A convergence property for quasisuperminimizers on metric measure spaces

Takayori ONO (小野太幹)
Fukuyama University (福山大学)

§1. Preliminaries

We assume that $X = (X, d, \mu)$ be a complete metric space with a metric $d$ and a positive Borel regular measure $\mu$ which is finite on a bounded set.

Let $u$ be a real valued function on $X$. A nonnegative Borel measurable function $g$ on $X$ is said to be an upper gradient of $u$ if for every rectifiable path $\gamma$ joining $x$ and $y$ in $X$,

$$|u(x) - u(y)| \leq \int_{\gamma} g \, ds. \quad (1.1)$$

The $p$-modulus of a family $\Gamma$ of paths in $X$ is defined by

$$\inf_{\rho} \int_{X} \rho^p \, d\mu,$$

where the infimum is taken over all nonnegative Borel measurable functions $\rho$ such that for all rectifiable paths $\gamma$ in $\Gamma$

$$\int_{\gamma} \rho \, ds \geq 1.$$

We say that a property holds for $p$-almost every path if the family of paths on which the property does not hold is of zero the $p$-modulus. If (1.1) holds for $p$-almost every path $\gamma$, then we say that $g$ is a $p$-weak upper gradient of $u$.

Let $1 < p < \infty$ and $L^p(X)$ be the space of functions $f$ on $X$ such that $|f|^p$ is integrable with respect to the measure $\mu$. A function $u$ belongs the space $\tilde{N}^{1,p}(X)$ if $u \in L^p(X)$ and $u$ has a $p$-weak upper gradient $g$ such that $g \in L^p(X)$. For a function $u \in \tilde{N}^{1,p}(X)$, we define

$$\|u\|_{\tilde{N}^{1,p}(X)} = \|u\|_{L^p(X)} + \inf_{g} \|g\|_{L^p(X)},$$
where the infimum is taken over all \( p \)-weak upper gradients of \( u \). For functions \( u, v \in \tilde{N}^{1,p}(X) \), we define the relation \( u \sim v \) if and only if \( ||u - v||_{\tilde{N}^{1,p}(X)} = 0 \). We define the Newtonian space \( N^{1,p}(X) = \tilde{N}^{1,p}(X)/\sim \) equipped with the norm \( ||\cdot||_{N^{1,p}(X)} \).

Following properties of the Newtonian spaces are known (see [S1]):

(i) \( N^{1,p}(X) \) is a Banach space.
(ii) Lipschitz functions are dense in \( N^{1,p}(X) \).
(iii) Every \( u \in N^{1,p}(X) \) has a unique minimal \( p \)-weak upper gradient \( g_u \in L^p(X) \) in the sense that for every \( p \)-weak upper gradient \( g \) of \( u \), \( g_u \leq g \mu\text{-a.e} \) in \( X \).

For a set \( E \) in \( X \), the \( p \)-capacity of \( E \) is defined by

\[
C_p(E) = \inf_u ||u||_{N^{1,p}(X)},
\]

where the infimum is taken over all \( u \in N^{1,p}(X) \) such that \( u = 1 \) on \( E \), and the Newtonian space with zero boundary values is defined by

\[
N_0^{1,p}(E) = \{u \in N^{1,p}(X) \mid C_p(\{x \in X \mid u(x) \neq 0\}) = 0\}.
\]

Let \( \Omega \) be an open subset in \( X \). If \( u \in N^{1,p}(E) \) for every measurable set \( E \subseteq \Omega \), we write \( u \in N_{1\text{oc}}^{1,p}(\Omega) \). For more various properties of Newtonian spaces, see [S1].

In addition, we assume following two conditions:

(I) The measure \( \mu \) is doubling, that is, there exists a constant \( C > 0 \) such that

\[
0 < \mu(2B) \leq C \mu(B)
\]

whenever \( B = B(x_0, r) = \{x \in X \mid d(x, x_0) < r\} \) is a ball in \( X \) and \( \lambda B = B(x_0, \lambda r) \) for \( \lambda \in \mathbb{R} \).

(II) \( X \) supports a weak \( (1, p) \)-Poincaré inequality, that is, there exist constants \( C > 0 \) and \( \lambda \geq 1 \) such that for all balls \( B \subset X \), all measurable functions \( f \) on \( X \) and all upper gradients \( g \) of \( f \),

\[
\frac{1}{\mu(B)} \int_B |f - f_B| d\mu \leq C(dim A B) \left( \frac{1}{\mu(\lambda B)} \int_{\lambda B} g^p d\mu \right)^{1/p},
\]

where \( f_B = \frac{1}{\mu(B)} \int_B f d\mu \).
In [B] there are various examples of spaces equipped with a doubling measure and supporting Poincaré inequality.

§2. Quasisuperminimizers

Let a constant $Q \geq 1$. A function $u \in N_{1oc}^{1,p}(\Omega)$ is said to be a $(Q,p)$-quasiminimizer in $\Omega$ if for all open $\Omega' \Subset \Omega$ and all $\varphi \in N_{0}^{1,p}(\Omega')$ we have

$$\int_{\Omega'} g_{u}^{p} d\mu \leq Q \int_{\Omega'} g_{u+\varphi}^{p} d\mu.$$  

A function $u \in N_{1oc}^{1,p}(\Omega)$ is said to be a $(Q,p)$-quasisuperminimizer if and only if $u$ is a $(Q,p)$-quasiminimizer and a $(Q,p)$-quasisubminimizer.

A $(Q,p)$-quasiminimizer (respectively, $(Q,p)$-quasisuperminimizer) has a continuous (respectively, lower semicontinuous) representative (see [KM1; Theorem 5.1], [KM2; Lemma 5.3] and [KS; Proposition 3.3 and Theorem 5.2]). If $u$ is a $(1,p)$-quasiminimizer (respectively, $(1,p)$-quasisuperminimizer), we say that $u$ is a minimizer (respectively, superminimizer). A continuous minimizer is said to be $p$-harmonic. Potential theory for $p$-harmonic functions on metric measure spaces has been studied in [C], [S2], [KM1], [BBS1] and [BBS2] etc.

If $u$ is a $(Q,p)$-quasisuperminimizer and $\lambda \geq 0$, $\tau$ are constants, then $\lambda u + \tau$ is a $(Q,p)$-quasisuperminimizer.

§3. A convergence property for quasisuperminimizers

In [KM2; Theorem 6.1] the following convergence result for quasisuperminimizers was established:

**Proposition.** Let $\Omega$ be an open set in $X$ and let $\{u_{n}\}$ be a nondecreasing sequence of $(Q,p)$-quasisuperminimizers in $\Omega$ and $u = \lim_{n \to \infty} u_{n}$. If either $u$ is locally bounded above or $u \in N_{loc}^{1,p}(\Omega)$, then $u$ is a $(Q,p)$-quasisuperminimizer in $\Omega$.

We can relax the condition in the above proposition as follows.

**Theorem.** Let $\Omega$ be an open set in $X$ and let $\{u_{n}\}$ be a nondecreasing sequence of $(Q,p)$-quasisuperminimizers in $\Omega$. If there is a function $f \in \ldots$
$N^{1,p}_{\text{loc}}(\Omega)$ such that $u_n \leq f \mu$-a.e. for all $n$, then $u = \lim_{n \to \infty} u_n$ is a $(Q,p)$-quaisuperminimizer in $\Omega$.

Let $\Omega$ be an open subset of $X$. A function $u : \Omega \to \mathbb{R} \cup \{\infty\}$ is said to be $(Q,p)$-quaisuperharmonic in $\Omega$ in the sense of [KM2] if

(i) $u$ is lower semicontinuous,
(ii) $u \not\equiv \infty$ in $\Omega$, and
(iii) there exist an exhaustion $\{\Omega_n\}$ of $\Omega$ and a nondecreasing sequence $\{u_n\}$ of $(Q,p)$-quaisuperminimizers in $\Omega_n$ such that $u = \lim_{n \to \infty} u_n^*$, where $u_n^*(x) = \text{ess lim inf}_{y \to x} u_n(y)$.

If $u$ is a $(Q,p)$-quaisuperminimizers, then $u$ has a $(Q,p)$-quaisuperharmonic representative (see [KM2; Proposition 7.2]).

From the above theorem the next corollary follows immediately.

**Corollary.** Let $\Omega$ be an open set in $X$ and let $u$ be a $(Q,p)$-quaisuperharmonic function in the sense of [KM2] in $\Omega$. If there is a function $f \in N^{1,p}_{\text{loc}}(\Omega)$ such that $u \leq f \mu$-a.e., then $u$ is a $(Q,p)$-quaisuperminimizers in $\Omega$.

**References**


[O] T. Ono, A convergence property for quasisuperminimizers, in preparation
