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On complementary spaces of the Lizorkin spaces

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§1. Introduction
Let $R^n$ be the $n$-dimensional Euclidean space. For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $x = (x_1, \cdots, x_n) \in R^n$, we let

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

The Schwartz space $S(R^n)$ is defined to be the class of all $C^\infty$-functions $\varphi$ on $R^n$ such that

$$p_{\alpha,\beta}(\varphi) = \sup_{x \in R^n} |x^\alpha D^\beta \varphi(x)| < \infty$$

for all multi-indices $\alpha$ and $\beta$. We introduce two kinds of the Lizorkin spaces $\Phi_1(R^n)$ and $\Phi_2(R^n)$. The Lizorkin space $\Phi_1(R^n)$ of the first kind is defined to be the class of all functions $\varphi \in S(R^n)$ which satisfy

$$\int_{R^n} \varphi(x) x^\alpha dx = 0$$

for any multi-index $\alpha$. The Lizorkin space $\Phi_2(R^n)$ of the second kind is defined to be the class of all functions $\varphi \in S(R^n)$ which satisfy

$$\int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_j, \cdots, x_n) x_j^{\ell} dx_j = 0$$

for $j = 1, \cdots, n$ and $\ell = 0, 1, 2, \cdots$. Clearly $\Phi_1(R^n) \supset \Phi_2(R^n)$.

An example of a function belonging to $\Phi_1(R^n)$ (resp. $\Phi_2(R^n)$) is $\mathcal{F}(e^{-|y|^2 - (1/|y|^2)})(x)$ (resp. $\mathcal{F}(e^{-|y|^2 - \sum_{j=1}^{n} 1/y_j^2})(x)$) where $\mathcal{F}\varphi$ is the Fourier transform of $\varphi$:

$$\mathcal{F}\varphi(x) = \int_{R^n} e^{-ixy} dy.$$
The Lizorkin spaces appeared in the theory of fractional derivatives, hypersingular integrals and Riesz potentials ([Sa2] and [SKM]). The properties of the Lizorkin spaces have studied by several authors. The denseness of the Lizorkin spaces in the Lebegue spaces was proved in O.I.Lizorkin [Li2] and S.G.Samko [Sa1]. Moreover P.I.Lizorkin [Li3] showed that the space $\Phi_1(R^n)$ is dense in the Sobolev spaces and T.Kurokawa [Ku] deals with the denseness of the space $\Phi_1(R^n)$ in the spaces of Beppo Levi type. The invariance of the space $\Phi_1(R^n)$ relative to Riesz potentials was noted by V.I.Semyanistyi [Se], P.I.Lizorkin [Li3] and S.Helgason [He]. T.Kurokawa [Ku] establish the invariance of the space $\Phi_1(R^n)$ relative to more general operators. In this note we are concerned with complementary spaces of $\Phi_1(R^n)$ and $\Phi_2(R^n)$ in $S(R^n)$. For a subspace $V \subset S(R^n)$, if a subspace $W \subset S(R^n)$ satisfies the condition

$$S(R^n) = V \oplus W,$$

then we call $W$ a complementary space of $V$ in $S(R^n)$ where the symbol $\oplus$ indicates a direct sum. In section 2 as a preparation we introduce dual functions of polynomials and tensor product functions, and study their properties. In section 3 we sketch our plan to give complementary spaces of $\Phi_1(R^n)$ and $\Phi_2(R^n)$ in $S(R^n)$.

§2. Dual functions of polynomials and tensor product functions

Let $h \in C^\infty(R^1)$ be a function which satisfies the conditions $0 \leq h(t) \leq 1$, $h(-t) = h(t)$ and

$$h(t) = \begin{cases} 1, & \text{for } |t| \leq 1/2 \\ 0, & \text{for } |t| \geq 1. \end{cases}$$

We fix the function $h(t)$. We denote by $A$ the set of sequences $a = \{a_j\}_{j=0,1,\ldots}$ which satify $0 < a_j \leq 1$ and $a_j \geq a_{j+1}$. For $a =
\{a_{j}\}_{j=0,1,\ldots} \in \mathcal{A} \text{ we put }

\eta_{j}^{a}(t) = \frac{t^{j}}{j!}h\left(\frac{t}{a_{j}}\right), \quad j = 0, 1, 2, \ldots

and

\theta_{j}^{a}(t) = \frac{i^{j}}{2\pi}\mathcal{F}\eta_{j}^{a}(t), \quad j = 0, 1, 2, \ldots.

Then \(\theta_{j}^{a} \in S(R^{1})\) and

\begin{equation}
\int_{-\infty}^{\infty} \theta_{j}^{a}(t)t^{k}dt = \begin{cases} 1, & k = j \\
0, & k \neq j \end{cases}, \quad j = 0, 1, 2, \ldots.
\end{equation}

Since \(\{\theta_{j}^{a}\}_{j=0,1,\ldots}\) satisfy (2.1), we call \(\{\theta_{j}^{a}\}_{j=0,1,\ldots}\) dual functions of polynomials associated with a sequence \(a \in \mathcal{A}\). For \(1 \leq p \leq n\) we denote by \(M_{p}\) the set of subsets of \(\{1, 2, \ldots, n\}\) which have \(p\) elements. For \(\{i_{1}, i_{2}, \ldots, i_{p}\} \in M_{p}\) we always assume that \(i_{1} < i_{2} \ldots < i_{p}\).

For multi-index \(\alpha = (\alpha_{1}, \ldots, \alpha_{n})\) and \(\{i_{1}, \ldots, i_{p}\} \in M_{p}\) the notation \((\{\alpha_{i_{1}}, \ldots, \alpha_{i_{p}}\})^{c}\) stands for

\(\{\alpha_{1}, \ldots, \alpha_{k_{n-p}}\}\)

where \(\{k_{1}, \ldots, k_{n-p}\} = \{1, \ldots, n\} - \{i_{1}, \ldots, i_{p}\}\). Similarly, for \(x = (x_{1}, \ldots, x_{n})\) we denote

\((\{x_{i_{1}}, \ldots, x_{i_{p}}\})^{c} = (x_{k_{1}}, \ldots, x_{k_{n-p}})\).

Moreover we denote

\((\{x_{i_{1}}, \ldots, x_{i_{p}}\})^{c}((\alpha_{i_{1}}, \ldots, \alpha_{i_{p}}))^{c} = x_{k_{1}}^{\alpha_{k_{1}}} \cdots x_{k_{n-p}}^{\alpha_{k_{n-p}}},\)

\((\{D_{i_{1}}, \ldots, D_{i_{p}}\})^{c}((\beta_{i_{1}}, \ldots, \beta_{i_{p}}))^{c} = D_{k_{1}}^{\alpha_{k_{1}}} \cdots D_{k_{n-p}}^{\alpha_{k_{n-p}}}.

Let \(\alpha, \beta\) be multi-indices and \(\{i_{1}, \ldots, i_{p}\} \in M_{p}\). For a function \(\varphi(\{x_{i_{1}}, \ldots, x_{i_{p}}\}) \in S(R^{n-p})\) we define

\[ p((\alpha_{i_{1}}, \ldots, \alpha_{i_{p}}), ((\beta_{i_{1}}, \ldots, \beta_{i_{p}}))^{c})(\varphi) = \sup_{(\{x_{i_{1}}, \ldots, x_{i_{p}}\})^{c} \in R^{n-p}} |(\{x_{i_{1}}, \ldots, x_{i_{p}}\})^{c}((\alpha_{i_{1}}, \ldots, \alpha_{i_{p}}))^{c}\]

\[ ((\{D_{i_{1}}, \ldots, D_{i_{p}}\})^{c}((\beta_{i_{1}}, \ldots, \beta_{i_{p}}))^{c})\varphi(\{x_{i_{1}}, \ldots, x_{i_{p}}\})^{c}|. \]
For $\{i_1, \cdots, i_p\} \in M_p$ we denote by $C^a_{i_1, \cdots, i_p}$ the set of $p$-multiple sequences of functions $\{\varphi_{s_1 \cdots s_p}(\{x_{i_1}, \cdots, x_{i_p}\^c})\}_{s_1, \cdots, s_p=0,1, \cdots} \subset \mathcal{S}(R^{n-p})$ which satisfy

$$\sum_{s_1, \cdots, s_p=0}^{\infty} p((\alpha_{i_1}, \cdots, \alpha_{i_p})^c), ((\beta_{i_1}, \cdots, \beta_{i_p})^c) (\varphi_{s_1 \cdots s_p}) a_{s_1} \cdots a_{s_p} < \infty$$

for all multi-indices $\alpha$ and $\beta$. We note that the sequence $\{\varphi_{s_1 \cdots s_n}(\{x_1, \cdots, x_n\^c})\}_{s_1, \cdots, s_n=0,1, \cdots}$ is a $n$-multiple sequence of numbers $\{b_{s_1 \cdots s_n}\}_{s_1, \cdots, s_n=0,1, \cdots}$ and

$$p((\alpha_1, \cdots, \alpha_n)^c), ((\beta_1, \cdots, \beta_n)^c) (b_{s_1 \cdots s_n}) = |b_{s_1 \cdots s_n}|.$$

Therefore

$$C^a_{1, \cdots, n} = \{\{b_{s_1 \cdots s_n}\}_{s_1, \cdots, s_n=0,1, \cdots} : \sum_{s_1, \cdots, s_n=0}^{\infty} |b_{s_1 \cdots s_n}| a_{s_1} \cdots a_{s_n} < \infty\}.$$

The basic fact is

**Lemma 1.** Let $\{i_1, \cdots, i_p\} \in M_p$. If a $p$-multiple sequence of functions $\{\varphi_{s_1 \cdots s_p}(\{x_{i_1}, \cdots, x_{i_p}\^c})\}_{s_1, \cdots, s_p=0,1, \cdots}$ belongs to $C^a_{i_1, \cdots, i_p}$, then the $p$-multiple series

$$\sum_{s_1, \cdots, s_p=0}^{\infty} \varphi_{s_1 \cdots s_p}(\{x_{i_1}, \cdots, x_{i_p}\^c}) \theta^a_{s_1}(x_{i_1}) \cdots \theta^a_{s_p}(x_{i_p})$$

converges in $\mathcal{S}(R^n)$.

We introduce two kinds of tensor product functions associated with $\{\theta^a_j\}$. If a function $f$ has the following form

$$(2.2) \quad f(x) = \sum_{s_1, \cdots, s_n=0}^{\infty} b_{s_1 \cdots s_n} \theta^a_{s_1}(x_1) \cdots \theta^a_{s_n}(x_n)$$

where $\{b_{s_1 \cdots s_n}\} \in C^a_{1, \cdots, n}$, then $f$ is called a tensor product function of the first kind associated with $\{\theta^a_j\}$. If a function $f$ which has the
form
(2.3)
\[ f(x) = \sum_{p=1}^{n} (-1)^{p} \sum_{\{i_1, \ldots, i_p\} \in M_p} \sum_{s_1, \ldots, s_p = 0}^{\infty} \lambda_{i_1, \ldots, i_p; s_1, \ldots, s_p}(\{x_{i_1}, \ldots, x_{i_p}\}^{c}) \]
\[ \theta_{s_1}^{a}(x_{i_1}) \cdots \theta_{s_p}^{a}(x_{i_p}) \]
satisfies the conditions
(i) \( \{\lambda_{i_1, \ldots, i_p; s_1, \ldots, s_p}\}_{s_1, \ldots, s_p = 0, 1, \ldots} \in C_{i_1, \ldots, i_p}^{a} \),
(ii) for \( 2 \leq p \leq n \), \( \{i_1, \ldots, i_p\} \in M_p \) and \( s_1, \ldots, s_p \geq 0 \),
\[ \lambda_{i_1, \ldots, i_p; s_1, \ldots, s_p}(\{x_{i_1}, \ldots, x_{i_p}\}^{c}) \]
\[ = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \lambda_{i_1, \ldots, i_p; s_1, \ldots, s_p}(\{x_{i_1}\}^{c}) x_{i_1}^{s_1} \cdots x_{i_p}^{s_p} dx_{i_1} \cdots dx_{i_p} \]
where \( \ell = 1, \ldots, p \), then we call \( f \) a tensor product function of the second kind associated with \( \{\theta_{j}^{a}\} \) where the symbol \( \sim \) indicates that the variable underneath is deleted. We denote by \( T_{1}^{a}(R^n) \) (resp. \( T_{2}^{a}(R^n) \)) the class of all tensor product functions of the first kind (resp. the second kind) associated with \( \{\theta_{j}^{a}\} \). By Lemma 1, we see that \( T_{1}^{a}(R^n), T_{2}^{a}(R^n) \subset S(R^n) \). A fundamental property of the tensor product functions is the following.

**Lemma 2.** (i) Let \( f \) be a tensor product function of the first kind with the form (2.2). Then
\[ \int_{R^n} f(x_1, \ldots, x_n)x_{t_1}^{t_1} \cdots x_{t_n}^{t_n} dx_{t_1} \cdots dx_{t_n} = b_{t_1 \cdots t_n} \]
for \( t_1, \ldots, t_n \geq 0 \).

(ii) Let \( f \) be a tensor product function of the second kind with the form (2.3). Then
\[ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_n)x_{k_1}^{t_1} \cdots x_{k_q}^{t_q} dx_{k_1} \cdots dx_{k_q} \]
\[ = \lambda_{k_1, \ldots, k_q; t_1, \ldots, t_q}(\{x_{k_1}, \ldots, x_{k_q}\}^{c}) \]
for $1 \leq q \leq n, \{k_1, \ldots, k_q\} \in M_q$ and $t_1, \ldots, t_q \geq 0$.

§3. Complementary spaces of the Lizorkin spaces

For $\{i_1, \ldots, i_p\} \in M_p, s_1, \ldots, s_p \geq 0$ and $\varphi \in S(R^n)$, we define

$$
\mu_{i_1, \ldots, i_p; s_1, \ldots, s_p} (\varphi) (\{x_{i_1}, \ldots, x_{i_p}\}^c) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi(x_1, \ldots, x_n) x_{i_1}^{s_1} \cdots x_{i_p}^{s_p} dx_{i_1} \cdots dx_{i_p}.
$$

Moreover, for $a \in \mathcal{A}$ and $\{i_1, \ldots, i_p\} \in M_p$ we set

$$
S_{i_1, \ldots, i_p}^a = \{ \varphi \in S(R^n) : \{\mu_{i_1, \ldots, i_p; s_1, \ldots, s_p} (\varphi) (\{x_{i_1}, \ldots, x_{i_p}\}^c)\}_{s_1, \ldots, s_p=0,1, \ldots} \in C_{i_1, \ldots, i_p}^a \}.
$$

and

$$
S^a(R^n) = \bigcap_{p=1}^{n} \bigcap_{\{i_1, \ldots, i_p\} \in M_p} S_{i_1, \ldots, i_p}^a (R^n).
$$

If $\varphi \in \Phi_1(R^n)$, then $\mu_{1, \ldots, n; s_1, \ldots, s_n} (\varphi) = 0$ for $s_1, \ldots, s_n \geq 0$. Hence $\Phi_1(R^n) \subset S_{1, \ldots, n}^a$ for any $a \in \mathcal{A}$. If $\varphi \in \Phi_2(R^n)$, then $\mu_{i_1, \ldots, i_p; s_1, \ldots, s_p} (\varphi) = 0$ for $1 \leq p \leq n, \{i_1, \ldots, i_p\} \in M_p$ and $s_1, \ldots, s_p \geq 0$. Hence $\Phi_2(R^n) \subset S^a(R^n)$ for any $a \in \mathcal{A}$. Moreover, By Lemma 2 (i), (ii) and the definitions of $T_1^a, T_2^a$ we see that $T_1^a(R^n) \subset S_{1, \ldots, n}^a(R^n)$ and $T_2^a(R^n) \subset S^a(R^n)$.

We introduce some operators. For $\varphi \in S_{1, \ldots, n}^a(R^n)$, we define

$$
T_{1, \ldots, n}^a \varphi(x) = \sum_{s_1, \ldots, s_n=0}^{\infty} \mu_{1, \ldots, n; s_1, \ldots, s_n} (\varphi) \theta_{s_1}^a(x_1) \cdots \theta_{s_n}^a(x_n)
$$

and

$$
U_{1, \ldots, n}^a \varphi = \varphi - T_{1, \ldots, n}^a \varphi.
$$

Further, for $\varphi \in S^a(R^n)$ we define

$$
T_j^a \varphi(x) = \sum_{s=0}^{\infty} \mu_{j; s_1, \ldots, s_n} (\varphi) (\{x_j\}^c) \theta_s^a(x_j), \quad j = 1, \ldots, n
$$
and

\[ U_j^a \varphi = \varphi - T_j^a \varphi. \]

Moreover

\[ U^a \varphi = U_1^a \cdots U_n^a \varphi. \]

We see that

\[ U^a \varphi = \varphi - \sum_{p=1}^{n} (-1)^{p+1} \sum_{\{i_1, \ldots, i_p\} \in M_p} T_{i_1, \ldots, i_p}^a \varphi \]

where

\[ T_{i_1, \ldots, i_p}^a \varphi(x) = \sum_{s_1, \ldots, s_p = 0}^{\infty} \mu_{i_1, \ldots, i_p; s_1, \ldots, s_p} (\varphi)(\{x_{i_1}, \ldots, x_{i_p}\}^c) \theta_{s_1}^a(x_{i_1}) \cdots \theta_{s_p}^a(x_{i_p}). \]

We put

\[ T^a = \sum_{p=1}^{n} (-1)^{p+1} \sum_{\{i_1, \ldots, i_p\} \in M_p} T_{i_1, \ldots, i_p}^a. \]

We establish properties of these operators which are necessary for decompositions of \( S_{1, \ldots, n}^a(R^n) \) and \( S^a(R^n) \). About ranges of these operators we have

**Lemma 3.**

(i) If \( \varphi \in S_{1, \ldots, n}^a(R^n) \), then \( T_{1, \ldots, n}^a \varphi, U_{1, \ldots, n}^a \varphi \in S_{1, \ldots, n}^a(R^n) \).

(ii) If \( \varphi \in S^a(R^n) \), then \( T^a \varphi, U^a \varphi \in S^a(R^n) \).

**Lemma 4.**

(i) \( \varphi \in S_{1, \ldots, n}^a(R^n) \), then \( U_{1, \ldots, n}^a \varphi \in \Phi_1(R^n) \).

(ii) If \( \varphi \in S^a(R^n) \), then \( U^a \varphi \in \Phi_2(R^n) \).

**Lemma 5.**

(i) \( \varphi \in S_{1, \ldots, n}^a(R^n) \), then \( T_{1, \ldots, n}^a \varphi \in T_{1}^a(R^n) \).

(ii) If \( \varphi \in S^a(R^n) \), then \( T^a \varphi \in T_{2}^a(R^n) \).

These operators become the identity operators on each proper subspace. In fact we have

**Lemma 6.**

(i) \( \varphi \in \Phi_1(R^n) \), then \( U_{1, \ldots, n}^a \varphi = \varphi \).
(ii) If $\varphi \in \Phi_2(R^n)$, then $U^a\varphi = \varphi$

**Lemma 7.** $\varphi \in T_1^a(R^n)$, then $T_1^a\varphi = \varphi$.

(ii) If $\varphi \in T_2^a(R^n)$, then $T^a\varphi = \varphi$.

Now we give decompositions of $S_{1,\ldots,n}^a(R^n)$ and $S^a(R^n)$.

**Theorem 8.** (i) $S_{1,\ldots,n}^a(R^n) = \Phi_1(R^n) \oplus T_1^a(R^n)$.
(ii) $S^a(R^n) = \Phi_2(R^n) \oplus T_2^a(R^n)$.

In order to give a decomposition of $S(R^n)$, we need a relation between $S(R^n)$ and $S^a(R^n)$ (or $S_{1,\ldots,n}^a(R^n)$). We have

**Lemma 9.** $S(R^n) = \bigcup_{a \in A} S^a(R^n)$, $S(R^n) = \bigcup_{a \in A} S_{1,\ldots,n}^a(R^n)$.

Taking Lemma 9 into account we put

$$T_1(R^n) = \bigcup_{a \in A} T_1^a(R^n), \quad T_2(R^n) = \bigcup_{a \in A} T_2^a(R^n).$$

Then we have

**Theorem 10.** (i) $S(R^n) = \Phi_1(R^n) \oplus T_1(R^n)$.
(ii) $S(R^n) = \Phi_2(R^n) \oplus T_2(R^n)$.

**References**


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