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On the discrete Morse flow as a numerical tool

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Dedicated to Professor Norio Kikuchi for his 60th birthday

1 Introduction

The discrete Morse flow (DMF) have been developed by Professor Norio Kikuchi (at Keio Univ.) for constructing solution of heat type equation related to a minimizing problem ([3], [4], [7], [8], [9], [10] and [11]). Kikuchi introduced the time-semidiscretized energy form below: For fixed positive constant $h$, consider the functional:

$$J_m(u) := \int_\Omega \frac{|u - u_{m-1}|^2}{h} dx + I(u),$$

(1.1)

for an elliptic functional $I(u)$. Here $\Omega(\subset \mathbb{R}^n)$ be a domain with Lipschitz boundary ($n \geq 1$). $I(u)$ is a functional which usually has the form

$$I(u) := \int_\Omega F(x, u, \nabla u) dx,$$

(1.2)

where $N \geq 1$ and $F(x, z, p) : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}$ with suitable regularity. Usually $F$ is required to have good properties such as the elliptic structure for the first variation (See [2] for example.) We determine a sequence $\{u_h^m\}$ as minimizers of $J_m$ in $\mathcal{K}$, called the DMF, inductively: Firstly, for an initial data $u_0 \in \mathcal{K}$ with $I(u_0) < \infty$, we define $u_1$, as a minimizer of $J_1$ in $\mathcal{K}$. The next function $u_2 \in \mathcal{K}$ is determined a minimizer of $J_2$ in $\mathcal{K}$, and so on. $\{u_h^m\}$ is an approximate solution of a heat equation (See section 2) which possibly describes the Morse (semi)flow for $I(u)$.

Historically, minimizing method for heat equation was firstly introduced by Rektorys ([14]). Kikuchi rediscovered this type of method and introduced the form (1.1). By Kikuchi’s rediscovery, many authors treated the problem of this type. For example, Bethuel et. al. treated harmonic mapping ([1]), Nagasawa and Omata treated free boundary problem to seek a stationary point of $I(u)$ without passing the limit of $h \to 0$ ([12]). Koji Kikuchi constructed a varifold solution for non-convex functional $I(u)$. Hyperbolic type problems are also studied. (See [6] and [7].) Recently, Omata developed a numerical method using minimizing method (See [13] and its references.)

2 Convergence and asymptotic behavior

The important estimate on this flow is based on the following property:

$$J_m^h(u_h^m) \equiv \int_\Omega \frac{|u_h^m - u_{m-1}^h|^2}{h} dx + I(u_h^m) \leq J_m^h(u_{m-1}^h) \equiv I(u_{m-1}^h),$$
and therefore we have \( \int_{\Omega} |u_{m}^{h} - u_{m-1}^{h}|^2 / h dx \leq I(u_{m-1}^{h}) - I(u_{m}^{h}). \) Summing up from \( m = 1 \) to \( M \), we have a estimate:

\[
I(u_{M}^{h}) + \sum_{m=1}^{M} \int_{\Omega} \frac{|u_{m}^{h} - u_{m-1}^{h}|^2}{h} dx \leq I(u_{0}). \tag{2.1}
\]

This estimate is a basic estimate of this flow, from which main properties are obtained.

For showing a convergence theory, we define an approximate solution of a heat equation.

**DEFINITION 2.1** We define functions \( \overline{u}^{h} \) and \( u^{h} \) on \( \Omega \times (0, \infty) \) by

\[
\overline{u}^{h}(x, t) = u_{m}^{h}(x), \quad u^{h}(x, t) = \frac{t-(m-1)h}{h} u_{m}^{h}(x) + \frac{mh-t}{h} u_{m-1}^{h}(x),
\]

for \((x, t) \in \Omega \times ((m-1)h, mh]\).

Firstly, we mention convergence theory when \( h \) tends to zero. By use of (2.1), if \( F \) is coercive in \( H^{1}(\Omega) \), we can easily obtain facts that the following norms are uniformly bounded with respect to \( h \):

\[
\left\| \frac{\partial u^{h}}{\partial t} \right\|_{L^{2}((0, \infty) \times \Omega)}, \left\| \nabla u^{h} \right\|_{L^{2}((0, \infty); L^{2}(\Omega))}, \left\| \nabla u^{h} \right\|_{L^{\infty}((0, \infty); L^{2}(\Omega))},
\]

\[
\left\| u^{h} \right\|_{L^{\infty}((0, \infty); L^{2}(\Omega))}, \left\| \nabla \overline{u}^{h} \right\|_{L^{\infty}((0, \infty); L^{2}(\Omega))}, \left\| u^{h} \right\|_{W^{1, 2}((0, \infty) \times \Omega)}, \text{for all } T > 0.
\]

By use of these estimates, we see that by a suitable choice of subsequence \( \{h_{j}\} (h_{j} \to 0, j \to \infty) \), approximate solutions may converge to a weak solution \( u \) (a limit function) in some topology. Thus we can say:

**THEOREM 2.2** If \( F \) satisfies good conditions, then a limit function is a weak solution, i.e.

\[
\int_{\Omega} u_{0} \eta(x, 0) dx = \int_{0}^{T} \int_{\Omega} D_{t} u \eta dx dt + \int_{0}^{T} \int_{\Omega} (F_{p_{\alpha}^{i}}(x, u, \nabla u) D_{\alpha} \eta^{i} + F_{z^{i}}(x, u, \nabla u) \eta^{i}) dx dt \tag{2.2}
\]

for all \( \eta \in \tilde{W}^{1, 1}((0, T) \times \Omega) \) with \( \eta(x, T) = 0 \), where \( \tilde{V}_{2}((0, T) \times \Omega) = \{ u \in L^{2}(Q_{T}), u_{x} \in L^{2}(Q_{T}); |u|_{Q_{T}} = \text{ess sup}_{0 \leq t \leq T} ||u(x, t)||_{L^{2}(\Omega)} + ||u_{x}||_{L^{2}(Q_{T})} < \infty \} \).

For above theorem, very strong conditions for \( F \) is requested, so some mathematicians are trying to relax them. (See [9] for example.)

Secondly, we can get a result on asymptotic behavior of the D.M.F. \( \{u_{m}\} \) as \( m \to \infty \). From (2.1), we easily have \( ||u_{m} - u_{m-1}||_{L^{2}(\Omega)} \to 0 \) as \( m \to \infty \). Again, by (2.1), for any subsequence \( \{u_{m_{j}}\} \subset \{u_{m}\} \), there exists a subsequence \( \{u_{m_{j_{\nu}}}\} \subset \{u_{m_{j}}\} \) and a function \( u_{\infty} \) on \( \Omega \) such that \( u_{m_{j_{\nu}}} \to u_{\infty} \) weakly in \( W^{1, 2}(\Omega) \), and strongly in \( L^{2}(\Omega) \), as \( \nu \to \infty \). Moreover, we have:

**THEOREM 2.3** If \( F \) satisfies good conditions, the limit function \( u_{\infty} \) is a minimizer of the functional

\[
J_{\infty}(u) = \int_{\Omega} \left( \frac{|u - u_{\infty}|^2}{h} + F(x, u, \nabla u) \right) dx
\]

in \( K \), hence, \( u_{\infty} \) satisfies \( -\int_{\Omega} (F_{p_{\alpha}^{i}}(x, u, \nabla u) D_{\alpha} \phi^{i} + F_{z^{i}}(x, u, \nabla u) \phi^{i}) dx = 0 \) for any \( \phi \in C_{0}^{\infty}(\Omega) \).

Sometimes regularity for \( u \) is obtained from this minimality ([12]). In other word, \( u_{\infty} \) is not a simple stationary point to \( I(u) \), but expected to have some regular properties.
3 Numerical Examples

By use of DMF, we can construct a simple algorithm (i.e. Nonlinear Optimization) for solving heat type equations numerically ([13]). Compared with the Gralerkin method, in some problem, it works well and usually the structure would be simpler. We show two types of examples.

3.1 Ginzburg-Landau energy

In this experiment, we treat \( u : B_1(0) \rightarrow \mathbb{R}^2 \)

\[
I(u) = \int_{B_1(0)} \left( |\nabla u|^2 + \frac{1}{\delta} (|u|^2 - 1) \right) dx.
\]

The boundary data is Dirichlet with 4 degrees and as a initial data we put 1 vortex with 4 degrees. The vortex splits into 4 vortices with 1 degree.

In this case, the boundary data is Neumann, and put 2 vortices with 1 degree as a initial data. They push each other and eventually they are going out.
3.2 A Free Boundary Problem

In these experiment, we treat \((u : B_1(0) \to \mathbb{R})\) the following free boundary \((\partial \{ u > 0 \})\) problem

\[
I(u) = \int_{B_1(0)} (|\nabla u|^2 + \chi_{u>0})\,dx.
\]

The difference of boundary data causes a different conclusions.

In this experiment boundary data is high enough and the limit function becomes constant.

\[
t = 0 \quad t = 0.1 \quad t = \infty
\]

In this experiment the boundary data is low so that the limit function keeps the free boundary.

\[
t = 0 \quad t = 0.04 \quad t = \infty
\]

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References


