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TWINTING CHARACTER FORMULA FOR DEMAZURE MODULES

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0. Introduction.

Let $\mathfrak{g}$ be a finite-dimensional complex semi-simple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$. Let $\Delta \subset \mathfrak{h}^*$ be the set of roots of $\mathfrak{g}$ relative to $\mathfrak{h}$. We choose the set of positive roots $\Delta_+$ such that the roots of $\mathfrak{b}$ are $-\Delta_+$. Let $\{\alpha_i \mid i \in I\}$ be the set of simple roots in $\Delta_+$, $\{h_i \mid i \in I\}$ the set of simple coroots in $\mathfrak{h}$, $A = (a_{ij})_{i,j \in I}$ the Cartan matrix with $a_{ij} = \alpha_j(h_i)$, and $W = \langle r_i \mid i \in I \rangle \subset GL(\mathfrak{h}^*)$ the Weyl group. We take and fix a Chevalley basis $\{e_\alpha, f_\alpha \mid \alpha \in \Delta_+\} \cup \{h_i \mid i \in I\}$ of $\mathfrak{g}$, and let $\mathfrak{h}_\mathbb{Z} = \sum_{i \in I} \mathbb{Z}h_i$.

A bijection $\omega$ (of order $N$) of the index set $I$ such that $\omega(\omega(i)\omega(j)) = a_{ij}$ for all $i, j \in I$ induces a unique automorphism $\omega$, called a (Dynkin) diagram automorphism, of the Lie algebra $\mathfrak{g}$ such that $\omega(e_{\alpha_i}) = e_{\alpha_{\omega(i)}}$, $\omega(f_{\alpha_i}) = f_{\alpha_{\omega(i)}}$, and $\omega(h_i) = h_{\omega(i)}$ for $i \in I$. We denote by $\langle \omega \rangle$ the cyclic subgroup (of order $N$) of $\text{Aut}(\mathfrak{g})$ generated by the diagram automorphism $\omega$. The restriction of $\omega$ to $\mathfrak{h}$ induces a transposed map $\omega^*: \mathfrak{h}^* \rightarrow \mathfrak{h}^*$, which stabilizes the integral weight lattice $\mathfrak{h}_\mathbb{Z}^* = \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbb{Z} \text{ for all } i \in I\} \simeq \text{Hom}(\mathfrak{h}_\mathbb{Z}, \mathbb{Z})$. We set $\mathfrak{g}^0 = \{x \in \mathfrak{g} \mid \omega(x) = x\}$, $\mathfrak{h}^0 = \{h \in \mathfrak{h} \mid \omega(h) = h\}$, $W^\omega = \{w \in W \mid \omega^*w = w\omega^*\}$, $\mathfrak{h}^0_\mathbb{Z} = \{\lambda \in \mathfrak{h}^* \mid \omega^*(\lambda) = \lambda\} \simeq (\mathfrak{h}^0)^*$, and $\mathfrak{h}^*_\mathbb{Z} = \{\lambda \in \mathfrak{h}_{\mathbb{Z}}^* \mid \omega^*(\lambda) = \lambda\}$.

Let $\hat{\mathfrak{g}}$ be the orbit Lie algebra, which is the dual complex semi-simple Lie algebra of the fixed point (semi-simple) subalgebra $\mathfrak{g}^0$ of $\mathfrak{g}$, i.e., a complex semi-simple Lie algebra with the opposite Dynkin diagram to that of $\mathfrak{g}^0$. Let $\hat{\mathfrak{h}}$ be the Cartan subalgebra of $\hat{\mathfrak{g}}$, $\hat{\mathfrak{b}} \supset \hat{\mathfrak{h}}$ the Borel subalgebra, and $\hat{\Delta}_+ \subset \hat{\mathfrak{h}}^*$ the set of positive roots chosen so that the roots of $\hat{\mathfrak{b}}$ are $-\hat{\Delta}_+$. Let $\{\hat{a}_i \mid i \in \hat{I}\}$ be the set of simple roots in $\hat{\Delta}_+$ and $\hat{W} = \langle \hat{r}_i \mid i \in \hat{I} \rangle \subset GL(\hat{\mathfrak{h}}^*)$ the Weyl group, where the index set $\hat{I}$ is a set of representatives of the $\omega$-orbits in $I$. It is known that there exist an isomorphism of groups $\Theta: \hat{W} \rightarrow W^\omega$ and a $C$-linear isomorphism $P_\omega: \mathfrak{h}^0 \rightarrow \hat{\mathfrak{h}}$ such that if $P_\omega^*: \hat{\mathfrak{h}}^* \rightarrow (\mathfrak{h}^0)^* \simeq (\mathfrak{h}^*)^0$ is the transposed map of $P_\omega$, then $\Theta(\hat{w})|_{(\mathfrak{h}^0)^0} = P_\omega^* \circ \hat{w} \circ (P_\omega^*)^{-1}$ for all $\hat{w} \in \hat{W}$. We set $w_i = \Theta(\hat{r}_i) \in W^\omega$ for $i \in \hat{I}$. In particular, $(W^\omega, \{w_i \mid i \in \hat{I}\})$ forms a Coxeter system.
For dominant $\lambda \in (\mathfrak{h}_Z^n)^0$, let $L(\lambda)$ be the simple $\mathfrak{g}$-module of highest weight $\lambda$. It admits a unique $\mathbb{C}$-linear $\langle \omega \rangle$-action such that $\omega \cdot (xv) = \omega(x)(\omega \cdot v)$ for each $x \in \mathfrak{g}$, $v \in L(\lambda)$ and such that $\omega \cdot v_\lambda = v_\lambda$, where $v_\lambda$ is a (nonzero) highest weight vector of $L(\lambda)$. So therefore does its dual module $L(\lambda)^* \simeq L(-w_0(\lambda))$ with $w_0$ the longest element in $W$. Let $\mathfrak{U}(\mathfrak{b})$ be the universal enveloping algebra of $\mathfrak{b}$, and for each $w \in W^\omega$, let $J_w(\lambda) = \mathfrak{U}(\mathfrak{b})v^*_w(\lambda) \subset L(\lambda)^*$ be Joseph's module of highest weight $-w(\lambda)$ in $L(\lambda)^*$, with $v^*_w(\lambda)$ a (nonzero) weight vector in $L(\lambda)^*$ of weight $-w(\lambda)$. Since $w \in W^\omega$, the weight vector $v^*_w(\lambda) \in L(\lambda)^*$ turns out to be fixed by the action of $\langle \omega \rangle$, and hence Joseph's module $J_w(\lambda) \subset L(\lambda)^*$ is $\langle \omega \rangle$-invariant. In the talk we will prove a formula of Demazure type for the twining character $\text{ch}^\omega(J_w(\lambda))$ of $J_w(\lambda)$ defined by

$$\text{ch}^\omega(J_w(\lambda)) = \sum_{\mu \in (\mathfrak{h}_Z^n)^0} \text{Tr}(\omega|_{J_w(\lambda)_\mu}) e(\mu)$$

in the group algebra $\mathbb{C}[(\mathfrak{h}_Z^n)^0]$ over $\mathbb{C}$ of $(\mathfrak{h}_Z^n)^0$ with basis $e(\mu)$, $\mu \in (\mathfrak{h}_Z^n)^0$. As a corollary, we will find a striking relation:

$$\text{ch}^\omega(J_w(\lambda)) = P_w^*(\text{ch}\widehat{J}_\mathfrak{g}(\widehat{\lambda})),$$

where $\hat{w} = \Theta^{-1}(w) \in \mathcal{W}$, $\hat{\lambda} = (P_w^*)^{-1}(\lambda) \in \mathfrak{h}^*$, and $\text{ch}\widehat{J}_\mathfrak{g}(\hat{\lambda}) \in \mathbb{C}[\hat{\mathfrak{h}}^*]$ is the ordinary character of Joseph's module $\widehat{J}_\mathfrak{g}(\hat{\lambda})$ of highest weight $-\hat{w}(\hat{\lambda})$ over the orbit Lie algebra $\widehat{\mathfrak{g}}$.  

Although our problem can be stated purely algebraically as above, it seems very difficult (at least for me) to solve it only by algebraic methods. Hence we resort to (algebro-) geometric methods. For that purpose, we introduce more notation. Let $G$ be a connected, simply connected semi-simple linear algebraic group over $\mathbb{C}$ with maximal torus $T$ and Borel subgroup $B \supset T$ such that $\text{Lie}(G) = \mathfrak{g}$, $\text{Lie}(T) = \mathfrak{t}$, and $\text{Lie}(B) = \mathfrak{b}$. Then the character group $\Lambda = \text{Hom}(T, GL_1)$ of $T$ may be identified with $h^*_Z$ by taking the differential at the identity element, i.e., by the map $\lambda \mapsto \text{d}\lambda$. For each $i \in I$ and $\lambda \in \Lambda$, we will write $\langle \lambda, \alpha_i^\vee \rangle = (\text{d}\lambda)(h_i)$, where $\alpha_i^\vee \in \text{Hom}(GL_1, T)$ is the coroot of $\alpha_i \in \Lambda$. There exists an automorphism of $G$ whose differential at the identity element coincides with the diagram automorphism $\omega$ of $\mathfrak{g}$ above. By abuse of notation, we will denote by $\omega$ this automorphism of $G$ and by $\langle \omega \rangle$ the cyclic subgroup (of order $N$) of $\text{Aut}(G)$ generated by $\omega$. We will also denote the induced action of $\omega \in \langle \omega \rangle$ on $\Lambda$ by the same letter $\omega$, and set $\Lambda^\omega = \{ \lambda \in \Lambda \mid \omega \cdot \lambda = \lambda \}$, $\Lambda_i^\omega = \{ \lambda \in \Lambda^\omega \mid \langle \lambda, \alpha_i^\vee \rangle \geq 0 \quad \text{for all} \ i \in I \}$.

By a $G \rtimes \langle \omega \rangle$-module $M$, we will mean a finite-dimensional rational $G$-module that admits a $\mathbb{C}$-linear $\langle \omega \rangle$-action such that $\omega \cdot (g m) = \omega(g)(\omega \cdot m)$ for each $g \in G$ and $m \in M$. Regarding the semi-direct product $G \rtimes \langle \omega \rangle$ of $G$ and $\langle \omega \rangle$ as a linear algebraic group, this is the same as a rational $G \rtimes \langle \omega \rangle$-module. Likewise for $B \rtimes \langle \omega \rangle$- and $T \rtimes \langle \omega \rangle$-modules. Let $\mathbb{C}[\Lambda^\omega]$ be the group algebra over $\mathbb{C}$ of $\Lambda^\omega$ with basis $e(\mu)$, $\mu \in \Lambda^\omega$. For a $T \rtimes \langle \omega \rangle$-module
we define the twining character $\text{ch}^\omega(V) \in \mathbb{C}[\Lambda^\omega]$ of $V$ to be

$$\text{ch}^\omega(V) = \sum_{\mu \in \Lambda^\omega} \text{Tr}(\omega|_{V_{\mu}}) e(\mu),$$

where $V_{\mu} = \{ v \in V \mid tv = \mu(t)v \text{ for all } t \in T \}$ is the $\mu$-weight space of $V$.

Recall that $W \simeq N_G(T)/T$, where $N_G(T)$ is the normalizer of $T$ in $G$. Fix $w \in W^\omega$, and let $X(w)$ be the associated Schubert variety over $\mathbb{C}$, which is the Zariski closure in the flag variety $G/B$ of the Bruhat cell $B^w B/B$, where $B$ denotes a right coset representative of $w$ in $N_G(T)$ fixed by $\omega \in \text{Aut}(G)$. If $M$ is a $B \rtimes \langle \omega \rangle$-module, then the $B$-equivariant $O_{X(w)}$-module $L_{X(w)}(M)$ associated to $M$ carries a structure of "$(B, \langle \omega \rangle)$-equivariant" (i.e., $B \rtimes \langle \omega \rangle$-equivariant) sheaf, so that its cohomology groups $H^\ast(X(w), L_{X(w)}(M))$ are $B \rtimes \langle \omega \rangle$-modules. (For the precise definition of a $(B, \langle \omega \rangle)$-equivariant sheaf, see our preprint [KN].)

For each $\lambda \in \Lambda^\omega$, we let $C_{\lambda}$ denote the one-dimensional $B \rtimes \langle \omega \rangle$-module on which $B$ acts via $\lambda$ through the quotient $B \to T$ and $\langle \omega \rangle$ trivially. We call $H^0(X(w), L_{X(w)}(C_{\lambda}))$ for $\lambda \in \Lambda^\omega_+$ a Demazure module. Joseph’s module $J_w(\lambda)$ admits a structure of $B \rtimes \langle \omega \rangle$-module, and we have an isomorphism of $B \rtimes \langle \omega \rangle$-modules

$$J_w(\lambda)^* \simeq H^0(X(w), L_{X(w)}(C_{\lambda})),
$$

where $J_w(\lambda)^*$ is the dual $B \rtimes \langle \omega \rangle$-module of $J_w(\lambda)$.

For $i \in \hat{I}$, we define the $\omega$-Demazure operator $\hat{D}_i$ to be a $\mathbb{C}$-linear endomorphism of $\mathbb{C}[\Lambda^\omega]$ such that

$$\hat{D}_i(e(\mu)) = \frac{e(\mu) - e(-s_i \beta_i) e(w_i(\mu))}{1 - e(-s_i \beta_i)}$$

for $\mu \in \Lambda^\omega$,

where $\beta_i = \sum_{k=0}^{N_i-1} \alpha_{\omega^k(i)}$ and $s_i = 2/ \sum_{k=0}^{N_i-1} a_i \omega^k(i)$ with $N_i$ the number of elements of the $\omega$-orbit of $i \in I$.

The following is our main result.

**Theorem 0.1.** Let $M$ be a finite-dimensional rational $B \rtimes \langle \omega \rangle$-module, $w \in W^\omega$, and let $w = w_1 w_2 \cdots w_n$ be a reduced expression in the Coxeter system $(W^\omega, \{w_i \mid i \in \hat{I}\})$. Then we have in $\mathbb{C}[\Lambda^\omega]$,

$$\chi^\omega_w(M) = \sum_{j \geq 0} (-1)^j \text{ch}^\omega(H^j(X(w), L_{X(w)}(M)))
$$

$$= \hat{D}_{i_1} \hat{D}_{i_2} \cdots \hat{D}_{i_n}(\text{ch}^\omega(M)).$$
In particular, for \( \lambda \in \Lambda_{\omega}^{\circ} \), we have

\[
\text{ch}^\omega(H^0(X(w), \mathcal{L}_{X(w)}(\mathbb{C} \lambda))) = \tilde{D}_{i_1} \tilde{D}_{i_2} \cdots \tilde{D}_{i_n}(e(\lambda)).
\]

There is thus revealed a striking relation between twining characters for \( \mathfrak{g} \) and ordinary characters for the orbit Lie algebra \( \widehat{\mathfrak{g}} \). Let \( \widehat{\mathfrak{h}}_{\mathbb{Z}} = \sum_{i \in I} \mathbb{Z} \widehat{h}_i \) and \( \widehat{\mathfrak{h}}_{\mathbb{Z}}^* = \text{Hom}(\widehat{\mathfrak{h}}_{\mathbb{Z}}, \mathbb{Z}) \subset \widehat{\mathfrak{h}}^* \).

For dominant \( \widehat{\lambda} \in \widehat{\mathfrak{h}}_{\mathbb{Z}}^* \), let \( \widehat{L}(\lambda) \) be the simple \( \widehat{\mathfrak{g}} \)-module of highest weight \( \widehat{\lambda} \), and for each \( \widehat{w} \in \overline{W} \), let \( \widehat{J}_{\widehat{w}}(\lambda) = \mathfrak{U}(\mathfrak{b}) \widehat{v}^*_{\mathfrak{g}(\lambda)} \subset \widehat{L}(\lambda)^* \) be Joseph's module of highest weight \( \pm \widehat{w}(\lambda) \), with \( \widehat{v}^*_{\mathfrak{g}(\lambda)} \in \widehat{L}(\lambda)^* \) a (nonzero) weight vector of weight \( \pm \widehat{w}(\lambda) \).

**Corollary 0.2.** Let \( \lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)^0 \) be dominant and \( w \in W^\omega \). We set \( \widehat{w} = \Theta^{-1}(w) \in \overline{W} \) and \( \widehat{\lambda} = (P_{\omega}^*)^{-1}(\lambda) \in \widehat{\mathfrak{h}}_{\mathbb{Z}}^* \). Then we have in \( \mathbb{C}[(\mathfrak{h}_{\mathbb{Z}}^*)^0] \),

\[
\text{ch}^\omega(J_w(\lambda)) = P_{\omega}^*(\text{ch}^{\omega}(\widehat{J}_{\widehat{w}}(\lambda))),
\]

where \( P_{\omega}^* \) on the right-hand side is a \( \mathbb{C} \)-algebra isomorphism \( \mathbb{C}[(\mathfrak{h}_{\mathbb{Z}}^*)^0] \sim \mathbb{C}[(\mathfrak{h}_{\mathbb{Z}}^*)^0] \) defined by \( P_{\omega}^*(e(\widehat{\mu})) = e(P_{\omega}^*(\widehat{\mu})) \) for each basis element \( e(\widehat{\mu}) \), \( \widehat{\mu} \in \widehat{\mathfrak{h}}_{\mathbb{Z}}^* \), of the group algebra \( \mathbb{C}[\widehat{\mathfrak{h}}_{\mathbb{Z}}^*] \) over \( \mathbb{C} \) of \( \widehat{\mathfrak{h}}_{\mathbb{Z}}^* \).

1. **Preliminaries.**

1.1. **Diagram automorphisms.** Let \( \mathfrak{g} \) be a finite-dimensional complex semi-simple Lie algebra with Cartan subalgebra \( \mathfrak{h} \) and Borel subalgebra \( \mathfrak{b} \supset \mathfrak{h} \). Let \( \Delta \subset \mathfrak{h}^* \) be the set of roots of \( \mathfrak{g} \) relative to \( \mathfrak{h} \). We choose the set of positive roots \( \Delta_+ \) such that the roots of \( \mathfrak{b} \) are \( -\Delta_+ \). Let \( \{\alpha_i \mid i \in I\} \) be the set of simple roots in \( \Delta_+ \), \( \{h_i \mid i \in I\} \) the set of simple coroots in \( \mathfrak{h} \), \( A = (a_{ij})_{i,j \in I} \) the Cartan matrix with \( a_{ij} = \alpha_j(h_i) \), and \( W = \langle r_i \mid i \in I \rangle \subset \text{GL}(\mathfrak{h}^*) \) the Weyl group. We take and fix a Chevalley basis \( \{e_\alpha, f_\alpha \mid \alpha \in \Delta_+ \cup \{h_i \mid i \in I\}\} \) of \( \mathfrak{g} \), and let \( \mathfrak{h}_Z = \sum_{i \in I} \mathbb{Z} h_i \).

We fix a bijection \( \omega: I \to I \) of the index set \( I \) such that

\[
a_{\omega(i), \omega(j)} = a_{ij} \quad \text{for all } i, j \in I.
\]

Let \( N \) be the order of \( \omega \), and \( N_i \) the number of elements of the \( \omega \)-orbit of \( i \in I \). This \( \omega \) can be extended in a unique way to an automorphism (also denoted by \( \omega \)) of order \( N \) of the Lie algebra \( \mathfrak{g} \) in such a way that

\[
\begin{align*}
\omega(e_{\alpha_i}) &= e_{\alpha_{\omega(i)}}, & i & \in I, \\
\omega(f_{\alpha_i}) &= f_{\alpha_{\omega(i)}}, & i & \in I, \\
\omega(h_i) &= h_{\omega(i)}, & i & \in I.
\end{align*}
\]
Note that the restriction of $\omega$ to the Cartan subalgebra $\mathfrak{h}$ induces a transposed map $\omega^* : \mathfrak{h}^* \to \mathfrak{h}^*$ such that $\omega^*(\lambda)(h) = \lambda(\omega(h))$ for $\lambda \in \mathfrak{h}^*$, $h \in \mathfrak{h}$. We set

$$(\mathfrak{h}^*)^0 = \{ \lambda \in \mathfrak{h}^* | \omega^*(\lambda) = \lambda \} \quad \text{and} \quad (\mathfrak{h}_\mathbb{Z}^*)^0 = \{ \lambda \in \mathfrak{h}_\mathbb{Z}^* | \omega^*(\lambda) = \lambda \},$$

where $\mathfrak{h}_\mathbb{Z}^* = \{ \lambda \in \mathfrak{h}^* | \lambda(h_i) \in \mathbb{Z} \}$ for all $i \in I \simeq \text{Hom}(\mathfrak{h}_\mathbb{Z}, \mathbb{Z})$. Note that the Weyl vector $\rho = (1/2) \cdot \sum_{\alpha \in \Delta^+} \alpha$ is in $(\mathfrak{h}_\mathbb{Z}^*)^0$.

### 1.2. Orbit Lie algebras.

We choose and fix a set $\hat{I}$ of representatives of the $\omega$-orbits in $I$, and set $\hat{A} = (\hat{a}_{ij})_{i,j \in \hat{I}}$, where $\hat{a}_{ij}$ is given by

$$\hat{a}_{ij} = s_j \times \sum_{k=0}^{N_j-1} a_{i,\omega^k(j)} \quad \text{for } i, j \in \hat{I} \quad \text{with} \quad s_j = \frac{2}{\sum_{k=0}^{N_j-1} a_{j,\omega^k(j)}} \quad \text{for } j \in \hat{I}.$$

Set for each $i \in \hat{I}$, $I_i = \{ \omega^k(i) | 0 \leq k \leq N_i - 1 \} \subset I$. We know that for each $i \in \hat{I}$,

$$\sum_{k \in I_i} a_{ik} = 1 \quad \text{or} \quad 2.$$

Moreover, there are only two possibilities:

(a) if $\sum_{k \in I_i} a_{ik} = 1$, then $N_i$ is even and the subgraph of the Dynkin diagram corresponding to the subset $I_i \subset I$ is of type $A_2 \times \cdots \times A_2$ (where $A_2$ appears $N_i/2$ times);

(b) if $\sum_{k \in I_i} a_{ik} = 2$, then the subgraph of the Dynkin diagram corresponding to the subset $I_i \subset I$ is totally disconnected and of type $A_1 \times \cdots \times A_1$ (where $A_1$ appears $N_i$ times).

The orbit Lie algebra associated to the diagram automorphism $\omega \in \text{Aut}(\mathfrak{g})$ is defined to be the complex semi-simple Lie algebra $\hat{\mathfrak{g}}$ associated to the Cartan matrix $\hat{A} = (\hat{a}_{ij})_{i,j \in \hat{I}}$ with the Cartan subalgebra $\hat{\mathfrak{h}}$, the Borel subalgebra $\hat{\mathfrak{b}} \supset \hat{\mathfrak{h}}$, the set of positive roots $\hat{\Delta}^+ \subset \hat{\mathfrak{h}}^*$ chosen so that the roots of $\hat{\mathfrak{b}}$ are $-\hat{\Delta}^+$, the set of simple roots $\{\hat{\alpha}_i | i \in \hat{I}\} \subset \hat{\mathfrak{h}}^*$, the set of simple coroots $\{\hat{\rho}_i | i \in \hat{I}\} \subset \hat{\mathfrak{h}}$, and the Weyl group $\hat{W} = \langle \hat{r}_i | i \in \hat{I} \rangle \subset GL(\hat{\mathfrak{h}}^*)$.

**Remark 1.2.1.** We can easily deduce that the orbit Lie algebra $\hat{\mathfrak{g}}$ is the dual complex semi-simple Lie algebra of the fixed point (semi-simple) subalgebra $\mathfrak{g}^0 = \{ x \in \mathfrak{g} | \omega(x) = x \}$ of $\mathfrak{g}$, i.e., a complex semi-simple Lie algebra which has the opposite Dynkin diagram to that of $\mathfrak{g}^0$. 
We set $\mathfrak{h}^0 = \{ h \in \mathfrak{h} \mid \omega(h) = h \}$. Then there exists a linear isomorphism $P_\omega: \mathfrak{h}^0 \rightarrow \hat{\mathfrak{g}}$ given by

$$P_\omega \left( \sum_{k \in I_i} h_k \right) = N_i \hat{h}_i \quad \text{for each } i \in \hat{I}.$$ 

This map $P_\omega: \mathfrak{h}^0 \rightarrow \hat{\mathfrak{g}}$ induces a transposed map $P_\omega^*: \hat{\mathfrak{g}}^0 \rightarrow \wedge \mathfrak{h}$ given by

$$P_\omega^*(_{k \in I} \sum h_k) = N_i h_i \wedge$$

for each $i \in I \wedge$. This map $P_\omega^*: \hat{\mathfrak{g}}^0 \rightarrow \mathfrak{h} \wedge$ induces an isomorphism $\overline{W}$ from the Weyl group $\overline{W}$ onto the group $W^\omega$ such that the following diagram commutes for each $\hat{w} \in \overline{W}$:

$$\begin{array}{ccc}
\hat{h}^* & \xrightarrow{P_\omega^*} & (\hat{h}^*)^0 \\
\downarrow \hat{w} & & \downarrow \hat{w}\mid(\mathfrak{h}^*)^0 \\
\hat{h}^* & \xrightarrow{P_\omega^*} & (\hat{h}^*)^0 
\end{array}$$

For each $i \in \hat{I}$, set $w_i = \Theta(\hat{r}_i) \in W^\omega$. Explicitly,

$$w_i = \begin{cases} 
\prod_{k=0}^{N_i/2-1} (r_{\alpha k(i)} r_{\alpha k+N_i/2(i)} r_{\alpha k(i)}) & \text{if } \sum_{k=0}^{N_i-1} a_i,\alpha k(i) = 1, \\
\prod_{k=0}^{N_i-1} r_{\alpha k(i)} & \text{if } \sum_{k=0}^{N_i-1} a_i,\alpha k(i) = 2.
\end{cases}$$

Hence each $w_i$ is the longest element of the subgroup $W_{I_i}$ of the Weyl group $W$ generated by the $r_k$'s for $k \in I_i$. Furthermore, $(W^\omega, \{ w_i \mid i \in \hat{I} \})$ forms a Coxeter system as $(\hat{W}, \{ \hat{r}_i \mid i \in \hat{I} \})$ is. We will denote the length function of the Coxeter system $(W, \{ r_i \mid i \in I \})$ (resp. $(W^\omega, \{ w_i \mid i \in \hat{I} \})$) by $\ell: W \rightarrow \mathbb{Z}_{\geq 0}$ (resp. $\hat{\ell}: W^\omega \rightarrow \mathbb{Z}_{\geq 0}$).

**Remark 1.2.2.** Note that the longest element $w_0 \in W$ belongs to $W^\omega$. In fact, we can easily show that the isomorphism $\Theta: \hat{W} \rightarrow W^\omega$ maps the longest element $\hat{w}_0 \in \hat{W}$ to the longest element $w_0 \in W$.

### 1.3. The $\omega$-Demazure operators.

Recall the ordinary Demazure operator $D_i$ for $i \in I$ on the group ring $\mathbb{Z}[\hat{\mathfrak{g}}] = \coprod_{\lambda \in \hat{\mathfrak{g}}} \mathbb{Z} e(\lambda)$:

$$D_i: e(\lambda) \mapsto \frac{e(\lambda) - e(-\alpha_i) e(r_i(\lambda))}{1 - e(-\alpha_i)}.$$
Let $\mathbb{C}[\hat{\mathfrak{h}}_\mathbb{Z}^\omega]$ be the group algebra over $\mathbb{C}$ of $\hat{\mathfrak{h}}_\mathbb{Z}$ with basis $e(\lambda)$, $\lambda \in \hat{\mathfrak{h}}_\mathbb{Z}$. Define likewise the Demazure operator $D_{\hat{r}_i}$, $i \in \hat{I}$, on $\mathbb{C}[\hat{\mathfrak{h}}_\mathbb{Z}^\omega]$ to be the $\mathbb{C}$-linear endomorphism of $\mathbb{C}[\hat{\mathfrak{h}}_\mathbb{Z}^\omega]$ given by

$$D_{\hat{r}_i}(e(\lambda)) = \frac{e(\lambda) - e(-\hat{\alpha}_i)\hat{r}_i(e(\lambda))}{1 - e(-\hat{\alpha}_i)}.$$ 

Then transfer $D_{\hat{r}_i}$ via $P_\omega^*$ onto the group algebra $\mathbb{C}[(\mathfrak{h}_\mathbb{Z}^\omega)^0]$ to define the $\omega$-Demazure operator

$$(1.3.1) \quad \hat{D}_i = P_\omega^* \circ D_{\hat{r}_i} \circ (P_\omega^*)^{-1} \quad \text{for } i \in \hat{I}. $$

Thus we can easily check the following.

**Lemma 1.3.1.** Let $i \in \hat{I}$. For each $\lambda \in (\mathfrak{h}_\mathbb{Z}^\omega)^0$, we have

$$\hat{D}_i(e(\lambda)) = \frac{e(\lambda) - e(-s_i\beta_i)\omega(w_i(\lambda))}{1 - e(-s_i\beta_i)},$$

and moreover

$$\hat{D}_i(e(\lambda)) = \begin{cases} 
  e(\lambda) + e(\lambda - s_i\beta_i) + \cdots + e(w_i(\lambda)) & \text{if } \lambda(h_i) \in \mathbb{Z}_{\geq 0}, \\
  0 & \text{if } \lambda(h_i) = -1, \\
  -\left( e(\lambda + s_i\beta_i) + e(\lambda + 2s_i\beta_i) + \cdots + e(w_i(\lambda + s_i\beta_i)) \right) & \text{if } \lambda(h_i) \in \mathbb{Z}_{\leq -2}. 
\end{cases}$$

**Remark 1.3.2.** Let $w = w_{i_1}w_{i_2}\cdots w_{i_n}$ be a reduced expression of $w \in W^\omega$ in the Coxeter system $(W^\omega, \{w_i \mid i \in \hat{I}\})$, i.e., $\hat{\ell}(w) = n$. We set $\hat{D}_w = \hat{D}_{i_1}\hat{D}_{i_2}\cdots\hat{D}_{i_n} \in \text{End}_\mathbb{C}(\mathbb{C}[(\mathfrak{h}_\mathbb{Z}^\omega)^0])$, which does not depend on the choice of the reduced expression of $w \in W^\omega$.

1.4. **Twining characters.** Let $G$ be a connected, simply connected semi-simple linear algebraic group over $\mathbb{C}$ with maximal torus $T$ and Borel subgroup $B \supset T$ such that $\text{Lie}(G) = \mathfrak{g}$, $\text{Lie}(T) = \mathfrak{h}$, and $\text{Lie}(B) = \mathfrak{b}$. Then the character group $\Lambda = \text{Hom}(T, GL_1)$ of $T$ may be identified with $\mathfrak{h}_\mathbb{Z}^\omega$ by taking the differential at the identity element, i.e., by the map $\lambda \mapsto d\lambda$. For each $i \in I$ and $\lambda \in \Lambda$, we will write $\langle \lambda, \alpha_i^\vee \rangle = (d\lambda)(h_i)$, where $\alpha_i^\vee \in \text{Hom}(GL_1, T)$ is the coroot of $\alpha_i \in \Lambda$. Let $\Lambda_+ = \{ \lambda \in \Lambda \mid \langle \lambda, \alpha_i^\vee \rangle \geq 0 \text{ for all } i \in I \}$ be the set of dominant weights of $\Lambda$.

There exists an automorphism of $G$ whose differential at the identity element coincides with the diagram automorphism $\omega$ of $\mathfrak{g}$. By abuse of notation, we will denote still by $\omega$ this automorphism of $G$ and by $\langle \omega \rangle$ the cyclic subgroup (of order $N$) of $\text{Aut}(G)$ generated by the $\omega$. Whenever there can be ambiguity, we will write $d\omega$ for the automorphism of $\mathfrak{g}$. Recall also that the Weyl group $W \subset GL(\mathfrak{h}^\ast)$ may be identified with $N_G(T)/T$, $N_G(T)$ the normalizer of $T$ in $G$. Each $w \in W^\omega$ lifts to an element of $N_G(T)$ fixed by
$\omega \in \text{Aut}(G)$, which will be denoted by $\dot{\omega}$. We will also denote the induced action of $\omega$ on $\Lambda$ by the same letter $\omega$, and set $\Lambda^\omega = \{ \lambda \in \Lambda \mid \omega \cdot \lambda = \lambda \}$, $\Lambda^\omega_+ = \Lambda^\omega \cap \Lambda_+$. Note that, under the identification $\Lambda \simeq \mathfrak{h}_\mathbb{Z}^* \subset \mathfrak{h}^*$, this action of $\omega$ on $\Lambda$ coincides with the restriction of $((d\omega)^{-1})^* = ((d\omega)^*)^{-1}$ to $\mathfrak{h}_\mathbb{Z}^*$.

By a $G \rtimes \langle \omega \rangle$-module $M$, we will always mean a finite-dimensional rational $G$-module that admits a $\mathbb{C}$-linear $\langle \omega \rangle$-action such that

$$\omega \cdot (gm) = \omega(g)(\omega \cdot m) \quad \text{for all } g \in G, \ m \in M.$$  

Regarding the semi-direct product $G \rtimes \langle \omega \rangle$ of $G$ and $\langle \omega \rangle$ as a linear algebraic group, this is the same as a finite-dimensional rational $G \rtimes \langle \omega \rangle$-module. Likewise for $B \rtimes \langle \omega \rangle$- and $T \rtimes \langle \omega \rangle$-modules. Let $\mathbb{C}[\Lambda^\omega]$ be the group algebra over $\mathbb{C}$ of $\Lambda^\omega$ with basis $e(\lambda)$, $\lambda \in \Lambda^\omega$. Let $M$ be a $T \rtimes \langle \omega \rangle$-module, and let

$$M = \prod_{\lambda \in \Lambda} M_\lambda \quad \text{with} \quad M_\lambda = \{ m \in M \mid t m = \lambda(t)m \quad \text{for all } t \in T \}$$

be the weight space decomposition with respect to $T$. Now we define the twining character $\text{ch}^\omega(M)$ of $M$ to be

$$\text{ch}^\omega(M) = \sum_{\lambda \in \Lambda^\omega} \text{Tr}(\omega|_{M_\lambda}) \in \mathbb{C}[\Lambda^\omega].$$

**Remark 1.4.1.** It easily follows that for each $t \in T$,

$$\text{Tr}((t, \omega) ; M) = \sum_{\lambda \in \Lambda^\omega} \text{Tr}(\omega|_{M_\lambda}) \lambda(t) \in \mathbb{C}$$

since $\omega \cdot M_\lambda = M_{\omega(\lambda)}$ for $\lambda \in \Lambda$.

1.5. **An important example.** Let $\lambda \in \Lambda^\omega_+$ and $L(\lambda)$ the simple rational $G$-module of highest weight $\lambda$. We can make $L(\lambda)$ into a $G \rtimes \langle \omega \rangle$-module as follows. Let $v_\lambda$ be a (nonzero) highest weight vector of $L(\lambda)$. If $\mathfrak{U}(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$, there is an isomorphism of $\mathfrak{U}(\mathfrak{g})$-modules

$$\mathfrak{U}(\mathfrak{g})/\mathfrak{J}(\lambda) \simeq L(\lambda) \quad \text{via} \ x \mapsto xv_\lambda,$$

where $\mathfrak{J}(\lambda)$ is the left ideal of $\mathfrak{U}(\mathfrak{g})$ given by

$$\mathfrak{J}(\lambda) = \sum_{i \in I} \left( \mathfrak{U}(\mathfrak{g})e_i + \mathfrak{U}(\mathfrak{g})(h_i - \langle \lambda, \alpha_i^\vee \rangle) + \mathfrak{U}(\mathfrak{g})f_i^{(\lambda, \alpha_i^\vee) + 1} \right).$$

Since $\lambda \in \Lambda^\omega$, the left ideal $\mathfrak{J}(\lambda)$ of $\mathfrak{U}(\mathfrak{g})$ is $\omega$-invariant, i.e., $d\omega$-invariant, and hence $L(\lambda)$ admits a structure of $\langle \omega \rangle$-module such that

$$\omega \cdot (x v_\lambda) = \left( (d\omega)(x) \right) v_\lambda \quad \text{for all } x \in \mathfrak{U}(\mathfrak{g}).$$
Therefore, the $L(\lambda)$ admits a structure of $G \rtimes \langle \omega \rangle$-module such that $\omega \cdot v_\lambda = v_\lambda$. Note that a $G \rtimes \langle \omega \rangle$-module structure on $L(\lambda)$ such that $\omega \cdot v_\lambda = v_\lambda$ is unique since $L(\lambda)$ is a cyclic $G$-module generated by $v_\lambda$.

On the other hand, for each $i \in \hat{I}$, we have $(P_\omega^*)^{-1}(\lambda)({\hat{h}_i}) = \lambda(h_i)$. Hence $\hat{\lambda} = (P_\omega^*)^{-1}(\lambda) \in \widehat{\mathfrak{h}}^*$ is dominant integral. If $\hat{L}(\lambda)$ is the simple $\widehat{\mathfrak{g}}$-module of highest weight $\hat{\lambda}$, we know that

$$
(1.5.1) \quad \text{ch}^\omega(L(\lambda)) = P_\omega^*(\text{ch} \hat{L}(\hat{\lambda})),
$$

where $P_\omega^*$ on the right-hand side is a $\mathbb{C}$-algebra isomorphism $\mathbb{C}[\mathfrak{h}_Z^*] \rightarrow \mathbb{C}[[\mathfrak{h}_Z]]$ defined by

$$
P_\omega^*(e(\bar{\mu})) = e(P_{\omega}^*(\bar{\mu})) \quad \text{for} \quad \bar{\mu} \in \mathfrak{h}_Z^*.
$$

Assume now that $J = I_i = \{\omega^k(i) \mid 0 \leq k \leq N_i - 1\} \subset I_i \in \hat{I}$, and let $P_J$ be the standard parabolic subgroup of $G$ associated to $J$. Let $\nu \in \Lambda^\omega$ with $\langle \nu, \alpha_i^\vee \rangle \geq 0$ (hence $\langle \nu, \alpha_j^\vee \rangle \geq 0$ for all $j \in J$). If $L_J(\nu)$ is the simple rational $P_j$-module of highest weight $\nu$, then it remains simple as a rational module over the Levi factor $L_J$ of $P_J$ with the unipotent radical $U_J$ of $P_J$ acting trivially. We can make $L_J(\nu)$ into a $P_J \rtimes \langle \omega \rangle$-module in the same way as $L(\lambda)$ above.

The following lemma is a first (but important) step towards our main result (Theorem 0.1).

**Lemma 1.5.1.** With the notation and assumption as above, we have in $\mathbb{C}[\Lambda^\omega]$,

$$
\text{ch}^\omega(L_J(\nu)) = \hat{D}_{i}(e(\nu)).
$$

2. **Proof of the main result.**

Since the proof of our main result is so simple and clear modulo some algebro-geometric arguments, we give a "detailed outline" of it in this section. Fix $w \in W^\omega$ and let $X(w)$ be the associated Schubert variety over $\mathbb{C}$, i.e., the Zariski closure of the Bruhat cell $BwB/B$ in the flag variety $G/B$. For a $B \rtimes \langle \omega \rangle$-module $M$, the $\omega$-Euler characteristic $\chi_w^\omega(M)$ is defined to be

$$
\chi_w^\omega(M) = \sum_{j \geq 0} (-1)^j \text{ch}^\omega(H^j(X(w), \mathcal{L}_{X(w)}(M))) \in \mathbb{C}[\Lambda^\omega].
$$

Here recall that, since $M$ is a $B \rtimes \langle \omega \rangle$-module, the $\mathcal{O}_{X(w)}$-module $\mathcal{L}_{X(w)}(M)$ associated to $M$ is a $(B, \langle \omega \rangle)$-equivariant sheaf, and hence the cohomology groups $H^j(X(w), \mathcal{L}_{X(w)}(M))$, $j \geq 0$, are $B \rtimes \langle \omega \rangle$-modules.
Let $w = w_{i_1} \cdots w_{i_n}$ be a reduced expression of $w \in W^\omega$ in the Coxeter system $(W^\omega, \{w_i \mid i \in I\})$, i.e., $\ell(w) = n$. Note that we have $\ell(w) = \ell(w_{i_1}) + \cdots + \ell(w_{i_n})$. We want to show that

$$\chi^\omega_w(M) = \hat{D}_{i_1} \cdots \hat{D}_{i_n}(\text{ch}^\omega(M)),$$

where $\hat{D}_j$ for $j = i_1, \ldots, i_n$ is the $\omega$-Demazure operator defined in §1.3. In particular, we will obtain a twining character formula of the Demazure module $H^0(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_\lambda))$ for $\lambda \in \Lambda^\omega$, where $\mathbb{C}_\lambda$ is the one-dimensional $B \rtimes \langle \omega \rangle$-module on which $B$ acts by the weight $\lambda$ through the quotient $B \to T$ and $\langle \omega \rangle$ trivially.

2.1. Formula for the $\omega$-Euler characteristics. Set $\hat{D}_w = \hat{D}_{i_1} \cdots \hat{D}_{i_n}$. Then we are to show

$$(2.1.1) \quad \chi^\omega_w(M) = \hat{D}_w(\text{ch}^\omega(M)).$$

Let us first make some reductions. Since both sides of (2.1.1) are additive in $M$, we may assume that $M$ is one-dimensional of weight $\mu \in \Lambda^\omega$ on which $\omega$ is acting by a scalar $\zeta^k$ for a primitive $N$-th root of unity $\zeta$ in $\mathbb{C}$ and $k \in \mathbb{Z}$. We will denote such $M$ by $\mathbb{C}_{\mu,k}$. Thus we are reduced to showing that

$$\chi^\omega_w(\mathbb{C}_{\mu,k}) = \hat{D}_w(\text{ch}^\omega(\mathbb{C}_{\mu,k})), $$

dewhere $\text{ch}^\omega(\mathbb{C}_{\mu,k}) = \zeta^k e(\mu)$.

Put for simplicity $z_j = w_{ij}$, $1 \leq j \leq n$. We have an isomorphism of $B \rtimes \langle \omega \rangle$-modules

$$(2.1.2) \quad H^s(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_{\mu,k})) \cong H^s(X(z_1, \ldots, z_n), \mathcal{L}_{X(z_1, \ldots, z_n)}(\mathbb{C}_{\mu,k})), $$

and for each $s$ with $1 \leq s \leq n - 1$, a $B \rtimes \langle \omega \rangle$-equivariant spectral sequence

$$(2.1.3) \quad H^s(X(z_s), \mathcal{L}(H^1(X(z_{s+1}, \ldots, z_n), \mathcal{L}(\mathbb{C}_{\mu,k}))))) \Rightarrow H^{s+j}(X(z_s, \ldots, z_n), \mathcal{L}(\mathbb{C}_{n,k})).$$

Here $X(z_s, \ldots, z_t)$ for $1 \leq s \leq t \leq n$ is the so-called Bott-Samelson variety, and $\mathcal{L}_{X(z_s, \ldots, z_t)}(\mathbb{C}_{\mu,k})$ is the sheaf of $\mathcal{O}_{X(z_s, \ldots, z_t)}$-modules associated to the $B \rtimes \langle \omega \rangle$-module $\mathbb{C}_{\mu,k}$. Note that, since $z_s, \ldots, z_t \in W^\omega$ and their right coset representatives $\hat{z}_s, \ldots, \hat{z}_t \in N_G(T)$ are fixed by $\omega \in \text{Aut}(G)$, the Bott-Samelson variety $X(z_s, \ldots, z_t)$ is an $\langle \omega \rangle$-invariant subvariety of $(G/B)^t \times (G/B)^{n-t+1}$. (The proofs of (2.1.2) and (2.1.3) are not so difficult, but rather long. For details, see our preprint [KN].)

Remark 2.1.1. There are several equivalent (or inequivalent) definitions of a Bott-Samelson variety, but in the talk, we stick to that of [Ja]:

$$X(y_1, \ldots, y_n) = \{(g_1 B, \ldots, g_n B) \in (G/B)^n \mid g_{i-1}^{-1} g_i \in \overline{B y_i B} \quad \text{for all } i\}$$

for $y_1, \ldots, y_n \in W$. If $J_i$ for $1 \leq i \leq n$ is a subset of $I$ and $z_{J_i}$ is the longest element of the subgroup $W_{J_i} = \langle r_k \mid k \in J_i \rangle$ of the Weyl group $W$, then the Bott-Samelson variety
$X(z_{J_{1}}, \ldots , z_{J_{n}})$ is smooth. Moreover, if we assume that $\ell(z_{J_{1}} \cdots z_{J_{n}}) = \ell(z_{J_{1}}) + \cdots + \ell(z_{J_{n}})$, then the restriction $\phi$ of the $n$-th projection $\pi_{n} : (G/B)^{n} \to G/B$ to the Bott-Samelson variety $X(z_{J_{1}}, \ldots , z_{J_{n}}) \subset (G/B)^{n}$ gives a Demazure-Hansen desingularization

$$
\phi: X(z_{J_{1}}, \ldots , z_{J_{n}}) \to X(z_{J_{1}} \cdots z_{J_{n}})
$$

of the Schubert variety $X(z_{J_{1}} \cdots z_{J_{n}})$. Note that the $\phi$ induces an isomorphism of suitable open and dense subvarieties.

Now it follows that

$$
\chi_{\omega}^\omega(C_{\mu, k}) = \sum_{j \geq 0} (-1)^{j} \mathrm{ch}^{\omega}(H^{j}(X(z_{1}, \ldots , z_{n}), \mathcal{L}(z_{1}, \ldots , z_{n}))(C_{\mu, k})) \quad \text{by (2.1.2)}
$$

$$
= \sum_{j \geq 0} (-1)^{j} \left( \sum_{i \geq 0} (-1)^{i} \mathrm{ch}^{\omega}(H^{j}(X(z_{1}), \mathcal{L}(H^{j}(X(z_{2}, \ldots , z_{n}), \mathcal{L}(C_{\mu, k})))) \right) \quad \text{by (2.1.3)}
$$

$$
= \sum_{j \geq 0} (-1)^{j} \chi_{\omega}^\omega(H^{j}(X(z_{2}, \ldots , z_{n}), \mathcal{L}_{X(z_{1}, \ldots , z_{n})(C_{\mu, k}))}).
$$

By induction on $n$, we may assume that $w = w_i$ for some $i \in \tilde{I}$ in proving (2.1.1). So put $J = I_i$ and let $P = P_J$ be the standard parabolic subgroup of $G$ associated to $J$. We are to show

$$(2.1.4) \quad \chi_{\omega}^\omega(C_{\mu, k}) = \hat{D}_{i}(\zeta^{k}e(\mu)).$$

Assume first that $\langle \mu, \alpha_{i}^{\vee} \rangle \geq 0$ (and hence that $\langle \mu, \alpha_{k}^{\vee} \rangle \geq 0$ for all $k \in J$). Let $L_{J}(\mu)$ be the simple rational $P_{J}$-module of highest weight $\mu$ admitting an $\langle \omega \rangle$-action as in §1.5, and let $\zeta^{k}$ be the one-dimensional trivial $P_{J}$-module with $\omega$ acting by the scalar $\zeta^{k}$.

**Lemma 2.1.2.** Let the notation and assumption be as above. Then we have the following isomorphism of $P_{J} \rtimes \langle \omega \rangle$-modules.

$$
H^{0}(P_{J}/B, \mathcal{L}_{P_{J}/B}(C_{\mu, k})) \simeq L_{J}(\mu) \otimes C \zeta^{k}.
$$

(This lemma is, in a sense, crucial to the proof of our main result. Although no one doubts the truth of this lemma, its complete proof would be rather long.)
Now we deduce that
\[
\chi_{w_{i}}^\omega(C_{\mu,k}) = \text{ch}^\omega(H^0(P/B, \mathcal{L}_{P/B}(C_{\mu,k}))) \quad \text{by Kempf’s vanishing theorem}
\]
\[
= \text{ch}^\omega(L_J(\mu) \otimes \zeta^k) \quad \text{by Lemma 2.1.2}
\]
\[
= \zeta^k \text{ch}^\omega(L_J(\mu))
\]
\[
= \hat{D}_i(e(\mu)) \quad \text{by Lemma 1.5.1}
\]
\[
= \hat{D}_i(\zeta^k e(\mu)).
\]

If \(\langle \mu, \alpha_i \rangle = -1\) (and hence \(\langle \mu, \alpha_k \rangle = -1\) for all \(k \in J\)), then both sides of (2.1.4) vanish.

Assume finally that \(\langle \mu, \alpha_i \rangle \leq -2\) (and hence \(\langle \mu, \alpha_k \rangle \leq -2\) for all \(k \in J\)). Set \(\rho_J = \frac{1}{2} \sum_{\alpha \in \Delta^+_J} \alpha\) with \(\Delta^+_J = \Delta_+ \cap \sum_{k \in J} \mathbb{Z} \alpha_k\) the positive root system of \(P_J\). By direct checking, using the \(T \times \langle \omega \rangle\)-module isomorphism \((\text{Lie}(P)/\text{Lie}(B))^* \simeq \bigoplus_{\alpha \in \Delta^+_J} \mathbb{C} f_{\alpha}\), we see that as \(B \times \langle \omega \rangle\)-modules,

\[
\bigwedge_{\mathbb{C}}^{\ell(w_i)}(\text{Lie}(P)/\text{Lie}(B))^* \simeq \mathbb{C}_{-20} \rho_J \otimes \mathbb{C}(-1)^{\ell(w_i)-1},
\]

where \(\ell(w_i) = \dim_{\mathbb{C}}(P/B)\) and \((-1)^{\ell(w_i)-1}\) is the one-dimensional \(B \times \langle \omega \rangle\)-module with \(B\) acting trivially and \(\omega\) by the scalar \((-1)^{\ell(w_i)-1}\). Then the \(B \times \langle \omega \rangle\)-equivariant Serre duality reads

(2.1.5)

\[
H^j(P/B, \mathcal{L}_{P/B}(C_{\mu,k}))^* \simeq H^{\ell(w_i)-j}(P/B, \mathcal{L}_{P/B}(C_{-2\rho_J,k}) \otimes \mathbb{C}(-1)^{\ell(w_i)-1})
\]

\[
\simeq \begin{cases} 
H^0(P/B, \mathcal{L}_{P/B}(C_{-2\rho_J,k}) \otimes \mathbb{C}(-1)^{\ell(w_i)-1} & \text{if } j = \ell(w_i), \\
0 & \text{otherwise (by Kempf).}
\end{cases}
\]

(The use above of the \(B \times \langle \omega \rangle\)-equivariant Serre duality is the most essential part of the proof of our main result.)

**Remark 2.1.3.** Put \(X = P/B\) and \(m = \dim_{\mathbb{C}}X\). Let \(\mathcal{M}\) be a \((B, \langle \omega \rangle)\)-equivariant \(\mathcal{O}_X\)-module that is locally free of finite rank over \(\mathcal{O}_X\). The \(B \times \langle \omega \rangle\)-equivariant Serre duality (see our preprint on Naito’s home page) asserts that, as \(B \times \langle \omega \rangle\)-modules,

\[
H^i(X, \mathcal{M}^\vee \otimes_X \Omega_X^m) \simeq H^{m-i}(X, \mathcal{M})^* \quad \text{for all } 0 \leq i \leq m,
\]

where \(\mathcal{M}^\vee = \mathcal{H}om_X(\mathcal{M}, \mathcal{O}_X)\) is the dual sheaf of \(\mathcal{M}\), \(\Omega_X^m = \bigwedge_X^m \Omega_X^1\) is the canonical sheaf on \(X\), and \(H^{m-i}(X, \mathcal{M})^*\) is the dual \(B \times \langle \omega \rangle\)-module of \(H^{m-i}(X, \mathcal{M})\). This Serre duality will be a consequence of the triviality of the \(B \times \langle \omega \rangle\)-action on the one-dimensional vector space \(H^m(X, \Omega_X^m)\). Since the triviality of the \(B\)-action on it is known, it remains to show the triviality of the \(\langle \omega \rangle\)-action. There are many ways to show it, but the way
we take here is (I think) purely algebro-geometric and elementary: first take a $P \ltimes \langle \omega \rangle$-equivariant closed immersion $\iota: X \to P = P(L(\lambda))$ for sufficiently dominant $\lambda \in \Lambda^\omega_+$; then use the fact that the full automorphism group $PGL(L(\lambda))$ of $P(L(\lambda))$ acts trivially on the one-dimensional vector space $H^l(P, \Omega^1_P)$ with $l = \dim_{\mathbb{C}} P$, where (though not so trivial)

$$H^l(P, \Omega^1_P) \simeq H^m(P, \iota_*\Omega^m_P) \simeq H^m(X, \Omega^m_X)$$

as $P \ltimes \langle \omega \rangle$-modules.

The proof of the following lemma is easy.

**Lemma 2.1.4.** Let $J$ be an $\omega$-invariant subset of $I$, $w_J$ the longest element of the Weyl group $W_J$ of $P_J$, and let $\nu \in \Lambda^\omega$ be such that $\langle \nu, \alpha_i^\vee \rangle \geq 0$ for all $k \in J$. Then we have the following isomorphism of $P_J \ltimes \langle \omega \rangle$-modules.

$$L_J(\nu)^* \simeq L_J(-w_J(\nu)).$$

The isomorphism (2.1.5) together with Lemmas 2.1.2 and 2.1.4 implies that, as $B \ltimes \langle \omega \rangle$-modules,

$$H^{\ell(w_J)}(P/B, \mathcal{L}_P/B(\mathbb{C}_{\mu,k})) \simeq (L_J(-\mu - 2\rho_J)^* \otimes_{\mathbb{C}} \zeta^k \otimes (1)^{\ell(w_J)} - 1) \simeq L_J(w_J(\mu + 2\rho_J)) \otimes_{\mathbb{C}} \zeta^k \otimes (1)^{\ell(w_J)} - 1.$$

Then, setting $\tilde{\mu} = (P_{\omega}^-)^{-1}(\mu)$,

$$\chi^\omega_{w_J}(C_{\mu,k}) = (-1)^{\ell(w_J)} \text{ch}^\omega(L_J(w_J(\mu + 2\rho_J)) \otimes_{\mathbb{C}} \zeta^k \otimes (1)^{\ell(w_J)} - 1) \text{ by (2.1.6)}$$

$$= -\zeta^k \text{ch}^\omega(L_J(w_J(\mu + 2\rho_J)))$$

$$= -\zeta^k \tilde{D}_i(e(\mu(\mu + 2\rho_J))) \text{ by Lemma 1.5.1}$$

$$= -\zeta^k \left( P_{\omega} \circ D_{\tilde{r}_i} \circ (P_{\omega}^*)^{-1} \right)(e(\mu(\mu + 2\rho_J)))$$

$$= -\zeta^k \left( P_{\omega} \circ D_{\tilde{r}_i} \right)(e(\tilde{\mu} + \tilde{\alpha}_i)) \text{ since } (P_{\omega}^*)^{-1}(2\rho_J) = \tilde{\alpha}_i$$

$$= -\zeta^k P_{\omega}(-D_{\tilde{r}_i}e(\tilde{\mu}))$$

$$= \zeta^k \left( \tilde{D}_i \circ P_{\omega} \right)(e(\tilde{\mu}))$$

$$= \zeta^k \tilde{D}_i(e(\mu))$$

$$= \tilde{D}_i(\zeta^k e(\mu)).$$

Thus in all cases (2.1.4) holds, and we are done.

If $\lambda \in \Lambda^\omega_+$, then for any Schubert variety $X(w)$,

$$H^j(X(w), \mathcal{L}_{X(w)}(\lambda)) = 0 \text{ for all } j \geq 1.$$
by the Demazure vanishing theorem of Andersen et al. Hence we have proved

**Theorem 2.1.5.** Let $M$ be a finite-dimensional rational $B \times \langle \omega \rangle$-module and $w \in W^\omega$. Then we have in $\mathbb{C}[\Lambda_\omega]$,

$$
\chi^\omega_w(M) = \sum_{j \geq 0} (-1)^j \text{ch}^\omega(H^j(X(w), \mathcal{L}_{X(w)}(M))) = \hat{D}_w(\text{ch}^\omega(M)),
$$

where $\hat{D}_w = \hat{D}_{i_1} \hat{D}_{i_2} \cdots \hat{D}_{i_n}$ for any reduced expression $w = w_{i_1} w_{i_2} \cdots w_{i_n}$ of $w \in W^\omega$ in the Coxeter system $(W^\omega, \{ w_i \mid i \in \tilde{I} \})$. In particular, for $\lambda \in \Lambda_\omega^+$, we have

$$
\text{ch}^\omega(H^0(X(w), \mathcal{L}_{X(w)}(C_\lambda))) = \hat{D}_w(e(\lambda)),
$$

where $\mathcal{C}_\lambda$ is the one-dimensional $B \times \langle \omega \rangle$-module on which $B$ acts by the weight $\lambda$ through the quotient $B \to T$ and $\omega$ trivially.

Theorem 2.1.5 above reveals that there exists a striking relation between the $\omega$-Euler characteristic $\chi^\omega_w(\mathcal{C}_\lambda) \in \mathbb{C}[(\mathfrak{h}_Z^*)^0]$ for $\mathfrak{g}$ and the ordinary Euler characteristic for the orbit Lie algebra $\hat{\mathfrak{g}}$. To state the relation, we need some notation. Recall that the orbit Lie algebra $\hat{\mathfrak{g}}$ is the dual complex semi-simple Lie algebra of the fixed point subalgebra $\mathfrak{g}^0 = \{ x \in \mathfrak{g} \mid \omega(x) = x \}$ of $\mathfrak{g}$. Let $\hat{G}$ be a connected, simply connected semi-simple linear algebraic group over $\mathbb{C}$ with maximal torus $\hat{T}$ and Borel subgroup $\hat{B} \supset \hat{T}$ such that $\text{Lie}(\hat{G}) = \hat{\mathfrak{g}}$, $\text{Lie}(\hat{T}) = \hat{\mathfrak{h}}$, and $\text{Lie}(\hat{B}) = \hat{\mathfrak{b}}$. For $\hat{w} \in \hat{W} \simeq N_{\hat{G}}(\hat{T})/\hat{W}$, we take a right coset representative $\hat{w} \in N_{\hat{G}}(\hat{T})$ of $\hat{w}$, and define the Schubert variety $\hat{X}(\hat{w})$ over $\mathbb{C}$ by

$$
\hat{X}(\hat{w}) = \overline{\hat{B}\hat{w}\hat{B}/\hat{B}} = \overline{\hat{B}\hat{w}\hat{B}}/\hat{B} \subset \hat{G}/\hat{B}.
$$

For each $\hat{\lambda} \in \hat{\mathfrak{h}}_Z^*$, we denote by $\mathcal{L}_{\hat{X}(\hat{w})}(\mathcal{C}_{\hat{\lambda}})$ the (locally free) $\hat{B}$-equivariant sheaf of $\mathcal{O}_{\hat{X}(\hat{w})}$-modules associated to the one-dimensional $\hat{B}$-module $\mathcal{C}_{\hat{\lambda}}$ on which $\hat{B}$ acts by the weight $\hat{\lambda}$ through the quotient $\hat{B} \to \hat{T}$.

Now we are ready to state the following

**Corollary 2.1.6.** Let $\lambda \in (\mathfrak{h}_Z^*)^0$ and $w \in W^\omega$. We set $\hat{w} = \Theta^{-1}(w) \in \hat{W}$ and $\hat{\lambda} = (P_\omega^*)^{-1}(\lambda) \in \hat{\mathfrak{h}}_Z^*$. Then we have in the algebra $\mathbb{C}[(\mathfrak{h}_Z^*)^0]$,

$$
\chi^\omega_w(\mathcal{C}_\lambda) = \sum_{j \geq 0} (-1)^j \text{ch}^\omega(H^j(X(w), \mathcal{L}_{X(w)}(\mathcal{C}_\lambda)))
$$

$$
= P_\omega^* \left( \sum_{j \geq 0} (-1)^j \text{ch} H^j(\hat{X}(\hat{w}), \mathcal{L}_{\hat{X}(\hat{w})}(\mathcal{C}_{\hat{\lambda}})) \right),
$$

where $\text{ch} H^j(\hat{X}(\hat{w}), \mathcal{L}_{\hat{X}(\hat{w})}(\mathcal{C}_{\hat{\lambda}})) \in \mathbb{C}[\hat{\mathfrak{h}}_Z^*]$ for $j \in \mathbb{Z}_{\geq 0}$ is the ordinary character of the $j$-th cohomology group $H^j(\hat{X}(\hat{w}), \mathcal{L}_{\hat{X}(\hat{w})}(\mathcal{C}_{\hat{\lambda}}))$ of $\hat{X}(\hat{w})$. 

This immediately follows from Theorem 2.1.5 and the ordinary Demazure character formula for the orbit Lie algebra \( \mathfrak{g} \).)

2.2. Joseph’s modules. Let us finally return to Joseph’s module \( J_w(\lambda) \), with \( w \in W^\omega \) and \( \lambda \in \Lambda^\omega_+ \). Thus let \( v^*_\lambda \) be a (nonzero) lowest weight vector of the dual module \( L(\lambda)^* \) (which is the dual element of a (nonzero) highest weight vector \( v_\lambda \) of \( L(\lambda) \)), and let \( \dot{w} \in N_G(T)^\omega \) representing \( w \in W^\omega \). Since \( v^*_\lambda \) is fixed by \( \omega \), so is \( \dot{w} v^*_\lambda \). Joseph’s module \( J_w(\lambda) \) of highest weight \(-w(\lambda)\) in \( L(\lambda)^* \) is defined to be

\[
J_w(\lambda) = \mathfrak{U}(b)(\dot{w}v^*_\lambda) \subset L(\lambda)^*,
\]

where \( \mathfrak{U}(b) \) is the universal enveloping algebra of \( b = \text{Lie}(B) \). Note that, since \( \omega \cdot (\dot{w}v^*_\lambda) = \dot{w}v^*_\lambda \), Joseph’s module \( J_w(\lambda) \) is a \( B \rtimes \langle \omega \rangle \)-submodule of \( L(\lambda)^* \). Moreover, since \( \dot{w}_0 v^*_\lambda \) is a (nonzero) highest weight vector of \( L(\lambda)^* \) fixed by \( \omega \), there is an isomorphism of \( G \rtimes \langle \omega \rangle \)-modules

\[
(2.2.1) \quad L(\lambda)^* \simeq L(-w_0(\lambda)),
\]

which enables us to regard \( J_w(\lambda) \) as a \( B \rtimes \langle \omega \rangle \)-submodule of \( L(-w_0(\lambda)) \). Then we obtain a short exact sequence of \( B \rtimes \langle \omega \rangle \)-modules

\[
0 \leftarrow J_w(\lambda)^* \leftarrow L(-w_0(\lambda))^* \leftarrow J_w(\lambda)^\perp \leftarrow 0,
\]

with \( J_w(\lambda)^\perp = \{ \phi \in L(-w_0(\lambda))^* \mid \phi(J_w(\lambda)) = 0 \} \). On the other hand, Lemma 2.1.2 for the case \( J = I \) combined with (2.2.1) yields an isomorphism of \( G \rtimes \langle \omega \rangle \)-modules

\[
H^0(G/B, \mathcal{L}_{G/B}(C_\lambda)) \simeq L(-w_0(\lambda))^*.
\]

Since the restriction map

\[
H^0(G/B, \mathcal{L}_{G/B}(C_\lambda)) \rightarrow H^0(X(w), \mathcal{L}_{X(w)}(C_\lambda))
\]

is known to be a \( (B \rtimes \langle \omega \rangle \)-equivariant) surjection, we obtain an isomorphism of \( B \rtimes \langle \omega \rangle \)-modules

\[
J_w(\lambda)^* \simeq H^0(X(w), \mathcal{L}_{X(w)}(C_\lambda)),
\]

or equivalently

\[
(2.2.2) \quad J_w(\lambda) \simeq H^0(X(w), \mathcal{L}_{X(w)}(C_\lambda))^*.
\]

We now define a \( \mathbb{C} \)-linear conjugation \( -: \mathbb{C}[\Lambda^\omega] \rightarrow \mathbb{C}[\Lambda^\omega] \) by

\[
\sum_{\mu \in \Lambda^\omega} a_\mu e(\mu) = \sum_{\mu \in \Lambda^\omega} a_\mu e(-\mu) \quad \text{with} \quad a_\mu \in \mathbb{C} \quad \text{for} \quad \mu \in \Lambda^\omega.
\]

Then we obtain the following theorem from the \( B \rtimes \langle \omega \rangle \)-module isomorphism (2.2.2).
Theorem 2.2.1. Let \( \lambda \in \Lambda_+^{\omega} \) and \( w \in W^{\omega} \). Then we have in \( \mathbb{C}[\Lambda^{\omega}] \),
\[
\text{ch}^{\omega}(J_w(\lambda)) = \overline{\text{ch}^{\omega}(H^0(X(w), \mathcal{L}x(w)(\mathbb{C}\lambda)))}.
\]

By combining Theorems 2.1.5 and 2.2.1, we obtain the following

Corollary 2.2.2. Let \( \lambda \in \Lambda_+^{\omega} \) and \( w \in W^{\omega} \). Then we have in \( \mathbb{C}[\Lambda^{\omega}] \),
\[
\text{ch}^{\omega}(J_w(\lambda)) = \hat{D}_w(e(\lambda)).
\]

Finally, by combining Corollary 2.1.6 and Theorem 2.2.1, we obtain a remarkable relation between the twining character \( \text{ch}^{\omega}(J_w(\lambda)) \) of Joseph’s module \( J_w(\lambda) \) for \( \mathfrak{g} \) and the ordinary character of Joseph’s module for the orbit Lie algebra \( \hat{\mathfrak{g}} \), which is the dual complex semi-simple Lie algebra of \( \mathfrak{g}^0 \). For each \( \hat{w} \in \overline{W} \), let
\[
\hat{J}_{\hat{w}}(\hat{\lambda}) = \mathfrak{U}(\hat{h})(\hat{w} \hat{\gamma}_{\hat{\lambda}}) \subset \hat{L}(\lambda)^*.
\]

Corollary 2.2.3. Let \( \lambda \in (\mathfrak{h}_\mathbb{Z})^0 \) be dominant and \( w \in W^{\omega} \). We set \( \hat{w} = \Theta^{-1}(w) \in \hat{W} \) and \( \hat{\lambda} = (P_w^*)^{-1}(\lambda) \in \hat{\mathfrak{h}}_\mathbb{Z}^* \). Then we have in \( \mathbb{C}[(\mathfrak{h}_\mathbb{Z})^0] \),
\[
\text{ch}^{\omega}(J_w(\lambda)) = P_w^* \left( \text{ch} \hat{J}_{\hat{w}(\hat{\lambda})}(\hat{\lambda}) \right).
\]

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