<table>
<thead>
<tr>
<th>Title</th>
<th>Sufficient conditions for starlikeness (Study on Inverse Problems in Univalent Function Theory)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Li, Jian-Lin; Owa, Shigeyoshi</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2001), 1192: 87-93</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2001-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/64773">http://hdl.handle.net/2433/64773</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Sufficient conditions for starlikeness

Jian-Lin Li and SHIGEYOSHI OWA

Abstract. The object of the present paper is to consider a sufficient condition for analytic functions in the open unit disk to be starlike.

1 Introduction.

Let $A$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z)$ in $A$ is said to be starlike of order $\alpha$ in $U$ if it satisfies

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} \left( \frac{zf''(z)}{f'(z)} + 1 \right) \right\} > \alpha \quad (z \in U).$$

We denote by $S^*(\alpha)$ the subclass of $A$ consisting of all functions $f(z)$ which are starlike of order $\alpha$ in $U$. We denote by $S^*(0) \equiv S^*$.

Lewandowski, Miller and Zlotkiewicz [1] have shown

Theorem A. If $f(z) \in A$ satisfies

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} \left( \frac{zf''(z)}{f'(z)} + 1 \right) \right\} > 0 \quad (z \in U),$$

then $f(z) \in S^*$.

Mathematics Subject Classification (1991): 30C45

Key Words and Phrases: Analytic, starlikeness.
Recently, Ramesha, Kumar and Padmanabhan [5] have given

Theorem B. If \( f(z) \in A \) satisfies

\[
\Re \left\{ \frac{zf'(z)}{f(z)} \left( \alpha \frac{zf''(z)}{f'(z)} + 1 \right) \right\} > 0 \quad (z \in U)
\]

for some \( \alpha (\alpha \geq 0) \), then \( f(z) \in S^* \).

On the other hand, Obradović [4] has proved

Theorem C. If \( f(z) \in A \) satisfies

\[
\left| \frac{zf''(z)}{f'(z)} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right| < \frac{1}{6} \quad (z \in U),
\]

then \( f(z) \in S^* \).

Further, more recently, Li and Owa [2] have derived

Theorem D. If \( f(z) \in A \) satisfies

\[
\left| \frac{zf''(z)}{f'(z)} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right| < \frac{3}{2} \quad (z \in U),
\]

then \( f(z) \in S^* \).

To derive our theorems, we need the following lemma due to Miller and Mocanu [3].

Lemma. Let \( \Omega \) be a set in the complex plane \( \mathbb{C} \). Suppose that \( \Phi \) is a mapping from \( \mathbb{C}^2 \times U \) to \( \mathbb{C} \) which satisfies \( \Phi(ix,y;z) \notin \Omega \) for \( z \in U \), and for all real \( x, y \) such that \( y \leq -(1 + x^2)/2 \). If the function \( p(z) \) is analytic in \( U \) with \( p(0) = 1 \) and \( \Phi(p(z),zp'(z);z) \in \Omega \) for all \( z \in U \), then \( \Re(p(z)) > 0 \) (\( z \in U \)).
2 Conditions for starlikeness

In this section, we derive some sufficient conditions for starlikeness, which are the improvements of the previous theorems. Our first result is contained in

**Theorem 1.** If \( f(z) \in A \) satisfies

\[
\text{Re}\left\{ \frac{zf'(z)}{f(z)} \left( \alpha \frac{zf''(z)}{f'(z)} + 1 \right) \right\} > -\frac{\alpha}{2} \quad (z \in U) \tag{2.1}
\]

for some \( \alpha (\alpha \geq 0) \), then \( f(z) \in S^* \).

**Proof.** Let us define the analytic function \( p(z) \) in \( U \) by

\[
p(z) = \frac{zf'(z)}{f(z)} = 1 + p_1 z + p_2 z^2 + \cdots. \tag{2.2}
\]

Making use of the logarithmic differentiations of both sides in (2.2), we know that

\[
\frac{zf'(z)}{f(z)} \left( \alpha \frac{zf''(z)}{f'(z)} + 1 \right) = \alpha z p'(z) + \alpha p(z)^2 + (1 - \alpha) p(z). \tag{2.3}
\]

Let \( \Omega = \{ w \in \mathbb{C} : \text{Re}(w) > -\alpha/2 \} \) and

\[
\Phi(z_1, z_2; z) = \alpha z_2 + \alpha z_1^2 + (1 - \alpha) z_1.
\]

Then from (2.1) and (2.3), we have \( \Phi(p(z), zp'(z); z) \in \Omega \) for all \( z \in U \). Further, we have

\[
\text{Re}\{ \Phi(ix, y; z) \} = \alpha y - \alpha x^2
\]

\[
\leq -\frac{\alpha}{2} - \frac{3}{2} \alpha x^2
\]

\[
\leq -\frac{\alpha}{2}.
\]

This shows that \( \Phi(ix, y; z) \in \Omega \). Therefore, by virtue of Lemma, we conclude that \( f(z) \in S^* \).

Letting \( \alpha = 1 \) in Theorem 1, we have
Corollary 1. If \( f(z) \in A \) satisfies
\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \left( \frac{zf''(z)}{f'(z)} + 1 \right) \right\} > -\frac{1}{2} \quad (z \in U),
\]
then \( f(z) \in S^* \).

Next we derive

**Theorem 2.** If \( f(z) \in A \) satisfies
\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \left( \alpha \frac{zf''(z)}{f'(z)} + 1 \right) \right\} > -\frac{\alpha^2}{4} (1 - \alpha) \quad (z \in U) \tag{2.4}
\]
for some \( \alpha (0 \leq \alpha < 2) \), then \( f(z) \in S^*(\alpha/2) \).

**Proof.** Define the function \( p(z) \) by
\[
\frac{zf'(z)}{f(z)} = \left(1 - \frac{\alpha}{2}\right) p(z) + \frac{\alpha}{2} \quad (z \in U). \tag{2.5}
\]
Then \( p(z) \) is analytic in \( U \) and \( p(z) = 1 + p_1 z + p_2 z^2 + \cdots \). Differentiating (2.6) logarithmically, we see that
\[
\frac{zf'(z)}{f(z)} \left( \alpha \frac{zf''(z)}{f'(z)} + 1 \right) = \alpha \left(1 - \frac{\alpha}{2}\right) zp'(z) + \alpha \left(1 - \frac{\alpha}{2}\right)^2 p(z)^2
\]
\[
+ \left(1 - \frac{\alpha}{2}\right) (\alpha^2 + 1 - \alpha)p(z) + \frac{\alpha^3}{4} + \frac{\alpha}{2}(1 - \alpha). \tag{2.6}
\]
Let us define
\[
\Omega = \left\{ w \in \mathbb{C} : \text{Re}(w) > -(1 - \alpha)\alpha^2 \right\}
\]
and
\[
\Phi(z_1, z_2; z) = \alpha \left(1 - \frac{\alpha}{2}\right) z_2 + \alpha \left(1 - \frac{\alpha}{2}\right)^2 z_1^2 + \left(1 - \frac{\alpha}{2}\right) (\alpha^2 - \alpha + 1) z_1 + \frac{\alpha^3}{4} + \frac{\alpha}{2}(1 - \alpha).
\]
Then by (2.4) and (2.6), we know that \( \Phi(p(z), zp'(z); z) \in \Omega \). Further, for all \( z \in U \) and for all real \( x, y \) such that \( y \leq -(1 + x^2)/2 \), we have
\[
\text{Re} \left\{ \Phi(ix, y; z) \right\} = \alpha \left(1 - \frac{\alpha}{2}\right) y - \alpha \left(1 - \frac{\alpha}{2}\right)^2 x^2 + \frac{\alpha^3}{4} + \frac{\alpha}{2}(1 - \alpha)\]
Thus, by applying Lemma, we have $\text{Re}(p(z)) > 0$ for $z \in U$, which, in view of (2.5), is equivalent to $f(z) \in S^{*}(\alpha/2)$.

If we take $\alpha - 1$ in Theorem 2, then we have

**Corollary 2.** If $f(z) \in A$ satisfies

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} \left( \frac{zf''(z)}{f'(z)} + 1 \right) \right\} > 0 \quad (z \in U),$$

then $f(z) \in S^{*}(1/2)$.

Finally, we consider

**Theorem 3.** If $f(z) \in A$ satisfies

$$\left| \frac{zf''(z)}{f'(z)} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right| < \rho \quad (z \in U),$$

where

$$\rho = \left( \frac{827 + 73\sqrt{73}}{288} \right)^{\frac{1}{2}} = 2.2443697\cdots,$$

then $f(z) \in S^{*}$.

**Proof.** Let the function $p(z)$ be defined by (2.2). Then it follows that

$$\frac{zf''(z)}{f'(z)} \left( \frac{zf'(z)}{f(z)} - 1 \right) = (p(z) - 1) \left( \frac{zp'(z)}{p(z)} + p(z) - 1 \right). \quad (2.7)$$

Letting $\Omega = \{w \in \mathbb{C}: |w| < \rho\}$ and

$$\Phi(z_1, z_2 : z) = (z_1 - 1) \left( \frac{z_2}{z_1} + z_1 - 1 \right),$$
we have $\Phi(p(z), zp'(z) : z) \in \Omega$. Further, for all $z \in U$, and for all real $x, y$ with $y \leq -(1 + x^2)/2$, $\Phi(p(z), zp'(z); z)$ satisfies

$$|\Phi(ix, y; z)| = \sqrt{(1 + x^2) \left(1 + \frac{(x^2 - y)^2}{x^2}\right)} = \sqrt{g(x^2, y)}, \quad (2.8)$$

where $t = x^2 > 0$ and $y \leq -(1 + t)/2$. Since

$$\frac{\partial g(t, y)}{\partial y} = 2 \frac{1 + t}{t} (y - t) < 0,$$

we have

$$g(t, y) \geq g\left(t, -\frac{1 + t}{t}\right) = \frac{(t + 1)^2(9t + 1)}{4t} \equiv h(t). \quad (2.9)$$

Further, since

$$h'(t) = \frac{(t + 1) \left(t + \frac{\sqrt{73} + 1}{36}\right) \left(t - \frac{\sqrt{73} - 1}{36}\right)}{4t^2},$$

we obtain

$$\min_{t>0} h(t) = h\left(\frac{\sqrt{73} - 1}{36}\right) = \frac{827 + 73\sqrt{73}}{288} = \rho^2. \quad (2.10)$$

This implies that $|\Phi(ix, y; z)| \geq \rho$. It follows from (2.8), (2.9) and (2.10) that $\Phi(ix, y; z) \notin \Omega$. An application of Lemma gives us that $\text{Re}(p(z)) > 0$ for $z \in U$. Thus we conclude that $f(z) \in S^*$. 

**References**


Jian-Lin Li
Department of Applied Mathematics
Northwestern Polytechnical University
Xi An, Shaan Xi 710072
People's Republic of China

Shigeyoshi Owa
Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577-8502
Japan