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Communication leading to Nash equilibrium

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To Shoji Koizumi on the occasion of his 77th birthday

ABSTRACT. A pre-play communication-process is presented which leads to a Nash equilibrium of a strategic form game. In the communication process each player predicts the other players' actions, and he/she communicates privately his/her conjecture through message according to a protocol. All the players receiving the messages learn and revise their conjectures. After a long round of the communications they reach a Nash equilibrium: We show that the profile of players' conjectures in the revision process leads a Nash equilibrium of a game in the long run if the protocol contains no cycle.

1. INTRODUCTION

The concept of Nash equilibrium (J.F. Nash [10]) has become central in game theory, economics and its related fields. Yet a little is known about the process by which players learn if they do. Recent papers by E. Kalai and E. Lehrer [6], J. S. Jordan [5] (and references in therein) indicate increasing interest in the mutual learning processes in games that leads to equilibrium.

They have studied the learning processes modeled by Bayesian updating: Each player starts with initial erroneous belief regarding the actions of all the other players. They show that if each player assigns a positive

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probability to the real action played by the others, their belief about the future actions of the others converge in the long run.

E. Kalai and E. Lehrer [6] studies two-player repeated games, and they show the two strategies converges to an ε-Nash equilibrium of the repeated game if the common prior belief satisfies a certain uniform condition. J. S. Jordan [5] investigates the general convergence result for strategic form games. R. B. Myerson [9] proposes the Bayesian games with mediated communication in which each player is asked to confidentially report his type to the meditator, after getting these reports, the meditator confidentially recommends an action to each player. He characterizes the acceptable correlated equilibria as a subclass of the correlated equilibria in the Bayesian games.

As for as Nash's fundamental notion of strategic equilibrium is concerned, R.J. Aumann and A. Brandenburger [1] gives epistemic conditions for Nash equilibrium. However it is not clear just what learning process leads to Nash equilibrium.

The present article aims to fill this gap. The pre-play communication process according to a protocol is proposed. It is a mutual learning that leads to a Nash equilibrium of a strategic form game such as a cheap talk proceeding as follows: The players start with the same prior distribution on a state-space. In addition they have private information which is given by a non-partitional structure. Each player communicates privately his/her belief about the other players' actions through messages, and accordingly the receiver of the message updates her/his belief. When a player communicates with another, the other players are not informed about the contents of the message. The players' predictions regarding the future beliefs converge in the long run, which lead to a Nash equilibrium of a game. Precisely, at every stage each player communicates privately not only his/her belief about the others' actions but also his/her rationality as messages according to a protocol, the receivers update their private information and revise their belief. Where each message is not required to become common-knowledge among all players. Then we prove:

Theorem. *In a communication process of a strategic form game according to a protocol with revisions of their beliefs about the other players' actions, their predictions induces a Nash equilibrium of the game in the long run if the protocol contains no cycle.*

This paper organizes as follows. Section 2 presents the communication process for a game according a protocol. In Section 3 we give the statement and proof of the theorem (Theorem(3.1)), assuming the technical result (Fundamental lemma (3.2)). Section 4 gives the proof of the lemma.

2. The Model

Let Ω be a non-empty set called a *state-space*, N a set of finitely many *players* $1, 2, \dots n$, and let 2^Ω be the family of all subsets of Ω. Each member of 2^Ω is called an *event* and each element of Ω called a *state*. Let μ be a probability measure on Ω which is common for all players.

2.1. **Information and Knowledge** (Samet [13], Binmore [3]). An *information structure* $(P_i)_{i\in N}$ is a class of mappings P_i of Ω into 2^Ω. It is called an *RT-information structure* if for every player i the two properties are true: For each ω of 2^Ω,

Ref: $\omega \in P_i(\omega)$;

Trn: $\xi \in P_i(\omega)$ implies $P_i(\xi) \subseteq P_i(\omega)$.

Given our interpretation, an player i for whom $P_i(\omega) \subseteq E$ knows, in the state ω, that some state in the event E has occurred. In this case we say that in the state ω the player i knows E. An i's *knowledge operator* is an operator K_i on 2^Ω such that K_iE is the set of states of Ω in which i knows that E has occurred; that is,

$$K_iE = \{\omega \in \Omega | P_i(\omega) \subseteqq E\}. \tag{1}$$

We note that the i's knowledge operator satisfies the following properties: For every E, F of 2^Ω,

N: $K_i\Omega = \Omega$ and $K_i\emptyset = \emptyset$;

K: $K_i(E \cap F) = K_iE \cap K_iF$;

T: $K_iF \subseteq F$;

4: $K_iF \subseteq K_iK_iF$.

The set $P_i(\omega)$ will be interpreted as the set of all the states of nature that i believes to be possible at ω, and K_iE will be interpreted as the set of states of nature for which i believes E to be possible. We will therefore call P_i an i's *possibility operator* on Ω and also will call $P_i(\omega)$ the i's *possibility set* at ω. An event E is said to be an i's *truism* if $E \subseteqq K_iE$

We should note that the RT-information structure P_i is uniquely determined by the knowledge operator K_i such that $P_i(\omega) = \bigcap_{\omega\in K_iE} E = \bigcap_{\omega\in T=K_iT} T$.

2.2. **Game and Knowledge** (Aumann and Brandenburger [1]). By a *game* G we mean a *finite* strategic form game $\langle N, (A_i), (g_i)\rangle$ with the following structure and interpretations: N is a finite set of players $\{1, 2, \dots, i, \dots n\}$ with $n \geq 2$, A_i is a finite set of i's *actions* (or i's pure strategies) and g_i is an i's *payoff-function* of A into $\mathbb{R}$, where A denotes the product $A_1 \times A_2 \times \cdots \times A_n$, A_{-i} the product $A_1 \times A_2 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_n$. We denote by g the n-tuple $(g_1, g_2, \dots g_n)$ and denote by a_{-i} the $(n-1)$-tuple $(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$ for a of A .

A probability distribution ϕ_i on A_{-i} is said to be an i's *overall conjecture* (or simply i's *conjecture*). For each player j other than i, this induces the marginal on j's actions; we call it an i's *individual conjecture* about j (or simply i's conjecture *about* j.) Functions on Ω are viewed like random variables in a probability space (Ω, μ). If $\mathbf{x}$ is a such function and x is a value of it, we denote by $[\mathbf{x} = x]$ (or simply by $[x]$) the set $\{\omega \in \Omega | \, \mathbf{x}(\omega) = x\}$.

An RT-information structure (P_i) with a common-prior μ yields the overall conjecture $\boldsymbol{\phi}_i$ defined by

$$\boldsymbol{\phi}_i(a_{-i}, \omega) = \mu([\mathbf{a}_i = a_i] | P_i(\omega));$$

it is viewed as a random variable of ϕ_i. We denote by $[\boldsymbol{\phi}_i = \phi_i]$ the intersection $\bigcap_{a_{-i} \in A_{-i}} [\boldsymbol{\phi}_i(a_{-i}) = \phi_i(a_{-i})]$ and denote by $[\phi]$ the intersection $\bigcap_{i \in N} [\boldsymbol{\phi}_i = \phi_i]$. Let $\mathbf{g}_i$ be a random variable of an i's payoff-function g_i and $\mathbf{a}_i$ a random variable of an i's action a_i. Where we assume that $[a_i] := [\mathbf{a}_i = a_i]$ is i's truism for every a_i of A_i. The pay-off functions $g = (g_1, g_2, \dots, g_n)$ is said to be *actually played* at a state ω if ω belongs to $[\mathbf{g} = g] := \bigcap_{i \in N} [\mathbf{g}_i = g_i]$. An i's action a_i is said to be *actual* at a state ω if ω belongs to the set $[\mathbf{a}_i = a_i]$.

An player i is said to be *rational* at ω if each i's actual action a_i maximizes the expectation of his actually played pay-off function g_i at ω when the other players actions are distributed according to his conjecture $\boldsymbol{\phi}_i(\omega)$: Formally, letting $g_i = \mathbf{g}_i(\omega)$ and $a_i = \mathbf{a}_i(\omega)$,

$$\mathrm{Exp}(g_i(a_i, \mathbf{a}_{-i}); \omega) \geqq \mathrm{Exp}(g_i(b_i, \mathbf{a}_{-i}); \omega)$$

for every b_i in A_i.[1] Let R_i denote the set of all the states at which an player i is rational, and R the intersection $\cap_{j \in N} R_j$.

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[1]The expectation Exp is defined by

$$\mathrm{Exp}(g_i(b_i, \mathbf{a}_{-i}); \omega) := \sum_{a_{-i} \in A_{-i}} g_i(b_i, a_{-i}) \; \boldsymbol{\phi}_i(\omega)(a_{-i}) \; .$$

2.3. **Protocol** (Parikh and Krasucki [11], Krasucki [7]). We assume that players communicate by sending *messages*. Let T be the time horizontal line $\{0, 1, 2, \cdots t, \cdots\}$.

A *protocol* is a mapping Pr of the set of non-negative integers into the Cartesian product $N \times N$ that assigns to each t a pair of players $(s(t), r(t))$ such that $s(t) \neq r(t)$. Here t stands for *time* and $s(t)$ and $r(t)$ are, respectively, the *sender* and the *receiver* of the communication which takes place at time t. We consider a protocol as the directed graph whose vertices are the set of all players N and such that there is an edge (or an arc) from i to j if and only if there are infinitely many t such that $s(t) = i$ and $r(t) = j$.

A protocol is said to be *fair* if the graph is strongly-connected; in words, every player in this protocol communicates directly or indirectly with every other player infinitely often. It is said to contain a *cycle* if there are players $i_1, i_2, \ldots, i_k$ with $k \geqq 3$ such that for all $m < k$, i_m communicates directly with i_{m+1}, and such that i_k communicates directly with i_1. The *period* of the protocol is the minimal number of all the natural number m such that $\Pr(t + m) = \Pr(t)$ for every t.

2.4. **Pre-play Communication.** By this we intuitively mean the learning process such that each player communicates privately his/her belief about the other players' actions through messages according to a protocol, and she/he updates her/his belief according to the message received. In addition, at every stage each player communicates privately not only his/her belief about the others' actions but also his/her rationality as messages, the receivers update their private information and revise their belief. When a player communicates with another, the other players are not informed about the contents of the message.

Formally, a *pre-play communication process* according to a protocol for a game G with revisions of players' conjectures is a tuple

$$\langle \Pr, (P_i^t)_{i\in N}, (\phi_i^t)_{i\in N} |\ t \in T \rangle$$

with the following structures: the players have a common-prior μ on a state-space Ω, a protocol $\Pr(t) = (s(t), r(t))$ satisfies the conditions that $r(t) = s(t+1)$ for every t and that the communications proceed in *rounds* (i.e. there exists a time m such that $\Pr(t) = \Pr(t + m)$ for all t.) An n-tuple $(\phi_i^t)_{i\in N}$ is a profile of i 's individual conjectures at time t. The i's information structure P_i^t at time t is the mapping of Ω into 2^Ω defined inductively as follows:

Set $P_i^0(\omega) = P_i(\omega)$. If $i = s(t)$ is a sender at t, $\phi_{s(t)}^t$ is the message sent by i to $j = r(t)$ at t. Assume that P_i^t is defined. It yields the overall conjecture $\phi_i^t(a_{-i},\omega) = \mu([\mathrm{a}_i = a_i]|P_i^t(\omega))$, whence we denote by R_i^t the set of all the state ω at which i is *rational* according to his conjecture $\phi_i^t(\omega)$: i.e., each i's actual action s_i maximizes the expectation of his pay-off function g_i being actually played at ω when the other players actions are distributed according to his conjecture $\phi_i^t(\omega)$ at time t.[2] Let Φ_i^t denote the partition of Ω that is decomposed into the components $\Phi_i^t(\omega)$ consisting of all the states ξ such that $\phi_i^t(\xi) = \phi_i^t(\omega)$. Denote by $\mathcal{G}_i$ the partition $\{[\mathrm{g}_i = g_i^t],\ \Omega \setminus [\mathrm{g}_i = g_i]\}$ of Ω, and R_i^t the partition $\{R_i^t, \Omega \setminus R_i^t\}$. Let W_i^t denote the join $\mathcal{G}_i \vee \Phi_i^t \vee \mathrm{R}_i^t$ that is the partition of Ω generated by $\mathcal{G}_i$, Φ_i^t and R_i^t.[3] Then P_i^{t+1} is defined as follows: If i is a receiver of a message at time $t+1$ then $P_i^{t+1}(\omega) = P_i^t(\omega) \cap W_{s(t)}^t(\omega)$. If not, $P_i^{t+1}(\omega) = P_i^t(\omega)$. It is of worth noting that $(P_i^t)_{i\in N}$ is an *RT*-information structure for every $t \in T$.

We require that the pre-play communication process satisfies the following two conditions: Let K_i^t be the knowledge operator corresponding to P_i^t by (1)[4];

(a) For each $i \in N$ and every $t \in T$, both $[\phi_i^t]$ and R_i^t are i's truisms :
(b) For every $t \in T$, the intersection $\bigcap_{i\in N} K_i^t([g_i] \cap [\phi_i^t] \cap R_i^t)$ is not empty.

The specification of (a) is that each player's conjecture and his/her rationality are truism, and the specification of (b) is that each player knows his/her pay-off, rationality and conjecture at every time t.

2.5. **Remark.** For every player i, the sequence of correspondences $\{P_i^t \mid t = 0,1,2,\dots\}$ is stationary in finitely many rounds. Furthermore so is the sequence of i's conjectures $\{\phi_i^t \mid t = 0,1,2,\dots\}$ in finitely many rounds. That is, there is a sufficiently large time $\tau \in T$ such that for every i, for all $\omega \in \Omega$ and for all $t \geq \tau$, $P_i^t(\omega) = P_i^\tau(\omega)$, and therefore $\phi_i^t = \phi_i^\tau$.

In fact, the sequence $\{P_i^t(\omega) \mid t = 0,1,2,\dots\}$ is a descending chain in 2^Ω. Since Ω is finite it immediately follows that there exists a time τ

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[2] Formally, letting $g_i = \mathrm{g}_i(\omega)$, $a_i = \mathrm{a}_i(\omega)$, the expectation at time t, Exp^t, is defined by

$$\mathrm{Exp}^t(g_i(b_i,\mathrm{a}_{-i});\omega) := \sum_{a_{-i}\in A_{-i}} g_i(b_i,a_{-i})\ \phi_i^t(\omega)(a_{-i}).$$

An player i is said to be rational according to his conjecture $\phi_i^t(\omega)$ at ω if for all b_i in A_i,

$$\mathrm{Exp}^t(g_i(a_i,\mathrm{a}_{-i});\omega) \geqq \mathrm{Exp}^t(g_i(b_i,\mathrm{a}_{-i});\omega).$$

[3] Therefore the component $W_i^t(\omega) = [g_i] \cap [\phi_i^t] \cap R_i^t$ if $\omega \in [g_i] \cap [\phi_i^t] \cap R_i^t$.

[4] That is, K_i^t is defined by $K_i^t E = \{\omega \in \Omega | P_i^t(\omega) \subseteqq E\}$.

such that $P_i^\tau(\omega) = P_i^{\tau+1}(\omega) = P_i^{\tau+2}(\omega) = \cdots$ for all ω. Hence for all $t \geq \tau$, we can observe that $\phi_i^t = \phi_i^\tau$ as required.

3. The Result

We now state and prove the main result:

3.1. **Theorem.** *Suppose that the players in a strategic form game have a common-prior. In a pre-play communication process according to a protocol for the game with revisions of their conjectures* $\{(\phi_i^t)_{i\in N} \mid t = 0, 1, 2, \dots\}$, *there exists a time* τ *such that for each* $t \geq \tau$, *the n-tuple* $(\phi_i^t)_{i\in N}$ *induces a Nash equilibrium of the game if one of the following conditions is true.*

(i) *The protocol contains no cycle.*
(ii) *Any two players communicate directly to each other:*

Assuming the result (Fundamental lemma) we complete the proof of the theorem: A non-empty event H is said to be P_i-*invariant* if for every ξ of H, $P_i(\xi)$ is contained in H.

3.2. **Fundamental lemma.** [5] *Let* $(P_i)_{i\in N}$ *be an RT-information structure with* μ *a common-prior. Let* X *be an event and* q_i *the player i's posterior of* X*; that is,* $q_i = \mu(X|P_i(\omega))$. *If there is an event* H *such that the following two conditions* (a), (b) *are true, then we obtain that* $\mu(X|H) = q_i$:

(a) H *is non-empty and it is* P_i*-invariant,*
(b) H *is contained in* $[q_i] := \{\omega \in \Omega \mid \mu(X|P_i(\omega)) = q_i\}$. □

We let τ be the time of T in Remark (2.5) and t an arbitrary element of T with $t \geq \tau$. Let ω_t be an state that belongs to $\bigcap_{i\in N} K_i^t([g_i]\cap[\phi_i^t]\cap R_i^t) \subseteqq \bigcap_{i\in N}([g_i]\cap[\phi_i^t]\cap R_i^t)$. The following result is the another key to proving Theorem (3.1):

3.3. **Proposition.** *In a pre-play communication process of a game with revisions of their conjectures* $\{(\phi_i^t)_{i\in N} \mid t = 0, 1, 2, \dots\}$, *if the protocol has no cycle then both the marginals of the conjectures* ϕ_i^t *and* ϕ_j^t *on* A_{-i-j} *must coincide; that is,* $\phi_i^t(a_{-i-j}) = \phi_j^t(a_{-i-j})$ *for all* $a \in A$.

Before proceeding with, we prove that

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[5] A similar result is implicitly appeared in D. Samet [12] (Theorem 7), and also it is explicitly appeared with the sketchy proof in T. Matsuhisa and K. Kamiyama [8] (Fundamental lemma). Here we shall give the detailed proof for its importance and for the readers' convenience.

3.3.1. *Lemma. In a rational pre-play communication process of a game with revisions of their conjectures, if a player i communicates his/her message directly to another player j then both the marginals of the conjectures ϕ_i^t and ϕ_j^t on A_{-i-j} must coincide; that is, $\phi_i^t(a_{-i-j}) = \phi_j^t(a_{-i-j})$ for all $a \in A$.*

Proof. We denote H by $[\phi_i^t] \cap [\phi_j^t]$ which is not empty because ω_t belongs to it. It can plainly be observed the two points: First that H is contained in $[\phi_i^t(a_{-i-j})] \cap [\phi_j^t(a_{-i-j})]$ for every $a \in A$ and secondly that H is both P_i^t-invariant and P_j^t-invariant by the definition of P_i^t. In view of Fundamental lemma (3.2) it follows that $\mu([a_{-i-j}]|H) = \phi_i^t(a_{-i-j}) = \phi_j^t(a_{-i-j})$, in completing the proof. □

3.3.2. *Proof of Proposition (3.3).* [6] We note that the protocol Pr has the property: If i and j are distinct players with $\mathrm{Pr}(t) = (i,j)$ then $\mathrm{Pr}(t+m) = (j,i)$ for some $m \in \mathbb{N}$, because Pr has no cycle. Viewing Lemma(3.3.1) we may assume the number of players in a pre-play communication process is at least three.

Suppose to the contrary that there exists at least one pre-play communication - process that is fair and contains no cycle with the property: There are two distinct players k, l such that $\phi_k^t(a_{-k-l}) \neq \phi_l^t(a_{-k-l})$ for some $a \in A$.

We can take one example such that the period of it is the minimal in all those of such pre-play communication processes. Since the protocol contains no cycle, there exists two players i, j such that i communicates his/her message directly to j and j sends his/her message directly back to j. It follows from the above lemma that the marginals of ϕ_i^t and ϕ_j^t on A_{-i-j} must coincide. By removing the pre-play communication process between i and j, we can modify the example into the new pre-play communication process whose number of players is lesser than that of players in the preceding pre-play communication process. The new protocol containing k, l as vertices is still fair and it contains no cycle such that $\phi_k^t(a_{-k-l}) \neq \phi_l^t(a_{-k-l})$ for some $a \in A$. This contradicts the minimality of the period of the first example, in completing the proof. □

3.4. **Proof of Theorem (3.1).** We denote by $\Gamma(i)$ the set of all the players that directly receive the message from i on N; i.e., $\Gamma(i) = \{\, j \in N \mid (i,j) = \mathrm{Pr}(t) \text{ for some } t \in T\}$. For any subset I of N denote $a_{-I} := (a_i)_{i \in N \setminus I}$.

[6]The discussion below follows the line of Krasucki [7].

3.4.1. *Proof for (i):* For each $i \in N$, we denote $[g_i] \cap [\phi^t] \cap R^t$ by F_i. It is noted that F_i is a non-empty P_i-invariant set because $\emptyset \neq \bigcap_{i\in N}([g_i] \cap [\phi_i^t] \cap R_i^t) \subseteq F_i$ and because $P_i^t(\omega) \subseteq F_i$ for every $\omega \in F_i$ by the definition. We observe the first point that for each $i \in N$, $j \in \Gamma(i)$ and for every $a \in A$,

$$\mu([a_{-j}] \mid F_i \cap F_j) = \phi_j^t(a_{-j}): \tag{2}$$

For, we note that $F_i \cap F_j \subseteq [\phi_j^t(a_{-j})]$ and $F_i \cap F_j$ is P_j-invariant because $j \in \Gamma(i)$. Hence by Fundamental lemma (3.2), we plainly obtain (2) as required. Then summing over a_i, we obtain that

$$\mu([a_i] \mid F_i \cap F_j) = \phi_j^t(a_i) \ \text{ for any } a \in A; \tag{3}$$

and therefore that $\phi_j^t(a_i)$ is independent of the choices of every $j \in \Gamma(i)$.

We set the probability distribution σ_i on A_i by $\sigma_i(a_i) := \phi_j^t(a_i)$, and the profile $\sigma = (\sigma_i)$. We observe the second point that for every $a \in \prod_{i\in N} \mathrm{Supp}(\sigma_i)$,

$$\phi_i^t(a_{-i}) = \sigma_1(a_1)\cdots\sigma_{i-1}(a_{i-1})\sigma_{i+1}(a_{i+1})\cdots\sigma_n(a_n): \tag{4}$$

In fact, viewing the definition of σ_i we shall show that $\phi_i^t(a_{-i}) = \prod_{k\in N\setminus\{i\}} \phi_i^t(a_k)$. To verify this it suffices to show that for every $k = 1, 2, \cdots, n$,

$$\phi_i^t(a_{-i}) = \phi_i^t(a_{-I_k}) \prod_{k\in I_k\setminus\{i\}} \phi_i^t(a_k): \tag{5}$$

We prove by induction on k. For $k = 1$ the result is immediate. Suppose it is true for $k \geq 1$. On noting the protocol is fair, we can take the sequence of sets of players $\{I_k\}_{1\leq k\leq n}$ with the following properties:

(a) $I_1 = \{i\} \subsetneqq I_2 \subsetneqq \cdots \subsetneqq I_k \subsetneqq I_{k+1} \subsetneqq \cdots \subsetneqq I_n = N$:

(b) For every $k \in N$ there is a player $i_{k+1} \in \bigcup_{j\in I_k} \Gamma(j)$ with $I_{k+1} \setminus I_k = \{i_{k+1}\}$.

We let take $j \in I_k$ such that $i_{k+1} \in \Gamma(j)$. Set $H_{i_{k+1}} := [a_{i_{k+1}}] \cap F_j \cap F_{i_{k+1}}$. We note that $H_{i_{k+1}}$ is not empty because $\sigma_i(a_i) = \phi_j^t(a_i) = \mu([a_i] \mid F_i \cap F_j) \gneqq 0$ in viewing of (3), and we note that $H_{i_{k+1}}$ is $P_{i_{k+1}}^t$- invariant which is included in $[\phi_{i_{k+1}}^t(a_{-j-i_{k+1}})]$. It immediately follows from Fundamental lemma (3.2) that $\mu([a_{-j-i_{k+1}}] \mid H_{i_{k+1}}) = \phi_{-j-i_{k+1}}^t(a_{-j})$. Dividing $\mu(F_j \cap F_{i_{k+1}})$ yields that

$$\mu([a_{-j}] \mid F_j \cap F_{i_{k+1}}) = \phi_{i_{k+1}}^t(a_{-j})\mu([a_{i_{k+1}}] \mid F_j \cap F_{i_{k+1}}).$$

In viewing of (2) and (3) it follows $\phi_j^t(a_{-j}) = \phi_{i_{k+1}}^t(a_{-j-i_{k+1}})\phi_j^t(a_{i_{k+1}})$; then summing over a_{I_k} we obtain $\phi_j^t(a_{-I_k}) = \phi_{i_{k+1}}^t(a_{-I_k-i_{k+1}})\phi_j^t(a_{i_{k+1}})$. It immediately follows from Proposition (3.3) that

$$\phi_i^t(a_{-I_k}) = \phi_i^t(a_{-I_k - i_{k+1}})\phi_i^t(a_{i_{k+1}}).$$

Viewing (4) we have just observed that

$$\phi_i^t(a_{-i}) = \phi_i^t(a_{-I_{k+1}}) \prod_{k \in I_{k+1} \setminus \{i\}} \phi_i^t(a_k),$$

as required.

Therefore each action a_i with $\phi_i^t(a_i) \gneqq 0$ for some $j \in \Gamma(i)$ maximizes g_i against ϕ_i^t because $a_i = \mathbf{a}_i(\omega_i)$, $g_i = \mathbf{g}_i(\omega_i)$ and $\phi_i^t = \boldsymbol{\phi}_i^t(\omega_i)$ at some state ω_i of $H_i = [a_i] \cap F_i \cap F_j$. Viewing (4) we conclude that each action a_i appearing with positive probability in σ_i maximizes g_i against the product of the distributions σ_l with $l \neq i$. This implies that the profile $\sigma = (\sigma_i)_{i \in N}$ is a Nash equilibrium of G, in completing the proof. □

3.4.2. *Proof for (ii):* For each agent i, we set $[g] \cap [\phi] \cap R$ by F and $[\mathbf{a}_i = a_i] \cap F$ by H_i. We note that F is non-empty and it is P_i-invariant because $\omega_t \in F$ and because any distinct two players communicate directly to each other. We can observe that F is a common-knowledge at ω_t[7]. While the conclusion follows by the similar discussion on Theorem B in Aumann and Brandenburger [1], we shall give the detail proof for completeness:

We set the probability distribution Q on A by $Q(a) = \mu([a]|F)$. Let $Q(a_i)$ denote the marginal of Q on A_i and $Q(a_{-i})$ denote the marginal of Q on A_{-i}. We define a probability distribution σ_j on A_j by $\sigma_j(a_j) = Q(a_j)$ for each j. Let $\text{Supp}(\sigma_j)$ denote the support of σ_j. We note that for every agent i, if a_j belongs to $\text{Supp}(\sigma_j)$ then $H_j := [\mathbf{a}_j = a_j] \cap F$ is non-empty and it is P_j-invariant: For it follows from $\sigma_j(a_j) > 0$ that $\mu([\mathbf{a}_j = a_j] \cap F) \neq 0$ and that H_j is non-empty. On noting that both F and $[a_j]$ are P_j-invariant, we can observe that H_j is also P_j-invariant.

We observe the point that: For every agent i, all conjectures ϕ_j^t with $j \neq i$ induces the same distribution σ_i on A_i. In fact, for every agent j and every a of A with a_{-j} of A_{-j} and a_j of $\text{Supp}(\sigma_j)$, we obtain by Fundamental Lemma that $\mu([a_{-j}]|H_j) = \phi_j^t(a_{-j})$ because $H_j \subseteqq [\boldsymbol{\phi}_j^t(a_{-j}) = \phi_j^t(a_{-j})]$. Dividing by $\mu(F)$ yields that $\mu([a]|F) = \phi_j^t(a_{-j})\mu([a_j]|F)$. This means that

$$Q(a) = \phi_j^t(a_{-j})Q(a_j). \tag{6}$$

Summing up over a_j we obtain that for every a_{-j} of A_{-j},

$$Q(a_{-j}) = \phi_j^t(a_{-j}). \tag{7}$$

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[7]An event F is said to be common-knowledge at ω if ω belongs to $\bigcap_{\{i_1,i_2,\cdots,i_k\} \subseteq N, k \in \mathbf{N}} K_{i_1}^t K_{i_2}^t \cdots K_{i_k}^t F$.

Therefore we can plainly observe that for each $i \neq j$, $\phi_j^t(a_i) = Q(a_i) = \sigma_i(a_i)$; that is, for all j the conjecture about i induced by ϕ_j^t is the same distribution σ_i which is independent of j.

By (6) and (7) it follows immediately that for every j and for all a_j of $\mathrm{Supp}(\sigma_j)$, $Q(a) = Q(a_{-j})Q(a_j)$. From this we can verify by induction on $j = 1, 2, \dots, n$ that the distribution ϕ_j^t is the product of σ_j ; that is,

$$\phi_j^t(a_{-j}) = \sigma_1(a_1) \cdots \sigma_{j-1}(a_{j-1})\sigma_{j+1}(a_{j+1}) \cdots \sigma_n(a_n). \tag{8}$$

Therefore we can observe that each action a_j with $\phi_j^t(a_j) = \sigma_j(a_j) > 0$ for some $i \neq j$ maximizes g_j against ϕ_j^t because $a_j = \mathbf{a}_j(\omega_j)$, $g_j = \mathbf{g}_j(\omega_j)$ and $\phi_j^t = \phi_j^t(\omega_j)$ at some state ω_j of H_j. By (8) we conclude that (σ_j) is a Nash equilibrium of G, in completing the proof. □

4. Proof of Fundamental Lemma

We define the equivalence relation $\sim$ on the state-space Ω by

$$\xi \sim \omega \qquad \text{if and only if} \qquad P_i(\xi) = P_i(\omega) \ .$$

We denote by $\Pi_i(\omega)$ the equivalence class of a state ω. Since H is P_i-invariant, it immediately follows that H is decomposed into a disjoint union of components $\Pi_i(\xi)$ for $\xi \in H$. We can observe that each component $\Pi_i(\xi)$ is μ-measurable. We set by $\mathcal{S}$ the class of all the components $\Pi_i(\xi)$ of H such that $\mu(X \mid \Pi_i(\xi)) = q_i$, and denote by S the union of all members of $\mathcal{S}$.

To prove the fundamental lemma it suffices to show that $S = H$. Suppose to the contrary that $S \neq H$, and therefore that S is properly contained in H. We observe the point that there exists a state $\omega_0 \in H \setminus S$ such that $P_i(\xi) \setminus S = P_i(\omega_0) \setminus S$ for every $\xi \in P_i(\omega_0) \setminus S$: For if not, noting that P_i satisfies both (Ref) and (Trn), we can plainly obtain an infinite sequence $\{\omega_n\}$ of states in H such that ω_{n+2} belongs to the set $P_i(\omega_{n+1}) \setminus S$ that is properly contained in $P_i(\omega_n) \setminus S \subseteqq H \setminus S$ for every $n = 0, 1, 2, \dots$, in contradiction to the assumption that Ω is finite as required. Therefore, we can verify that $\Pi_i(\omega_0) = P_i(\omega_0) \setminus S$, and since $\omega_0 \in H \subseteqq [q_i]$ we conclude that $\Pi_i(\omega_0) \in \mathcal{S}$, in final contradiction. This establishes the fundamental lemma. □

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