} B$ for each $i < \omega$;
(ii) $CB_0...B_i,B_0...B_{i-1}BX \leq CB_0...B_i BX \in K$ for each $i < \omega$;
(iii) $XB \cap B_i = B_j \cap B_i = A$ for each $j < i < \omega$.

Proof: Suppose that $\{B_i\}_{i \leq n}$ has been defined. By (ii), $CB_0...B_n \leq CB_0...B_n BX \in K$, and so we have $CB_0...B_n \leq CB_0...B_n B \in K$. By amalgamation, we can take a copy $B_{n+1}$ of $B$ over $CB_0...B_n$ such that $CB_0...B_n BX, CB_0...B_n B_{n+1} \leq CB_0...B_n B_{n+1} BX \in K$. Hence (i) and (ii) hold. On the other hand, $B_{n+1} \cap B_i = A$ for each $i \leq n$, since $B_{n+1} \cong_{CB_0...B_n} B$. So, to see that (iii) holds, it is enough to show that $B_{n+1} \cap XB = A$. Let $B' = B_{n+1} \cap XB$. First, suppose that $B' = B_{n+1}$. Then we have $B_{n+1} \subset BX$, and so $CB_{n+1} \leq CBX$, since $\delta(XB/BC_{n+1}) = \delta(XB/C) - \delta(B_{n+1}/C) = \delta(XB/C) - \delta(B/C) = \delta(X/BC) < 0$. This contradicts our choice of $B_{n+1}$. Hence we have $B' \neq B_{n+1}$. We have to see that $B' = A$. This can be shown as follows: By our choice of $B_{n+1}$, we have $CB_0...B_n BX \leq CB_0...B_n B_{n+1} BX$, and so $B' \leq B_{n+1}$. Since $B$ is minimal and $B' \neq B_{n+1}$, we have $B' = A$. (End of Proof of Claim 1)

Claim 2: $B_j \leq B_0...B_i B \in K$ for $j \leq i < \omega$

Proof: We prove by induction on $i$. By (ii) of claim 1, $B_0...B_i B \leq B_0...B_{i+1} B$. By induction hypothesis, we have $B_j \leq B_0...B_i B$ for $j \leq i$. Hence $B_j \leq B_0...B_{i+1} B$ for $j \leq i$. So, it is enough to show that $B_{i+1} \leq B_0...B_{i+1} B$. By induction hypothesis again, we have $B \leq B_0...B_i B$. From (i) of claim 1, it follows that $B_{i+1} \leq B_0...B_{i+1}$. By (ii) of claim 1, $B_0...B_{i+1} \leq B_0...B_{i+1} B$. Hence we have $B_{i+1} \leq B_0...B_{i+1} B$. (End of Proof of Claim 2)

We show that $\text{tp}(B/A)$ is non-algebraic. By claim 2, we can assume that $B,B_j \leq BB_0...B_i \leq M$ for each $i,j$ with $j \leq i < \omega$. So we have $\text{tp}(B_j/A) = \text{tp}(B/A)$ for each $j < \omega$. By (iii) of claim 1, $B_j$'s are pairwise distinct. Hence $\text{tp}(B/A)$ is not algebraic.

**Proposition 9** Let $A \leq B \leq M$ and $A = \text{acl}(A) \cap B$. Then $\text{acl}(A) \perp A B$ and $\text{acl}(A) \cup B \leq M$.

**Proof** We can assume that $A,B$ are finite. We will show that $A^* \perp A B$ and $A^* B \leq M$ for any finite $A^* \leq \text{acl}(A)$ with $A \subset A^*$. Take $A = A_0 \leq A_1 \leq ... \leq A_n = A^*$ with $A_{i+1}$ minimal over $A_i$ for each $i < n$. Then it is enough to show
that $A_i \perp_{A_0} B$ and $A_i B \leq \mathcal{M}$ for each $i \leq n$. (Proof: We prove by induction on $i$. Clearly $A_i \leq A_{i+1}, A_{i+1} \cap A_i B = A_i$ and $\text{tp}(A_{i+1}/A_i)$ is algebraic. By induction hypothesis, $A_i B \leq \mathcal{M}$. So we have $A_{i+1} \perp_{A_i} B$ and $A_{i+1} B \leq \mathcal{M}$ by lemma. By induction hypothesis, $A_i \perp_{A_0} B$, and hence $A_{i+1} \perp_{A_0} B$.)

**Theorem 10** Let $B, C \leq \mathcal{M}$ and $A = B \cap C$. Then the following are equivalent:

(i) $\text{tp}(B/C)$ does not fork over $A$;
(ii) $B \perp_{A} C$ and $BC \cup \text{acl}(A) \leq \mathcal{M}$.

**Proof** By proposition 9, $B \cup \text{acl}(A), C \cup \text{acl}(A) \leq \mathcal{M}$. So, by fact 7, (i) is equivalent to $B \perp_{\text{acl}(A)} C$ and $BC \cup \text{acl}(A) \leq \mathcal{M}$. Therefore, proving that (i) and (ii) are equivalent, it is enough to show that $B \perp_{\text{acl}(A)} C$ if and only if $B \perp_{A} C$. We can assume that $A, B, C$ is finite. Take any finite $A^* \leq \text{acl}(A)$ with $BC \cap \text{acl}(A) \subset A^*$. Then we will show that $B \perp_{A^*} C$ if and only if $B \perp_{A} C$. Let $B' = B \cap A^*, C' = C \cap A^*$.

$(\Rightarrow)$ Since $\text{tp}(A^*/B'C')$ is algebraic, we have $A^* \perp_{B'C'} BC$. So, from $B \perp_{A^*} C$ it follows that $B \perp_{B'C'} C$. On the other hand, since $\text{tp}(B'/C')$ and $\text{tp}(C'/A)$ are algebraic, we have $B' \perp_{C'} C$ and $B \perp_{A} C'$. Hence we have $B \perp_{A} C$.

$(\Leftarrow)$ By $B \perp_{A} C$, we have $B \perp_{B'C'} C$. On the other hand, since $\text{tp}(A^*/B'C')$ is algebraic, we have $A^* \perp_{B'C'} BC$. Hence $B \perp_{A^*} C$.

**Corollary 11** Let $L$ be a countable relational language and $K$ a class of finite $L$-structures that is derived from a predimension $\delta$. Then there is no $K$-generic structure that is superstable but not $\omega$-stable.

**Proof** Suppose that a theory $T$ of a $K$-generic structure is superstable. Take any countable model $N$ of $T$.

Claim: For any $p \in S(N)$ there is finite $A \subset N$ such that $p$ does not fork over $A$ and $p|A$ is stationary.

Proof: Take a realization $\bar{b}$ of $p$. By superstability, there is finite $X \subset N$ such that $p$ does not fork over $X$. Let $B = \text{cl}(X\bar{b})$ and $A = B \cap N$. Clearly $p$ does not fork over $A$. We show that $\text{tp}(\bar{b}/A)$ is stationary. Take any $\bar{b}'$ such that $\text{tp}(\bar{b}'/A) = \text{tp}(\bar{b}/A)$ and $\text{tp}(\bar{b}'/N)$ does not fork over $A$. Let $B' = \text{cl}(\bar{b}'A)$. Then $\text{tp}(B/N)$ and $\text{tp}(B'/N)$ do not fork over $A$. Since $\text{tp}(\bar{b}/A) = \text{tp}(\bar{b}'/A)$, we have $B \cong_A B'$. Note that $B \cap N = B' \cap N = A$. By theorem, $B \perp_A N, B' \perp_A N$ and $BN, B'N \leq \mathcal{M}$. In particular, $BN \cong B'N$. It follows that $\text{tp}(BN) = \text{tp}(B'N)$ and hence $\text{tp}(b/N) = \text{tp}(\bar{b}'/N)$. (End of Proof of Claim)

By claim, we have $|S(N)| \leq \aleph_0 \cdot |S(T)| = \aleph_0$. Hence $T$ is $\omega$-stable.

**Reference**

Faculty of Business Administration
Hosei University
2-17-1, Fujimi, Chiyoda
Tokyo, 102-8160
JAPAN
ikeda@hosei.ac.jp