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Kyoto University
A Quantization of Conjugacy Classes of Matrices

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1. Introduction

Let $A$ be an element of the space $M(n, \mathbb{C})$ of square matrices of size $n$ with components in $\mathbb{C}$. Then the conjugacy class containing $A$ is the algebraic variety $V_A = \bigcup_{g \in G} \text{Ad}(g)A$ by denoting $G = GL(n, \mathbb{C})$ and $\text{Ad}(g)A = gAq^{-1}$. Under the $G$-action on $M(n, \mathbb{C})$, we will study a quantization of $V_A$ interpreted as follows:

For the defining equations of $V_A$ or the $G$-invariant defining ideal of $V_A$ in the ring of polynomial functions of $M(n, \mathbb{C})$, we will associate left invariant differential operators on $G$ or an ideal $J_A$ of the ring of the left invariant differential operators on $G$. The Lie algebra $\mathfrak{g}$ of $GL(n, \mathbb{C})$ is identified with $M(n, \mathbb{C})$ and we identify the left invariant differential operators on $G$ with the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. Then our quantization of $V_A$ is a $U(\mathfrak{g})$-homomorphism of $U(\mathfrak{g})/J_A$ to a suitable $U(\mathfrak{g})$ module $M$. Note that the quantization of $V_A$ becomes a representation space of a real form $G_R$ of $G$ if $M$ is a function space on a homogeneous space of $G_R$ or a space of sections of a $G_R$-homogeneous vector bundle.

\[ V_A = \bigcup_{g \in G} \text{Ad}(g)A \quad \longrightarrow \quad \text{G-invariant defining ideal of } V_A \]

\[
\begin{array}{c}
\vdots \\
\text{Representations of } U(\mathfrak{g}) \text{ or } G_R \\
\rightarrow \text{Ideal of } U(\mathfrak{g})
\end{array}
\]

In §2 we introduce a homogenized universal enveloping algebra $U^\epsilon(\mathfrak{g})$ to study our quantization together with “the classical limit” ($\epsilon = 0$). We construct generators of $J_A$ from the generalized Capelli operators introduced by [O2] which can be considered as quantizations of minors and we show in Theorem 2.8 that they generate the annihilator of a generalized Verma module induced from a character of a parabolic subalgebra of $\mathfrak{g}$. When $\epsilon = 0$ and $A$ is a nilpotent matrix, the corresponding result is Tanisaki’s conjecture [Ta], which is solved by Weyman [We]. In particular, if $A$ is a regular nilpotent matrix, the result is due to Kostant [Ko].
In §3 we examine how the annihilator determines the difference between the
generalized Verma module and the Verma module, which is important for applica-
tions. For example, the theorem on boundary value problems for symmetric spaces
studied in [O2, Theorem 5.1] is improved by the generator system defined in this
note.

We can also quantize the minimal polynomial of $V_A$ from which we can construct
another generator system of the annihilator. This is valid for other general reductive
Lie algebras and is studied in another paper [O3].

2. Elementary divisors

The Lie algebra $\mathfrak{g}$ of $G = GL(n, \mathbb{C})$ is identified with $M(n, \mathbb{C})$ and also with
the space of left $G$-invariant holomorphic vector fields on $G$. Then $\mathfrak{g}$ is spanned by
$E_{ij}$ for $1 \leq i \leq n$ and $1 \leq j \leq n$ where $E_{ij}$ is the fundamental matrix unit whose
$(p, q)$-component equals $\delta_{ip}\delta_{jq}$ and

\[
E_{ij} = \sum_{\nu=1}^{n} x_{\nu i} \frac{\partial}{\partial x_{\nu j}}
\]

with the coordinate $(x_{ij}) \in G$. Then $\mathfrak{g}$ is naturally a $(\mathfrak{g}, G)$-module.

Using the non-degenerate symmetric bilinear form $\langle X, Y \rangle = \text{Trace}(XY)$ on
$M(n, \mathbb{C}) \times M(n, \mathbb{C})$ we identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$. The dual basis $\{E_{ij}^*\}$ of $\{E_{ij}\}$ is
given by $E_{ij}^* = E_{ji}$. For simplicity, we will denote $E_i = E_{ii}$ and $e_i = E_{i*}^i$.

**DEFINITION 2.1.** The *homogenized universal enveloping algebra* $U^\epsilon(\mathfrak{g})$ of $\mathfrak{g}$ is
defined by

\[
U^\epsilon(\mathfrak{g}) = \left( \sum_{k=0}^{\infty} \otimes^k \mathfrak{g} \right) / \langle X \otimes Y - Y \otimes X - \epsilon[X, Y]; X, Y \in \mathfrak{g} \rangle
\]

and the subalgebra of $G$-invariants in $U^\epsilon(\mathfrak{g})$ is denoted by $U^\epsilon(\mathfrak{g})^G$. Here $\epsilon$ is a
complex number (or an element commuting with $\mathfrak{g}$) and the denominator is the
span as a two-sided ideal of the numerator, the tensor algebra of $\mathfrak{g}$.

Note that $U^\epsilon(\mathfrak{g})$ is naturally a $(\mathfrak{g}, G)$-module induced from the tensor algebra.
$U^1(\mathfrak{g})$ and $U^0(\mathfrak{g})$ are the universal enveloping algebra $U(\mathfrak{g})$ and the symmetric
algebra $S(\mathfrak{g})$ of $\mathfrak{g}$, respectively. If $\epsilon \neq 0$, the map defined by $E_{ij} \mapsto \epsilon E_{ij}$ gives an
algebra isomorphism of $U^\epsilon(\mathfrak{g})$ onto $U(\mathfrak{g})$.

The residue class of the element $X_1 \otimes X_2 \otimes \cdots \otimes X_m$ ($X_j \in \mathfrak{g}$) in $U^\epsilon(\mathfrak{g})$ will
be denoted by $X_1X_2\cdots X_m$ and the image of $\sum_{k=0}^{m} \otimes^k \mathfrak{g}$ in $U^\epsilon(\mathfrak{g})$ is denoted by
$U^\epsilon(\mathfrak{g})^{(m)}$. 
For an ordered partition \( \{n_1', \ldots, n_L'\} \) of a positive integer \( n \) into \( L \) positive integers put

\[
\begin{align*}
\Theta &= \{n_1, n_2, \ldots, n_L\}, \\
\iota_\Theta(\nu) &= j \quad \text{if } n_{j-1} < \nu \leq n_j \quad (1 \leq \nu \leq n).
\end{align*}
\]

The ordered partition of \( n \) is expressed by the set \( \Theta \) of strictly increasing positive integers ending at \( n \). Define Lie subalgebras \( \mathfrak{n}_\Theta, \overline{\mathfrak{n}}_\Theta \) and \( \mathrm{m}_\Theta \) by the span of \( E_{ij} \) with \( \iota_\Theta(i) > \iota_\Theta(j), \iota_\Theta(i) < \iota_\Theta(j) \) and \( \iota_\Theta(i) = \iota_\Theta(j) \), respectively, and put \( \mathfrak{p}_\Theta = \mathfrak{m}_\Theta + \mathfrak{n}_\Theta \).

We denote \( \mathfrak{m}_\Theta^k = \sum_{1 \leq i < j \leq n} \mathbb{C}E_{ij} \), \( \mathfrak{n} = \sum_{1 \leq i < j \leq n} \mathbb{C}E_{ij}, \overline{\mathfrak{n}} = \sum_{1 \leq j < i \leq n} \mathbb{C}E_{ij} \), and \( \mathfrak{p}_\Theta = \mathfrak{m}_\Theta + \mathfrak{n} \). We remark that \( \mathfrak{p}_\Theta = \{X \in \mathfrak{g}; \langle X, \mathrm{Y} \rangle = 0 \ (\forall Y \in \mathfrak{n}_\Theta)\} \).

Fix \( \lambda = (\lambda_1, \ldots, \lambda_L) \in \mathbb{C} \) and define a closed subset of \( \mathfrak{p} \):

\[
A_{\Theta, \lambda} = \sum_{j=1}^{n} \lambda_{\iota_\Theta(j)} E_j + \mathfrak{n}_\Theta
\]

where

\[
A_{\Theta, \lambda} = \begin{pmatrix}
\lambda_1 I_{n_1} & 0 \\
A_{21} & \lambda_2 I_{n_2} \\
A_{31} & A_{32} & \lambda_3 I_{n_3} \\
& \vdots & \ddots & \ddots \\
A_{L1} & A_{L2} & A_{L3} & \cdots & \lambda_L I_{n_L}
\end{pmatrix} ; A_{ij} \in M(n_i', n_j'; \mathbb{C})
\]

Here \( I_m \) denotes the identity matrix of size \( m \) and \( M(k, \ell; \mathbb{C}) \) denotes the space of matrices of size \( k \times \ell \) with components in \( \mathbb{C} \). The generic element of \( A_{\Theta, \lambda} \) corresponds to a unique Jordan's canonical form and any Jordan's canonical form is obtained by this correspondence with a suitable choice of \( \Theta \) and \( \lambda \).

The set \( \bigcup_{g \in G} \text{Ad}(g)A_{\Theta, \lambda} \) is a closed algebraic variety of \( M(n, \mathbb{C}) \) because any element of \( M(n, \mathbb{C}) \) can be transformed into an element in \( \mathfrak{p} \) under the Ad-action of the unitary group \( U(n) \). Then if \( \epsilon = 0 \), for \( f \in U^0(\mathfrak{g}) = S(\mathfrak{g}) \) we have

\[
f(\bigcup_{g \in G} \text{Ad}(g)A_{\Theta, \lambda}) = 0 \iff (\text{Ad}(g)f)(A_{\Theta, \lambda}) = 0 \quad (\forall g \in G) \]

\[
\iff \text{Ad}(g)f \in J_{\Theta}^\epsilon(\lambda) \quad (\forall g \in G) \]

\[
\iff f \in \text{Ann}_G(M_{\Theta}^\epsilon(\lambda))
\]
where
\[ J^\ominus_\epsilon(\lambda) = \sum_{X \in \mathfrak{p}^\ominus} U^\epsilon(\mathfrak{g})(X - \lambda \ominus (X)), \]
\[ M^\ominus_\epsilon(\lambda) = U^\epsilon(\mathfrak{g})/J^\ominus_\epsilon(\lambda), \]
\[ \text{Ann} \left( M^\ominus_\epsilon(\lambda) \right) = \{ D \in U^\epsilon(\mathfrak{g}); DM^\epsilon_\ominus(\lambda) = 0 \}, \]
\[ \text{Ann}_G \left( M^\epsilon_\ominus(\lambda) \right) = \{ D \in U^\epsilon(\mathfrak{g}); \text{Ad}(g)D \in \text{Ann} \left( M^\epsilon_\ominus(\lambda) \right) (\forall g \in G) \} \]
and the character \( \lambda^\ominus \) of \( \mathfrak{p}^\ominus \) is defined by
\[ \lambda^\ominus(Y + \sum_{k=1}^{L} x_k) = \sum_{1k=}^{L} \lambda_k \text{Trace}(x_k) \]
for \( X_k \in \mathfrak{m}^\ominus_k \) and \( Y \in \mathfrak{n}_\ominus \).

When \( \epsilon = 1 \), \( M^\ominus(\lambda) = M^1_\ominus(\lambda) \) is a generalized Verma module induced from the character \( \lambda^\ominus \) of \( \mathfrak{m}^\ominus \), which is a quotient of the Verma module
\[ M(\lambda^\ominus) = U(\mathfrak{g})/J(\lambda^\ominus) \]
with
\[ J^\epsilon(\lambda^\ominus) = \sum_{X \in \mathfrak{p}} U^\epsilon(\mathfrak{g})(x - \lambda^\ominus (X)) \] and \( J(\lambda^\ominus) = J^1(\lambda^\ominus) \).

In general we will omit the superfix \( \epsilon \) if \( \epsilon = 1 \).

**Proposition 2.2.**
\[ \text{Ann}_G \left( M^\ominus_\epsilon(\lambda) \right) = \text{Ann} \left( M^\epsilon_\ominus(\lambda) \right) \text{ if } \epsilon \neq 0, \]
\[ \text{Ann}_G \left( M^\epsilon_\ominus(\lambda) \right) = \bigcap_{g \in G} \text{Ad}(g)J^\ominus_\epsilon(\lambda). \]

**Proof.** We may assume \( \epsilon \neq 0 \) to prove the proposition.

Let \( D \in \text{Ann} \left( M^\epsilon_\ominus(\lambda) \right) \). Then for \( X \in \mathfrak{g} \) and \( v \in M^\ominus_\epsilon(\lambda) \), \( XD - DX)v = X(Dv) - D(Xv) = 0 \) and therefore \( XD - DX \in \text{Ann} \left( M^\epsilon_\ominus(\lambda) \right) \). Since \( XD - DX = \epsilon \text{ad}(X)D \) in \( U^\epsilon(\mathfrak{g}) \), \( \text{ad}(X)D \in \text{Ann} \left( M^\epsilon_\ominus(\lambda) \right) \) and therefore \( \text{Ad}(g)D \in \text{Ann} \left( M^\epsilon_\ominus(\lambda) \right) \) for \( g \in G \).

Put \( I = \bigcap_{g \in G} \text{Ad}(g)J^\ominus_\epsilon(\lambda) \). Since \( \text{Ann}(M^\epsilon_\ominus(\lambda)) \subset J^\ominus_\epsilon(\lambda) \), \( \text{Ann}_G \left( M^\epsilon_\ominus(\lambda) \right) \subset I \).

For \( P \in U^\epsilon(\mathfrak{g}) \), \( IP = PI \equiv 0 \mod J^\epsilon_\ominus(\lambda) \) because \( I \) is a two-sided ideal of \( U^\epsilon(\mathfrak{g}) \), which means \( I \subset \text{Ann} \left( M^\epsilon_\ominus(\lambda) \right) \).

**Definition 2.3.** Define the polynomials and an integer
\[ d^\epsilon_m(x) = d^\epsilon_m(x; \Theta, \lambda) = \prod_{j=1}^{L} (x - \lambda_j - n_{j-1})^{(n_j+m-n)}, \]
\[ d_m = d_m(\Theta) = \deg_x d^\epsilon_m(x; \Theta, \lambda) = \sum_{j=1}^{L} \max\{n_j + m - n, 0\}, \]
\[ e^\epsilon_m(x) = e^\epsilon_m(x; \Theta, \lambda) = d^\epsilon_m(x)/d^\epsilon_{m-1}(x), \]
\[ q^\epsilon(x) = q^\epsilon(x; \Theta, \lambda) = \prod_{j=1}^{L} (x - \lambda_j - n_{j-1})^{(n_j+m-n)}. \]
by putting

$$z^{(l)} = \begin{cases} 
  z(z - \epsilon) \cdots (z - (\ell - 1)\epsilon) & \text{if } \ell > 0, \\
  1 & \text{if } \ell \leq 0 
\end{cases}$$

and call $d_{n}^{\epsilon}(x)$, $q_{n}^{\epsilon}(x)$ and $\{e_{m}^{\epsilon}(x); 1 \leq m \leq n\}$ the characteristic polynomial, the minimal polynomial and the elementary divisors of $M_{\ominus}^{\epsilon}(\lambda)$, respectively.

**Remark 2.4.** i) The set $\{e_{m}^{\epsilon}(x)\}$ recovers $\{d_{m}^{\epsilon}(x)\}$ because $e_{m}^{\epsilon}(x) \in \mathbb{C}[x]e_{\tau}^{\epsilon}(\mathrm{L}x^{-}\epsilon)$.

ii) For the generic element $A$ of $J_{\ominus}^{0}(\lambda)$, the greatest common divisor of $\epsilon$-minors of the matrix $xI_{n} - A$ equals $d_{m}^{0}(x)$ and therefore when $\epsilon = 0$, the above definition coincides with that in the linear algebra.

iii) The meaning of the minimal polynomial for $\epsilon \neq 0$ will be clear in [O3].

Now we introduce quantized minors.

**Definition 2.5.** For set of indices $I = \{i_{1}, \ldots, i_{m}\}$ and $J = \{j_{1}, \ldots, j_{?n}\}$ with $i_{\mu}, j_{\nu} \in \{1, \ldots, n\}$, define a generalized Capelli operator (cf. [O2])

$$\det^{\epsilon}(x; E_{IJ}) = \det((x + (\nu - m)\epsilon)\delta_{ij}\delta^{\mu}\nu + E_{ij}\delta^{\mu}\nu), 1 \leq \nu \leq \mu \leq ?n \leq n$$
in $U^{\epsilon}(\mathfrak{g})[x]$ by the column determinant:

$$\det \begin{pmatrix} 
  A_{\mu\nu} 
\end{pmatrix}_{1 \leq \mu \leq ?n} = \sum_{\sigma \in \mathfrak{S}_{m}} \mathrm{sgn}(\sigma) A_{\sigma(1)} A_{\sigma(2)} \cdots A_{\sigma(m)}.$$

**Proposition 2.6.** The Capelli operators satisfy

$$\det^{\epsilon}(x; E_{\sigma(I)\sigma'(J)}) = \det^{\epsilon}(x; E_{IJ}) \quad \text{for } \sigma, \sigma' \in \mathfrak{S}_{m},$$

$$\mathrm{ad}(E_{ij}) \det^{\epsilon}(x; E_{IJ}) = D_{1} - D_{2}$$

where

$$\sigma(I) = \{i_{\sigma(1)}, \ldots, i_{\sigma(m)}\}, \quad \sigma'(J) = \{j_{\sigma'(1)}', \ldots, j_{\sigma'(m)}'\},$$

$$D_{1} = \begin{cases} 
  \det^{\epsilon}(x; E_{i_{1}, \ldots, i_{\mu-1}, j, i_{\mu+1}, \ldots, i_{m}}) & \text{if there exists only one } i_{\mu} \text{ with } i_{\mu} = j, \\
  0 & \text{otherwise,} 
\end{cases}$$

$$D_{2} = \begin{cases} 
  \det^{\epsilon}(x; E_{I\{\nu-1}, i, j_{\nu}, \ldots, j_{1}}) & \text{if there exists only one } j_{\nu} \text{ with } j_{\nu} = i, \\
  0 & \text{otherwise.} 
\end{cases}$$

**Proof.** When $\epsilon = 1$, (2.15) and (2.16) are proved by [O2, Lemma 2.2 and Proposition 2.4]. Combining this with the definition of $U^{\epsilon}(\mathfrak{g})$, we have the proposition. \qed

**Definition 2.7.** Under Definition 2.3 and Definition 2.5, put

$$\det^{\epsilon}(x; E_{IJ}) = h_{IJ}(x)d_{m}^{\epsilon}(x) + r_{IJ}^{d_{m}-1}x^{d_{m}-1} + \cdots + r_{IJ}^{1}x + r_{IJ}^{0}$$
in $U^\epsilon(\mathfrak{g})[x]$ with $h_{IJ}[x] \in U^\epsilon(\mathfrak{g})[x]$ and $r_{IJ}^j \in U^\epsilon(\mathfrak{g})^{(m-j)}$ for $j = 0, \ldots, d_m - 1$ and define the two-sided ideal of $U^\epsilon(\mathfrak{g})$:

$$I_\ominus^\epsilon(\lambda) = \sum_{n=1}^n \sum_{\# J = m} \sum_{j=0}^{d-1} U^\epsilon(\mathfrak{g}) r_{IJ}^j$$

(2.18)

Note that if $m \leq n - \max\{n'_1, \ldots, n'_L\}$ the summand equals 0 because $d_m = 0$. Moreover note that $\{r_{IJ}^j\}$ with $\# I = n$ are in $U^\epsilon(\mathfrak{g})^G$. In particular, if $\Theta = \{1, 2, \ldots, n\}$, then $p_\Theta = p$ and $I_\ominus^\epsilon(\lambda)$ is generated by suitable $n$ elements in $U^\epsilon(\mathfrak{g})^G$.

Now we can state the main result in this section and we call $r_{IJ}^j$ quantized Tanisaki generators of $\text{Ann}_G(M_\ominus^\epsilon(\lambda))$. In the case when $\epsilon = \lambda = 0$, $d_m^0(x; \Theta, 0) = x^{d_m}$ and the generators are introduced by [Ta].

**Theorem 2.8.** Under the notation (2.5) and (2.18)

$$\text{Ann}_G(M_\ominus^\epsilon(\lambda)) = I_\ominus^\epsilon(\lambda).$$

If all the roots of $d_n^\epsilon(x) = 0$ are simple, which is equivalent to say that the infinitesimal character of $M_\ominus^\epsilon(\lambda)$ is regular (cf. Remark 2.14), then

$$\text{Ann}_G(M_\ominus^\epsilon(\lambda)) = \sum_{k=1}^L \sum_{\# I = \# J = n+1-n_k} U^\epsilon(\mathfrak{g}) D_{IJ}^\epsilon(\lambda_k + n_{k-1} \epsilon).$$

(2.19)

Here for $I = \{i_1, \ldots, i_m\}$ and $J = \{j_1, \ldots, j_m\}$ we put

$$D_{IJ}^\epsilon(x) = (-1)^m \det \epsilon(x; E_{IJ}) = \det \left( E_{i_\mu j_\nu} - (x + (\nu - m) \epsilon) \delta_{i_\mu j_\nu} \right)_{1 \leq \mu \leq m, 1 \leq \nu \leq m}.$$

(2.20)

If all the roots of $d_n^\epsilon(x) = 0$ are simple, (2.19) holds modulo the ideal generated by $\text{Ann}_G(M_\ominus^\epsilon(\lambda)) \cap U^\epsilon(\mathfrak{g})^G$.

When $\epsilon = 0$, (2.19) holds if $\lambda_i \neq \lambda_j$ for $1 \leq i < j \leq L$ and the last statement above holds if $\lambda_i \neq \lambda_j$ for $1 \leq i < j \leq L$ satisfying $n'_i > 1$ and $n'_j > 1$.

**Remark 2.9.** Let $\{\lambda'_1, \ldots, \lambda'_k\}$ be the set of the roots of $d_n^\epsilon(x) = 0$ and let $m_k$ be the multiplicity of the root $\lambda'_k$. Here $d_m = m_1 + \cdots + m_k$ and $\lambda'_\mu \neq \lambda'_\nu$ if $1 \leq \mu < \nu \leq k$. Then

$$\sum_{j=0}^{d_m-1} \mathbb{C} r_{IJ}^j = \sum_{i=1}^k \sum_{j=1}^{m_i} \mathbb{C} \left( \frac{d^{j-1}}{dx^{j-1}} D_{IJ}^\epsilon(x) \right)_{x=\lambda'_i}$$

(2.21)

for $\# I = \# J = m$.

The rest of this section will be devoted to the proof of this theorem. First we will examine the image of our minors under the Harish-Chandra homomorphism.

Define the map $\omega$ of $U^\epsilon(\mathfrak{g})$ to $S(\mathfrak{a}) = U^\epsilon(\mathfrak{a})$ by

$$D - \omega(D) \in U^\epsilon(\mathfrak{g})n + \mathfrak{n} U^\epsilon(\mathfrak{n} + \mathfrak{a}).$$

(2.22)
Fix $I = \{i_1, \ldots, i_m\}$ and $J = \{j_1, \ldots, j_m\}$ with $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ and $1 \leq j_1 < j_2 < \cdots < j_m \leq n$. Then \[O2, Corollary 2.11\] in the case $\varepsilon = 1$ shows

\[(2.23) \quad \omega(D^I_J(x)) = \begin{cases} 0 & \text{if } I \neq J, \\ \prod_{\nu=1}^m (E_{i_\nu} - x + (\nu - 1)\varepsilon) & \text{if } I = J \end{cases}
\]

under the notation in Theorem 2.8. Introducing the algebra isomorphism

\[(2.24) \quad \omega(P) = \overline{\omega(P)}.
\]

Then $\omega$ defines the Harish-Chandra isomorphism of $U^\varepsilon(\mathfrak{g})^G$ onto the algebra $\mathfrak{S}(\mathfrak{a})^W$ of $\mathfrak{S}_n$-invariants in $S(\mathfrak{a})$. Here we note that if $I = \{i_1 < i_2 < \cdots < i_m\}$,

\[(2.25) \quad \overline{\omega}(P) = \omega(P).
\]

Then $\omega$ defines the Harish-Chandra isomorphism of $U^\varepsilon(\mathfrak{g})^G$ onto the algebra $\mathfrak{S}(\mathfrak{a})^W$ of $\mathfrak{S}_n$-invariants in $S(\mathfrak{a})$. Here we note that if $I = \{i_1 < i_2 < \cdots < i_m\}$,

\[(2.26) \quad \overline{\omega}(D^I_J(x)) = \prod_{\nu=1}^m (E_{i_\nu} - x + (\nu - 1 + (\nu - 1)\varepsilon)\epsilon).
\]

Since $D^I_J(x) \in U^\varepsilon(\mathfrak{g})^G[x]$ (cf. Proposition 2.6), it is clear that the coefficients of $D^I_J(x)$ as a polynomial of $x$ generate the algebra $U^\varepsilon(\mathfrak{g})^G$.

**Lemma 2.10.** Let $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be a triangular decomposition of a reductive Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. Here $\mathfrak{n}$ and $\mathfrak{n}$ are nilpotent subalgebras of $\mathfrak{g}$ and $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{n}$ is a Borel subalgebra of $\mathfrak{g}$. For an element $D$ of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$, we define $\omega(D) \in S(\mathfrak{a})$ so that

\[(2.27) \quad D - \omega(D) \in U(\mathfrak{g})\mathfrak{n} + \mathfrak{n}U(\mathfrak{n} + \mathfrak{a}).
\]

For a subspace $V$ of $U(\mathfrak{g})$

\[(2.28) \quad \langle \omega(V) \rangle_{S(\mathfrak{a})} = \sum_{p \in \omega(V)} S(\mathfrak{a})p.
\]

Then if $\text{ad}(\mathfrak{g})V \subset V$, we have

\[(2.29) \quad \omega(PDQ) \in \langle \omega(V) \rangle_{S(\mathfrak{a})} \quad \text{for any } P, Q \in U(\mathfrak{g}), \text{ and any } D \in V.
\]

**Proof.** Let $\{X_1, \ldots, X_N\}$, $\{Y_1, \ldots, Y_N\}$ and $\{H_1, \ldots, H_M\}$ be the basis of $\mathfrak{n}$, $\mathfrak{n}$ and $\mathfrak{a}$, respectively. Then $\{Y^\alpha H^\beta X^\gamma = Y_1^\alpha \cdots Y_N^\alpha H_1^\beta \cdots H_M^\beta X_1^\gamma \cdots X_N^\gamma; \alpha \in \mathbb{N}^N, \beta \in \mathbb{N}^M, \gamma \in \mathbb{N}^N\}$ with $\mathbb{N} = \{0,1,2,\ldots\}$ is a Poincare-Birkhoff-Witt’s basis of $U(\mathfrak{g})$.

Let $D \in V$. The assumption implies $PDQ \in U(\mathfrak{g})V$ and therefore we may assume $Q = 1$ in (2.29). Since $XD = \text{ad}(X)D + DX \in V + U(\mathfrak{g})\mathfrak{n}$ for $X \in \mathfrak{n}$, we have $X^\gamma D \in V + U(\mathfrak{g})\mathfrak{n}$. On the other hand, $Y^\alpha H^\beta D - Y^\alpha H^\beta \omega(D) \in Y^\alpha H^\beta (\mathfrak{n}U(\mathfrak{n} + \mathfrak{a}) + U(\mathfrak{g})\mathfrak{n}) \subset \mathfrak{n}U(\mathfrak{n} + \mathfrak{a}) + U(\mathfrak{g})\mathfrak{n}$ and therefore $\omega(Y^\alpha H^\beta D) = H^\beta \omega(D)$ if $\alpha = 0$.
and 0 otherwise. Hence $\omega(Y^\alpha H^\beta X^\gamma D) \in \langle \omega(V) \rangle_{S(a)}$ and $\omega(PD) \in \langle \omega(V) \rangle_{S(a)}$ for $P \in U(\mathfrak{g})$.

**Lemma 2.11.** Under the notation in Lemma 2.10, fix $H_\Theta \in \mathfrak{a}$ so that the condition $\text{ad}(H_\Theta)Y = c_Y Y$ with $c_Y \in \mathbb{C}$ and $Y \in \mathfrak{n} \setminus \{0\}$ means that $c_Y \neq 0$. Suppose $\text{ad}(H_\Theta)n \neq \{0\}$. Let $m_\Theta$ be the centralizer of $H_\Theta$ in $\mathfrak{g}$ and let $n_\Theta$ and $\bar{n}_\Theta$ be subspaces spanned by the elements $Y$ in $\mathfrak{n}$ and $\bar{n}$, respectively, satisfying $\text{ad}(H_\Theta)Y = c_Y Y$ with $c_Y \neq 0$. Then $p_\Theta = m_\Theta \oplus n_\Theta$ be a Levi decomposition of a parabolic subalgebra $p_\Theta$ containing $\mathfrak{p}$. Let $a_\Theta$ denote the center of $m_\Theta$. For an element $\lambda$ of the dual $a_\Theta^*$ of $a_\Theta$ we define a character $\lambda_\Theta$ of $p_\Theta$ so that $\lambda_\Theta(n_\Theta + [m_\Theta,m_\Theta]) = 0$ and $\lambda_\Theta(H) = \lambda(H)$ for $H \in a_\Theta$. Suppose there exist $D_1(\lambda), \ldots, D_m(\lambda)$ in $U(\mathfrak{g})[\lambda]$ so that

\begin{align}
(2.30) & \quad \text{ad}(X)D_k(\lambda) \in \sum_{j=1}^m U(\mathfrak{g})[\lambda]D_j(\lambda) \quad \text{for } X \in \mathfrak{g} \text{ and } k = 1, \ldots, m, \\
(2.31) & \quad D_k(\lambda) \in \sum_{X \in \mathfrak{p}} U(\mathfrak{g})[\lambda](X - \lambda_\Theta(X)) + \bar{n}U(\mathfrak{g})[\lambda] \quad \text{for } k = 1, \ldots, m.
\end{align}

Then $D_k(\lambda) \in \sum_{X \in \mathfrak{p}_\Theta} U(\mathfrak{g})[\lambda](X - \lambda_\Theta(X))$ and therefore $D_k(\lambda) \in \text{Ann}(M_\Theta(\lambda))$ for $k = 1, \ldots, m$ under the same notation as in the case $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$.

**Proof.** Retain the notation in the proof of Lemma 2.10. We may assume $\{Y_1, \ldots, Y_{N'}\}$ is a basis of $\bar{n}_\Theta$ for a suitable $N'$. We note that for $D \in U(\mathfrak{g})[\lambda]$

\begin{equation}
D \equiv \sum_{\alpha \in N'^{+}} c_\alpha(D; \lambda)Y^\alpha \mod \sum_{X \in p_\Theta} U(\mathfrak{g})[\lambda](X - \lambda_\Theta(X)).
\end{equation}

Here $c_\alpha(D; \lambda) \in \mathbb{C}[\lambda]$ are uniquely determined by $D$ because of the decomposition $U(\mathfrak{g}) = U(\bar{n}_\Theta) \oplus U(\mathfrak{p}_\Theta)$.

Put $I = \sum_{k=1}^m U(\mathfrak{g})D_k(\lambda)U(\mathfrak{g})$ and $I_\lambda = \sum_{H \in a_\Theta} S(a)[\lambda](H - \lambda(H))$ and suppose $D \in I$. Then (2.31) implies $\omega(D_k(\lambda)) \in I_\lambda$ for $k = 1, \ldots, m$ and therefore $\omega(PD_k(\lambda)Q) \in I_\lambda$ for $P, Q \in U(\mathfrak{g})$ by Lemma 2.10 which implies $c_0(D; \lambda) = \omega(D)(\lambda) = 0$. Hence $IM_\Theta(\lambda)$ is a proper $\mathfrak{g}$-submodule of $M_\Theta(\lambda)$ for any fixed $\lambda \in a_\Theta^*$.

Since $M_\Theta(\lambda)$ is an irreducible $\mathfrak{g}$-module for a generic $\lambda$ (if the infinitesimal character of the Verma module with the highest weight which equals to the weight $Y^\alpha$ with $\alpha \neq 0$ plus $\lambda$ is different from that of $M_\Theta(\lambda)$, then $M_\Theta(\lambda)$ is irreducible), $IM_\Theta(\lambda) = 0$ for a generic $\lambda$. Hence $c_\alpha(D; \lambda) = 0$ for $\alpha \in \mathbb{N}^{N'}$ and $IM_\Theta(\lambda) = 0$ for any $\lambda$.

The following remark is clear from the argument in the proof of Lemma 2.11.

**Remark 2.12.** i) Let $\ell$ be a positive integer and let $r(\lambda, \epsilon)$ be a polynomial function of $(\lambda, \epsilon) \in \mathbb{C}^{\ell+1}$ valued in $U^*(\mathfrak{g})$. If $r(\lambda, \epsilon) \in \text{Ann}_G(M_\Theta(\lambda))$ for generic $(\lambda, \epsilon)$, then $r(\lambda, \epsilon) \in \text{Ann}_G(M_\Theta(\lambda))$ for any $(\lambda, \epsilon)$. 


ii) Let \( p \) be a suitable polynomial function of \( \mathbb{C}^\ell \) to \( \mathfrak{a}_\oplus^* \). Replacing \( D_k(\lambda) \), \( U(\mathfrak{g})[\lambda] \) and \( \lambda \) by \( D_k(\mu), U(\mathfrak{g})[\mu] \) and \( \mu \), respectively, in Lemma 2.11, we have the same conclusion if \( M_\Theta(p(\mu)) \) is irreducible for generic \( \mu \in \mathbb{C}^\ell \).

**Remark 2.13.** Use the notation in Lemma 2.10. Let \( \lambda \in \mathfrak{a}^* \) and consider the Verma module \( M(\lambda) = U(\mathfrak{g})/(U(\mathfrak{g})n + \sum_{H \in \mathfrak{a}} U(\mathfrak{g})(H - \lambda(H))) \). Then
\[
P_\lambda = \{ D \in U(\mathfrak{g}); \omega(D)(\lambda) = \omega(\text{ad}(X)D)(\lambda) = 0 (\forall X \in \mathfrak{g}) \}
\]
is the annihilator \( \text{Ann}(L(\lambda)) \) of the unique irreducible quotient \( L(\lambda) \) of \( M(\lambda) \). Here we identify \( S(\mathfrak{a}) \) with the space of polynomial functions of \( \mathfrak{a}^* \). This may be also considered to be a quantization of the conjugacy class of semisimple matrices.

**Proof.** Lemma 2.10 proves that \( P_\lambda \) is a two-sided ideal of \( U(\mathfrak{g}) \). Since the assumption means that the projection of \( P_\lambda L(\lambda) \) into the highest weight space of \( L(\lambda) \) vanishes, \( P_\lambda L(\lambda) = 0 \) because of the irreducibility of \( L(\lambda) \). On the other hand, if \( DL(\lambda) = 0, D \in U(\mathfrak{g})n + \sum_{H \in \mathfrak{a}} U(\mathfrak{g})(H - \lambda(H)) \) and therefore \( \omega(D)(\lambda) = 0 \). Since \( \text{Ann}(L(\lambda)) \) is a two-sided ideal of \( U(\mathfrak{g}) \), we have \( \text{Ann}(L(\lambda)) \subset P_\lambda \). \( \square \)

**Remark 2.14.** Define \( \rho \in \mathfrak{a}^* \) by \( \rho(X) = \frac{1}{2} \text{Trace} \text{ad}(H)|_n \) and \( w.\lambda = w(\lambda + \rho) - \rho \) for the element \( w \) of the Weyl group \( W \) of the pair \( (\mathfrak{g}, \mathfrak{a}) \). Then the infinitesimal character of the highest weight module \( M(\lambda) \) is parametrized by \( W \lambda \). We say that the infinitesimal character is regular if \( w.\lambda \neq \lambda \) for any \( w \in W \) with \( w \neq e \).

If \( \mathfrak{g} = \mathfrak{g}(n, \mathbb{C}) \), then
\[
\rho = \left(-\frac{n-1}{2} + (1 - 1)\right)e_1 + \cdots + \left(-\frac{n-1}{2} + (n - 1)\right)e_n,
\]
\( W \simeq \mathfrak{S}_n \) and
\[
w\left(\sum_{j=1}^{\ell} \mu_j e_j\right) = \sum_{j=1}^{\ell} \mu_j e_{w^{-1}(j)} = \sum_{j=1}^{\ell} \mu_{w(j)} e_j \quad \text{for } (\mu_1, \ldots, \mu_\ell) \in \mathbb{C}^\ell \text{ and } w \in W.
\]
In \( U^\ell(\mathfrak{g}) \), \( \rho \) changes into \( \rho^\ell = \epsilon \rho \) and the infinitesimal character of \( M_\Theta(\lambda) \) equals that of \( M^\ell(\lambda_\Theta) \). Hence the infinitesimal character is regular if and only if all the roots of \( d^\ell_n(x) = 0 \) are simple because the set of roots is \( \{\lambda_\nu + \frac{n-1}{2}; \nu = 1, \ldots, n\} \) by putting
\[
\lambda_\Theta + \rho^\ell = \lambda_1 e_1 + \cdots + \lambda_\ell e_\ell.
\]

**Lemma 2.15.** Let \( I = \{i_1, \ldots, i_m\} \) and \( J = \{j_1, \ldots, j_{m-1}\} \) be sets of positive numbers with \( m > 0, i_1 < i_2 < \cdots < i_m \) and \( j_1 < j_2 < \cdots < j_{m-1} \). Then there exists a positive number \( \mu \leq m \) such that \( \# \{j \in J; j < i_\mu\} = \mu - 1 \) and \( i_\mu \notin J \).

**Proof.** Suppose \( m > 1 \) since the lemma is clear when \( m = 1 \). If \( j_{m-1} < i_m \), we can put \( \mu = m \). If \( j_{m-1} \geq i_m \), we can reduce to the case when \( I = \{i_1, \ldots, i_{m-1}\} \) and \( J = \{j_1, \ldots, j_{m-2}\} \). \( \square \)

Retain the notation in Theorem 2.8. Fix \( k \) with \( 1 \leq k \leq L \) and put \( m = n+1-n_k^\ell \) and \( J = \{1, 2, \ldots, n\} \setminus \{n_{k-1}+1, n_{k-1}+2, \ldots, n_k\} \). Note that \( \#J = m-1 \).
For $I = \{i_1, \ldots, i_m\}$ with $1 \leq i_1 < \cdots < i_m \leq n$, choose an integer $\mu$ as in Lemma 2.15. Then $n_{k+1} < i_\mu \leq n_k$ and $\#\{1,2,\ldots, n_{k-1}\} = \mu - 1$, from which we have $\mu = n_{k+1} + 1$ and $\lambda(E_{i_\mu}) - (\lambda_k + n_{k-1} \epsilon) + (\mu - 1) \epsilon = 0$ and therefore (2.23) and Proposition 2.6 show

$$\omega(D_{IJ}^\epsilon(\lambda_k + n_{k-1} \epsilon)) \in \sum_{H \in \alpha} S(a)(H - \lambda(H))$$

(2.36) if $\# I = \# J = n+1-n'_k$.

Denoting

$$J(m,x) = \sum_{\# I = \# J = m} \mathbb{C}D_{IJ}^\epsilon(x),$$

the basis of $J(n+1-n'_k, \lambda_k + n_{k-1} \epsilon)$ satisfies the assumption in Lemma 2.11 for $\epsilon = 1$ and therefore

$$J(n+1-n'_k, \lambda_k + n_{k-1} \epsilon) \subset \text{Ann}_G(M_\epsilon^\lambda)$$

(2.38) for $\epsilon = 1$. But this holds for any $\epsilon$ because of Remark 2.12 i) with the isomorphism between $U(\mathfrak{g})$ and $U^\epsilon(\mathfrak{g})$.

Now the Laplace expansions of $D_{IJ}^\epsilon(x)$ with respect to the first and the last column show (cf. \cite[Proposition 2.6 i]{O2})

$$J(m+1, \lambda) + J(m+1, \lambda + \epsilon) \subset U^\epsilon(\mathfrak{g})J(m, \lambda)$$

(2.39) if $m < n$

and therefore

$$J(m+1, \lambda) + J(m+1, \lambda + \epsilon) \subset U^\epsilon(\mathfrak{g})J(m, \lambda)$$

(2.40) for $0 \leq i \leq j \leq n'_k - 1$. When $\epsilon = 0$, it is obvious by the Laplace expansion of $D_{IJ}^0(x)$ that

$$\left(\frac{d^i}{dx^i} D_{IJ}^0(x)\right)|_{x=\lambda_k} = 0$$

for $\# I = \# J = n+1-n'_k + j$ with $0 \leq i \leq j \leq n'_k - 1$.

Hence if $c \in \mathbb{C}$ satisfies $d_m^\epsilon(c; \lambda) = 0$, then $\det_m^\epsilon(c; E_{IJ}) \in I_\epsilon^\lambda(\lambda)$ for $\# I = \# J = m$ by denoting

$$I_\epsilon^\lambda(\lambda)' = \sum_{k=1}^L U^\epsilon(\mathfrak{g})J(n+1-n'_k, \lambda_k + n_{k-1} \epsilon).$$

We have proved

$$I_\epsilon^\lambda(\lambda)' \subset I_\epsilon^\lambda(\lambda)$$

and $I_\epsilon^\lambda(\lambda)' = I_\epsilon^\lambda(\lambda)$ if all the root of $d_m^\epsilon(x; \lambda) = 0$ are simple for $m = 1, \ldots, n$ (cf. Remark 2.9). Hence it follows from Remark 2.12 i) that

$$I_\epsilon^\lambda(\lambda) \subset \text{Ann}_G(M_\epsilon^\lambda).$$

(2.43) Note that the element $r_{IJ}^\epsilon$ for $\# I = n$ in (2.17) are contained in $J^\epsilon(\lambda_\Theta)$ because they are in the center $U^\epsilon(\mathfrak{g})^G$ of $U^\epsilon(\mathfrak{g})$ and $U^\epsilon(\mathfrak{g})^G \equiv \mathbb{C}$ mod $J^\epsilon(\lambda_\Theta)$. 

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Thus we have only to show $I^\Theta_\Theta(\lambda) \subseteq \text{Ann}_G(M^\Theta_\Theta(\lambda))$ to complete the proof of Theorem 2.8. We can prove this for generic $\lambda$ with $\epsilon \neq 0$ using the result in the next section (cf. [O3]) or Theorem 2.21 but we reduce it to the claim

(2.44) $I^0_\Theta(0) = \text{Ann}_G(M^0_\Theta(0)).$

For $\epsilon = \lambda = 0$, this is conjectured by [Ta] and is proved by [We]. In this case $r^I_{IJ} \in S(\mathfrak{g})$ are of homogeneous polynomials of $\mathfrak{g}^*$ with degree $\#I - j$. Here we note that $\det^\epsilon(x; E_{IJ})$ is homogeneous of degree $\#I$ with respect to $(\mathfrak{g}, \epsilon, \lambda)$, which is well-defined under any choice of Poincare-Birkhoff-Witt basis because of the homogenized universal enveloping algebra.

Let $S(\mathfrak{g})_m$ be the space of homogeneous elements of $S(\mathfrak{g})$ with degree $m$. Then $U^\epsilon(\mathfrak{g})^{(m)} / U^\epsilon(\mathfrak{g})^{(n'-1)} \simeq S(\mathfrak{g})_m$ and for $D \in U^\epsilon(\mathfrak{g})^{(m)}$, we denote by $\sigma_m(D)$ the corresponding element in $S(\mathfrak{g})_m$. Note that $\sigma_{\#I-j}(r^I_{IJ})$ in (2.17) does not depend on $\lambda$ and $\epsilon$. Hence

(2.45) $I^\Theta_\Theta(0) = \sum_{m=n+1}^{\infty} \max\{n\ldots n_l\} \sum_{\#I=m}^{d_m-1} \sum_{j=0}^{\#I-j} S(\mathfrak{g}) \sigma_{m-j}(r^I_{IJ})$

Put $R^\epsilon(\lambda)^{(m)} = \text{Ann}_G(M^\Theta_\Theta(\lambda)) \cap U^\epsilon(\mathfrak{g})^{(m)}$ and $D \in R^\epsilon(\lambda)^{(m)} \setminus R^\epsilon(\lambda)^{(m-1)}$. We will prove $D \in I^\Theta_\Theta(\lambda)$ by the induction on $m$. Since (2.10) implies $\text{Ad}(g)D \equiv 0 \mod U^\epsilon(\mathfrak{g})^{(m-1)} p_\Theta + U^\epsilon(\mathfrak{g})^{(m-1)}$, we have

(2.46) $\sigma_m(D)(\text{Ad}(g)n_\Theta) = 0 \quad (\forall g \in G)$

and $\sigma_m(D) \in I^\Theta_\Theta(0)$. Hence it follows from (2.44) and (2.45) that there exist homogeneous elements $p^I_{IJ} \in S(\mathfrak{g})$ satisfying $\sigma_m(D) = \sum p^I_{IJ} \sigma_{\#I-j}(r^I_{IJ})$. Here $r^I_{IJ}$ are generators of $I^\Theta_\Theta(\lambda)$ appeared in (2.17) and $\deg(p^I_{IJ}) + \#I - j = m$ if $p^I_{IJ} \neq 0$. Let $P^I_{IJ} \in U^\epsilon(\mathfrak{g})^{(m-\#I+j)}$ with $\sigma_{\#I-j}(P^I_{IJ}) = p^I_{IJ}$ and put $D' = \sum P^I_{IJ} D^I_{IJ}$. Then $D' \in I^\Theta_\Theta(\lambda)$ and $D - D' \in R^\epsilon(\lambda)^{(m-1)}$ and therefore we have $D - D' \in I^\Theta_\Theta(\lambda)$ by the hypothesis of the induction. Thus we have completed the proof of Theorem 2.8. □

**Remark 2.16.** The procedure to deform $\lambda$ to 0 under the classical limit $\epsilon = 0$ is studied by [BK].

In the proof of Theorem 2.8 we have shown the following, which is proved by [BB] together with the fact that it is not valid for a generalized Verma module of a general semisimple Lie algebra induced form a character of a parabolic subalgebra.

**Corollary 2.17.** The graded ring $\text{gr}(\text{Ann}_G(M^\Theta_\Theta(\lambda))) = \bigoplus_{m=0}^{\infty} (\text{Ann}_G(M^\Theta_\Theta(\lambda)) \cap U^\epsilon(\mathfrak{g})^{(m)}) / (\text{Ann}_G(M^\Theta_\Theta(\lambda)) \cap U^\epsilon(\mathfrak{g})^{(m-1)})$ equals the defining ideal of the closure of the nilpotent conjugacy class of the generic element $A_{\epsilon, 0}$ of the form (2.4). In particular it is a prime ideal and does not depend on $(\lambda, \epsilon)$. 
Corollary 2.18. The following two conditions are equivalent.

\[(2.47) \quad \text{Ann}_G(M_\Theta^\epsilon(\lambda)) \supset \text{Ann}_G(M_\Theta^\epsilon(\lambda')).\]

\[(2.48) \quad d_m^\epsilon(x; \Theta, \lambda) \in \mathbb{C}[x]d_m^\epsilon(x; \Theta', \lambda') \quad \text{for } m = 1, \ldots, n.\]

**Proof.** It is obvious that the latter condition implies the former. Hence suppose the first condition. Let \(f_m(x)\) be the least common multiple of \(d_m^\epsilon(x; \Theta, \lambda)\) and \(d_m^\epsilon(x; \Theta', \lambda')\). Then if \(\#I = \#J = m\), \(\det^\epsilon(x; E_{IJ}) \in U^\epsilon(\mathfrak{g})f_m(x) \mod \mathbb{C}[x] \otimes \text{Ann}_G(M_\Theta^\epsilon(\lambda))\). Applying \(\sigma_m\) to this relation as in the proof of Theorem 2.8, we have \(\det^0(x; E_{IJ}) \in S(\mathfrak{g})x^\deg(f_m) \mod \mathbb{C}[x] \otimes \text{Ann}_G(M_\Theta^0(0))\) because of the homogeneity with respect to \((\mathfrak{g}, \epsilon, \lambda)\). Let \(A_{\Theta,0}\) be the generic element of the form (2.4) and let \(J_\Theta\) be the maximal ideal of \(S(\mathfrak{g})\) corresponding to \(A_{\Theta,0}\). Considering under modulo \(J_\Theta\), we can conclude that all the \(m\)-minors of the matrix \((x - A_{\Theta,0})\) are in \(\mathbb{C}[x]x^\deg(f_m)\). On the other hand, \(x^{d_m(\Theta)}\) is the greatest common devisors of \(m\)-minors of \((x - A_{\Theta,0})\) and therefore \(\deg f_m(x) \leq d_m(\Theta) = \deg d_m^\epsilon(x; \Theta, \lambda)\) and we have the latter condition. \(\square\)

Remark 2.19. If \(\epsilon = 0\), Corollary 2.18 gives the closure relation in the conjugacy classes of the matrices.

Remark 2.20. The following theorem is a part of a conjecture proposed by [O1] for the general symmetric pair. The case in this note corresponds to the pair \((GL(n, \mathbb{C}), U(n))\). In the case of the classical limit \(\epsilon = \lambda = 0\), the following theorem is obtained by [DP] and [Ta].

Theorem 2.21. Let \(W_\Theta\) be the Weyl group of \(\mathfrak{m}_\Theta\) and let \(W = W(\Theta)W_\Theta\) be the decomposition of \(W = S_n\) so that \(W(\Theta)\) be the set of the representatives of \(W/W_\Theta\) with the minimal length. Then the common zeros of \(\omega(\text{Ann}_G(M_\Theta^\epsilon(\lambda)))\) coincides with the set \(\{w\lambda; w \in W(\Theta)\}\) counting their multiplicities.

In particular, the space \(S(\mathfrak{a})/\omega(\text{Ann}_G(M_\Theta^\epsilon(\lambda)))\) is naturally a representation space of \(W\) which is isomorphic to \(\text{Ind}_W^1\) id.

**Proof.** Under the notation \((2.35)\)

\[\tilde{\lambda}_\nu = \lambda_{\epsilon(\nu)} - \frac{n-1}{2} + (\nu - 1) \quad \text{for } \nu = 1, \ldots, n.\]

and

\[\tilde{\omega}(D_{ij}^\epsilon)(\lambda_k + n_{k-1} \epsilon) = \prod_{\mu = 1}^m (E_{i_\mu} - \lambda_k + \frac{n-1}{2} - n_{k-1} + \mu - i_\mu)\epsilon).\]

Fix \(k\) with \(1 \leq k \leq L\) and \(w \in W(\Theta)\). Put \(m = n + 1 - n'\), \(K = \{n_{k-1} + 1, \ldots, n_k\}\), \(K^c = \{1, \ldots, n\} \setminus K\) and \(J = w(K^c)\). For \(I = \{i_1, \ldots, i_m\}\) with \(1 \leq i_1 < \cdots < i_m \leq n\), choose \(\mu\) as in Lemma 2.15 and put \(\ell = w^{-1}(i_\mu)\). Then \(\ell \in K\) and \(\{\nu \in K^c; w(\nu) < i_{\mu}\} = \mu - 1\), which implies \(\#\{\nu \in K; w(\nu) < i_{\mu}\} = i_{\mu} - \mu\). On the other hand, since the condition \(n_{k-1} < \nu < \nu' \leq n_k\) means \(w(\nu) < w(\nu')\),
we have \( \nu \in K; w(\nu) < i_\mu \) = \( \{n_{k-1} + 1, n_{k-1} + 2, \ldots, \ell - 1\} \) and therefore \( \ell - n_{k-1} - 1 = i_\mu - \mu \) and

\[
\bar{\lambda}_\ell - \lambda_k + \left(\frac{n_{k-1}}{2} - n_{k-1} + \mu - i_\mu\right)\varepsilon = (\ell - n_{k-1} + \mu - i_\mu)\varepsilon = 0.
\]

Since \( \bar{\lambda}_\ell \) is the \( i_\mu \)-th component of \( (\bar{\lambda}_{w(1)}, \ldots, \bar{\lambda}_{w(n)}) \), we can conclude that \( \bar{\omega}(D_{II}(\lambda_k + n_{k-1}\varepsilon)) \) vanishes at \( w(\lambda_\Theta + \rho^\epsilon) \), which is equivalent to the condition that \( \omega(D_{II}(\lambda_k + n_{k-1}\varepsilon)) \) vanishes at \( w.\lambda_\Theta \). Hence if \( \lambda \) is generic, \( \omega(I_\Theta^\epsilon(\lambda)) \) vanishes at \( w.\lambda_\Theta \) for \( w \in W(\Theta) \) and therefore for any \( \lambda \in \mathbb{C}^L \) because of the continuity. In particular, \( \dim S(\mathfrak{a})/\omega(I_\Theta^\epsilon(\lambda)) \geq \#W(\Theta) \) for generic \( \lambda \) and therefore for any \( \lambda \) by the same reason.

Since \( \omega(I_\Theta^\epsilon(\lambda)) \) are generated by homogeneous polynomials of \( (a, \lambda, \epsilon) \) and [Ta, Theorem 1] shows \( \dim S(\mathfrak{a})/\omega(I_\Theta^\epsilon(0)) = \#W(\Theta) \), we have \( \dim S(\mathfrak{a})/\omega(I_\Theta^\epsilon(\lambda)) \leq \#W(\Theta) \). Thus we can conclude \( \dim S(\mathfrak{a})/\omega(I_\Theta^\epsilon(\lambda)) = \#W(\Theta) \) and the theorem follows from this. In fact, the last claim is clear because \( I_\Theta^\epsilon(\lambda) \) is \( W \)-invariant. \( \square \)

3. Generalized Verma modules

In this section we study the necessary and sufficient condition on \( \lambda \in \mathbb{C}^L \) so that

\[
J_\Theta^\epsilon(\lambda) = \text{Ann}_G (M_\Theta^\epsilon(\lambda)) + J^\epsilon(\lambda_\Theta).
\]

Note that it is clear by the definition that \( J_\Theta^\epsilon(\lambda) \supset \text{Ann}_G (M_\Theta^\epsilon(\lambda)) + J^\epsilon(\lambda_\Theta) \) and

\[
\text{Ann}_G (M_\Theta^\epsilon(\lambda)) = \text{Ann}_G (U^\epsilon(\mathfrak{g})/(\text{Ann}_G (M_\Theta^\epsilon(\lambda)) + J^\epsilon(\lambda_\Theta))).
\]

In general it is proved by [BG] and [Jo] that for \( \mu \in \mathfrak{a}^* \) the map

\[
\{I; I \text{ is the two sided ideal of } U(\mathfrak{g}) \text{ with } I \supset \text{Ann} (M(\mu))\}
\]

\[
\exists I \mapsto I + J(\mu) \in \{J; J \text{ is the left ideal of } U(\mathfrak{g}) \text{ with } J \supset J(\mu)\}
\]

is injective if \( \mu \) is dominant:

\[
2\frac{(\mu + \rho, \alpha)}{\langle \alpha, \alpha \rangle} \notin \{-1, -2, \ldots\} \text{ for any positive root } \alpha \text{ for the pair } (\mathfrak{n}, \mathfrak{a}).
\]

Moreover the map is surjective if \( \mu \) is regular, that is,

\[
\langle \mu + \rho, \alpha \rangle \neq 0 \text{ for any root } \alpha \text{ for the pair } (\mathfrak{n}, \mathfrak{a})
\]

and dominant. Hence in our case with \( \epsilon \neq 0 \), (3.1) is valid if \( \lambda_\Theta + \rho^\epsilon \) is regular and dominant:

\[
\bar{\lambda}_j - \bar{\lambda}_i \notin \{0, -\epsilon, -2\epsilon, \ldots\} \text{ for } 1 \leq i < j \leq n.
\]

For \( \mu \in \mathfrak{a}^* \) and \( D \in U^\epsilon(\mathfrak{g}) \) let \( \gamma(\mu; D) \) denote the unique element in \( U^\epsilon(\mathfrak{g}) \) with \( D \equiv \gamma(\mu; D) \mod J^\epsilon(\mu) \). For a basis \( \{R_j\} \) of an \( \text{ad}(\mathfrak{g}) \)-invariant subspace \( V \) of \( U^\epsilon(\mathfrak{g}) \) we note that

\[
\gamma(\mu; \sum P_j R_j) \in \sum U^\epsilon(\mathfrak{g}) \gamma(\mu; R_j) \text{ for } P_j \in U^\epsilon(\mathfrak{g}).
\]
Let $R_-$ denote the set of weights of $U^e(\mathfrak{h})$ with respect to $\mathfrak{a}$. Then

$$R_- = \{ \sum_{i=1}^{n} m_i e_i; \ m_i \in \mathbb{Z}, \ \sum m_i = 0 \text{ and } m_1 \geq m_2 \geq \cdots \geq m_n \} \setminus \{0\}.$$ 

Suppose $R_j \in U^e(\mathfrak{g})$ are weight vectors and $U^e(\mathfrak{g})V + J^e(\mu) \neq U^e(\mathfrak{g})$. Since $\gamma(\mu; R_j)$ has the weight which equals that of $R_j$, $\gamma(\mu; R_j) = 0$ if the weight of $R_j$ is not in $R_-$. Moreover since $E_{ii+1}$ has a maximal weight $e_i - e_{i+1}$ in $R_-$ for any integer $i$ with $1 \leq i < n$,

$$E_{ii+1} \in U^e(\mathfrak{g})V + J^e(\mu) \Leftrightarrow \mathbb{C}E_{ii+1} = \sum_{\text{the weight of } R_j = e_i - e_{i+1}} \mathbb{C}\gamma(\mu; R_j).$$  

The key to studying the condition for (3.1) is the following argument used in [O2, proof of Theorem 5.1].

Fix positive integers $k$, $\overline{i}$ and $\overline{j}$ satisfying $1 \leq k \leq L$ and $n_{k-1} < \overline{i} < \overline{j} \leq n_k$. Let $I = \{i_m, \ldots, i_1\}$ and $J = \{j_m, \ldots, j_1\}$ be a set of positive numbers such that

$$1 \leq i_1 < i_2 < \cdots < i_m \leq n,$$

$$i_\nu = j_\nu \text{ if } \nu \neq \ell,$$

$$i_\ell = \overline{i} < j_\ell = \overline{j} < i_{\ell+1}$$

with a suitable $1 \leq \ell \leq m$. Define non-negative integers

$$m' = n - m,$$

$$a_j' = n'_j - \# \{ \nu; \ n_{j-1} < i_\nu \leq n_j \},$$

$$a_j = n_j - \# \{ \nu; \ i_\nu \leq n_j \} = a'_1 + \cdots + a'_j, \ a_0 = 0,$$

$$b = \# \{ \nu; \ n_{k-1} < i_\nu < \overline{i} \},$$

$$b' = \# \{ \nu; \ \overline{j} < i_\nu \leq n_k \}.$$

Then

$$1 \leq a_L = m' \leq n - 2, \ 1 \leq a'_k = n'_k - b - b' - 1,$$

$$0 \leq a'_j \leq n'_j - \delta_{kj}, \ 0 \leq b \leq \overline{i} - n_{k-1} + 1, \ 0 \leq b' \leq n_k - \overline{j}$$

and we have

$$\det^e(x; E_{IJ}) \equiv \prod_{\nu=\ell+1}^{m} (x - E_{i\nu} - (\nu - 1)e) \cdot E_{\overline{i}\overline{j}}$$

$$\cdot \prod_{\nu=1}^{\ell-1} (x - E_{i\nu} - (\nu - 1)e) \mod U^e(\mathfrak{g})\mathfrak{n}$$

$$\equiv \frac{\prod_{j=1}^{L} p_{IJ}^j(x)}{s_{IJ}(x)} E_{ij} \mod J^e(\lambda_{\Theta})$$
by putting

\begin{align}
(3.13) \quad p_{IJ}^j(x) &= (x - \lambda_j - (n_{j-1} - a_{j-1})\epsilon)^{(n_j' - a_j)}, \\
    s_{IJ}(x) &= x - \lambda_k - (n_{k-1} - a_{k-1} + b)\epsilon.
\end{align}

Hence it follows from (2.17) that

\begin{align}
(3.14) \quad \sum_{i=0}^{d_m-1} C_{\overline{I}\overline{J}}^i \equiv \begin{cases} \mathbb{C}E_{\overline{I}\overline{J}}^i \mod J^\epsilon(\lambda) & \text{if } \prod_{j=1}^L p_{IJ}^j(x) \notin \mathbb{C}[x]s_{IJ}(x)d_m^\epsilon(x), \\
0 \mod J^\epsilon(\lambda) & \text{otherwise.}
\end{cases}
\end{align}

Since \((n_j' - a_j' - a_j') - (n_j' - m') = m' - a_j \geq m' - a_L \geq 0\), we can define polynomials

\[ \tilde{p}_{IJ}^j(x) = \frac{p_{IJ}^j(x)}{(x - \lambda_j - n_{j-1}\epsilon)^{(n_j' - m')}}. \]

Then the condition \( \prod_{j=1}^L p_{IJ}^j(x) \in \mathbb{C}[x]s_{IJ}(x)d_m^\epsilon(x) \) is equivalent to the existence of \( j \) with

\[ (3.15) \quad \tilde{p}_{IJ}^j(x) \in \mathbb{C}[x]s_{IJ}(x). \]

If \( \epsilon \neq 0 \), the condition (3.15) is equivalent to the condition that \( \nu \) is an integer satisfying

\[ (3.16) \quad 0 \leq \nu \leq n_j' - a_j' - 1 \quad \text{and} \quad (\nu < a_{j-1} \text{ or } \nu \geq a_{j-1} + n_j' - m') \]

by denoting

\[ (3.17) \quad \lambda_j + (n_{k-1} - a_{k-1} + b)\epsilon = \lambda_j + (n_{j-1} - a_j + \nu)\epsilon. \]

If \( \epsilon = 0 \), it is equivalent to

\[ (3.18) \quad \lambda_j = \lambda_k \quad \text{and} \quad a_j' < m'. \]

Put \( I = \{n, n-1, \ldots, n_{k+1}, n_{k-1}, n_{k-1}-1, \ldots, 1\} \) and \( J = \{n, n-1, \ldots, n_{k+1}, n_{k-1}, n_{k-1}-1, \ldots, 1\} \). Then

\[ m' = n_k' - 1, \quad b = b' = 0, \quad a_k' = n_{k-1}' - 1, \quad a_j' = 0 \quad \text{and} \quad n_j' - a_j' - 1 = n_j' - 1 \quad \text{if} \quad j \neq k. \]

Suppose (3.15) holds. Then \( j \neq k \) because \( \tilde{p}_{IJ}^k(x) = 1 \). Since

\[ \begin{cases} a_{j-1} - 1 = -1 < 0 \quad \text{and} \quad a_{j-1} + n_j' - m' = n_j' - n_k' + 1 & \text{if } j < k, \\
    a_{j-1} - 1 = n_k' - 2 \quad \text{and} \quad a_{j-1} + n_j' - m' = n_j' + n_j' - a_j' - 1 & \text{if } j > k,
\end{cases} \]

the condition (3.16) is equivalent to

\[ \max\{0, n_j' - n_k' + 1\} \leq \nu' \leq n_j' - 1 \quad \text{if } j < k, \]

\[ 1 - n_k' \leq \nu' \leq \min\{n_j' - n_k', -1\} \quad \text{if } k < j \]

with

\[ \nu' = (\nu - a_{j-1}) - (b - a_k) = \begin{cases} \nu & \text{if } j < k, \\
    \nu - n_k' + 1 & \text{if } k < j.
\end{cases} \]
Hence (3.15) is equivalent to the condition (cf. Remark 2.14)

\[ \Lambda_k \cap \Lambda_j \neq \emptyset, \quad \Lambda_k \subsetneq \Lambda_j \quad \text{and} \quad (\mu \in \Lambda_j, \mu' \in \Lambda_k \setminus \Lambda_j \Rightarrow (\mu' - \mu)(k - j) > 0) \]

with \( \Lambda_i = \{ \lambda_{\nu}; n_{i-1} < \nu \leq n_i \} = \{ \lambda_i + ((\nu - 1) - \frac{n_{i-1}}{2}) \epsilon; n_{i-1} < \nu \leq n_i \} \)

if \( \epsilon \neq 0, \)

\( \lambda_j = \lambda_k \) and \( n_k' > 1 \) if \( \epsilon = 0. \)

Thus we have the following theorem.

**Theorem 3.1.** i) Fix \( k \) with \( 1 \leq k \leq L. \) Recall \( m^k_{\varnothing} = \sum_{n_{k-1} < i \leq n_k} CE_{ij}. \)

Then

\[ \text{Ann}_G (M^k_{\varnothing}(\lambda)) + J^\epsilon(\lambda_{\varnothing}) \supset m^k_{\varnothing} \cap \overline{n} \]

if and only if (3.19) does not hold for \( j = 1, \ldots, L. \)

ii) The equality (3.1) is valid if and only if (3.19) does not hold for \( j = 1, \ldots, L \)

and \( k = 1, \ldots, L, \) which is equivalent to the condition

\[ \min \overline{\Lambda}_i > \min \overline{\Lambda}_j \quad \text{or} \quad \max \overline{\Lambda}_i > \max \overline{\Lambda}_j \quad \text{or} \quad \Lambda_i \cap \Lambda_j = \emptyset \quad \text{or} \quad \Lambda_i = \Lambda_j \]

\[ \lambda_i \neq \lambda_j \quad \text{or} \quad n_i' = n_j' = 1 \]

for \( 1 \leq i < j \leq L. \)

Here \( \overline{\Lambda}_i = \{ \text{Re} \mu; \mu \in \Lambda_i \} \) etc. In particular (3.1) is valid if the infinitesimal character of \( M^k_{\varnothing}(\lambda) \) is regular.

**Proof.** We have only to prove that (3.20) is not valid if (3.19) holds for a suitable \( j. \) Suppose there exists \( j = j_o \) which satisfies (3.19). Fix such \( j_o \) and continue the argument just before the theorem. Put \( j = \overline{i} + 1 \) and suppose (3.15) does not valid for \( j = k. \) Then if \( \epsilon \neq 0, \quad \nu = b \) in (3.17) and since \( 0 \leq b \leq n_k' - a_k' - 1 \)

and (3.16) is not valid with \( j = k, \) we have

\[ a_{k-1} \leq b < a_{k-1} + n_k' - m' \quad \text{and} \quad m' > n_k' \quad \text{if} \quad \epsilon \neq 0. \]

On the other hand, if \( \epsilon = 0, \) we have \( a_k' = m' \) because \( a_k' \leq a_L = m'. \)

First consider the case when \( j_o < k. \) Put \( \ell = \lambda_k + n_{k-1} - \lambda_{j_o} - n_{j_o-1}, \overline{i} = n_{k-1} + 1 \) and \( \overline{j} = \overline{i} + 1. \) Then \( b = 0. \) If \( \epsilon \neq 0, \quad a_{k-1} = a_{j_o} = 0 \) because of (3.22) and it follows from (3.19) that

\[ 0 \leq \ell < n_{j_o} \quad \text{and} \quad \ell + n_k' > n_{j_o}. \]

In this case putting \( j = j_o \) in (3.17) we have \( \nu = \ell \) and then \( 0 \leq \nu, \quad n_j' - n_{j_o} + 1 \leq \nu \)

and \( \nu \leq n_j' - 1 \) in (3.16), which implies \( \overline{p}_{IJ}^{j_o}(x) \in \mathbb{C}[x]s(x). \) We have this relation also in the case when \( \epsilon = 0 \) because \( \deg \overline{p}_{IJ}(x) = n_{j_o}' - a_{j_o}' - (n_{j_o} - m') = m' - a_{j_o}' \geq m' - (m' - a_k') = a_k' > 0. \) Thus we can conclude \( r_{IJ}^{j_o} \equiv 0 \quad \text{mod} \quad J^\epsilon(\lambda_{\varnothing}) \) if the weight
of \(r_{IJ}^j\) is \(e_i - e_{i+1}\). Note that the weight of \(r_{\{i_1,\ldots,i_m\}\{j_1,\ldots,j_m\}}\) is \(\sum_{\nu=1}^m e_{i\nu} - e_{j\nu}\). Hence \(E_{i+1}^{\overline{i}} \not\in \text{Ann}_G(M_{\Theta}^0(\lambda)) + J^\epsilon(\lambda_{\Theta})\) because of (3.8).

Lastly consider the case when \(k < j_0\). If \(\epsilon = 0\), the same argument as in the case when \(j_0 < k\) works and therefore we may assume \(\epsilon \neq 0\). Let \(\ell = \lambda_{j_0} + n_{j_0-1} - \lambda_k - n_{k-1}\), \(\overline{i} = n_k - 1\) and \(\overline{j} = n_k\). Then similarly we have

\[
1 \leq \ell < n'_k, \quad n'_k \leq \ell + n'_{j_0}, \quad b' = 0, \quad a'_k = n'_k - b - 1
\]

and \(a_k = a'_k + a_{k-1} > (n'_k - b - 1) + (b - n'_k + m') = m' - 1\) by (3.22). Since \(a_k \leq a_L = m'\), we have \(a_k = a_{j_0} = a_{j_0-1} = m'\) and \(a'_{j_0} = 0\). Putting \(j = j_0\) in (3.17), we have \(\nu = -\ell - a_{k-1} + b + a_{j_0-1} = a'_k - \ell + b = n'_k - \ell - 1\) and therefore \(0 \leq \nu\) and \(\nu \leq n'_{j_0} - 1 = n'_{j_0} - a'_{j_0} - 1\) and \(\nu < n'_k - 1 \leq m' = a_{j_0-1}\) in (3.16). Hence \(\overline{p}_I^{\Theta}(x) \in \mathbb{C}[x]S_{\tau}(x)\) and thus \(E_{i+1}^{\overline{i}} \not\in \text{Ann}_G(M_{\Theta}^0(\lambda)) + J^\epsilon(\lambda_{\Theta})\) as in the previous case.

\[\square\]

**EXAMPLE 3.2.** Suppose \(n = 3\), \(\Theta = \{2, 3\}\) and \(\lambda = (\lambda_1, \lambda_2)\). Then

\[
\begin{align*}
d_1^1(x) &= 1, \quad d_2^2(x) = x - \lambda_1, \quad d_3^3(x) = (x - \lambda_1)(x - \lambda_1 - \epsilon)(x - \lambda_2 - 2\epsilon), \\
J^\epsilon(\lambda_{\Theta}) &= \sum_{3 \geq i > j \geq 1} U(\mathfrak{g})E_{ij} + U(\mathfrak{g})(E_1 - \lambda_1) + U(\mathfrak{g})(E_2 - \lambda_1) + U(\mathfrak{g})(E_3 - \lambda_2), \\
J^\epsilon(\lambda) &= J^\epsilon(\lambda_{\Theta}) + U^\epsilon(\mathfrak{g})E_{12}.
\end{align*}
\]

Since

\[
D^\epsilon_{I,J}(x) = (E_{i_1,j_1} - (x - \epsilon)\delta_{i_1,j_1})(E_{i_2,j_2} - x\delta_{i_2,j_2}) - (E_{i_2,j_1} - (x - \epsilon)\delta_{i_2,j_1})(E_{i_1,j_2} - x\delta_{i_1,j_2})
\]

for \(I = \{i_1 > i_2\}\) and \(J = \{j_1 > j_2\}\), we have

\[
\begin{align*}
D^\epsilon_{\{21\}\{21\}}(\lambda_1) &= (E_2 - \lambda_1 + \epsilon)(E_1 - \lambda_1) - E_{12}E_{21} \equiv 0, \\
D^\epsilon_{\{32\}\{32\}}(\lambda_1) &= (E_3 - \lambda_1 + \epsilon)(E_2 - \lambda_1) - E_{23}E_{32} \equiv 0, \\
D^\epsilon_{\{31\}\{31\}}(\lambda_1) &= (E_3 - \lambda_1 + \epsilon)(E_1 - \lambda_1) - E_{13}E_{31} \equiv 0, \\
D^\epsilon_{\{21\}\{32\}}(\lambda_1) &= E_{23}E_{12} - E_{13}(E_2 - \lambda_1) \equiv E_{23}E_{12}, \\
D^\epsilon_{\{21\}\{31\}}(\lambda_1) &= E_{23}(E_1 - \lambda_1) - E_{13}E_{21} \equiv 0, \\
D^\epsilon_{\{32\}\{21\}}(\lambda_1) &= E_{32}E_{21} - (E_2 - \lambda_1 + \epsilon)E_{31} \equiv 0, \\
D^\epsilon_{\{32\}\{31\}}(\lambda_1) &= (E_3 - \lambda_1 + \epsilon)E_{21} - E_{23}E_{31} \equiv 0, \\
D^\epsilon_{\{31\}\{21\}}(\lambda_1) &= E_{32}(E_1 - \lambda_1) - E_{12}E_{31} \equiv 0, \\
D^\epsilon_{\{31\}\{32\}}(\lambda_1) &= (E_3 - \lambda_1 + \epsilon)E_{12} - E_{13}E_{32} \equiv (\lambda_2 - \lambda_1 + \epsilon)E_{12}.
\end{align*}
\]
Here the above $\equiv$ is considered under modulo $J^\epsilon(\lambda_\Theta)$. Note that

\[
(3.24) \quad \text{Ann}_G( M(\Theta^\epsilon(\lambda)) = \sum_{3 \geq j_1 > j_2 \geq 1} \sum_{3 \geq j_1 > j_2 \geq 1} U^\epsilon(\mathfrak{g}) D_{\{i_1 i_2\}\{j_1 j_2\}}(\lambda_1) + \sum_{D \in U^\epsilon(\mathfrak{g})^G} U^\epsilon(\mathfrak{g})(D - \omega(D)(\lambda_\Theta)).
\]

Hence if $\lambda_1 \neq \lambda_2 + \epsilon$ which is equivalent to (3.21), we have (3.1).

Suppose $\lambda_1 = \lambda_2 + \epsilon$. Then since $\text{ad}(\mathfrak{p})(E_{32}E_{12}) \subset J^\epsilon(\lambda_\Theta)$, we have

\[
(3.25) \quad J^\epsilon_0(\lambda) = U^\epsilon(\mathfrak{n})E_{12} \oplus J^\epsilon(\lambda_\Theta) \supset \text{Ann}_G( M(\Theta(\lambda)) + J^\epsilon(\lambda_\Theta) = U^\epsilon(\mathfrak{n})E_{23}E_{12} \oplus J^\epsilon(\lambda_\Theta) \supset J^\epsilon(\lambda_\Theta).
\]

If $\epsilon \neq 0$, the above inclusion relation gives a Jordan-Hörder sequence of $M^\epsilon(\lambda_\Theta)$ and

\[
(3.26) \quad J^\epsilon_0(\lambda)/(\text{Ann}_G( M(\Theta(\lambda)) + J^\epsilon(\lambda_\Theta)) \simeq M^\epsilon_\Theta(\lambda')
\]

with $\Theta' = \{1, 3\}$ and $\lambda' = (\lambda_1 + \epsilon, \lambda_1 - \epsilon)$. Note that $\rho^\epsilon = (-\epsilon, 0, \epsilon)$, $\lambda_\Theta + \rho^\epsilon = (\lambda_1 - \epsilon, \lambda_1, \lambda_1)$, $\lambda_\Theta - \lambda_\Theta = \epsilon(e_1 - e_2)$, $(1, 2, 3) \lambda_\Theta = \lambda_\Omega$, and $\text{Ann}_G( M(\Theta(\lambda))$ is the unique two-sided proper ideal of $U(\mathfrak{g})$ which is larger than $U(\mathfrak{g})(J(\lambda_\Theta) \cap U(\mathfrak{g})^G)$.

References


